4 Boundary value problems

- Goal: Solve Poisson's equation
$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \tag{182}$$
for Φ, subject to non-trivial boundary conditions.

- The following subsections present techniques for doing this.

4.1 Laplace's equation

- In regions of space where the charge density $\rho = 0$, Poission's equation reduces to Laplace's equation
$$\nabla^2 \Phi = 0 \tag{183}$$

- Solutions to Laplace's equation are the 'smoothest' possible functions that satisfy the boundary conditions, having no local maxima or minima. This statements will be made more precise below.

- In 1-dimension, Laplace's equation is simply
$$\frac{d^2\Phi}{dx^2} = 0 \tag{184}$$

- The most general solution is a straight line:
$$\Phi(x) = mx + b \tag{185}$$
where the constants m and b are fixed by boundary conditions.

- Suppose the region of interest is the interval $x \in [x_1, x_2]$. Then possible boundary conditions are:

  (i) Specify Φ at both of the endpoints x_1 and x_2.

  (ii) Specify Φ and $d\Phi/dx$ at *one* of the endpoints, x_1 or x_2.

  Note that specifying $d\Phi/dx$ at *both* endpoints will either be redundant (if they have the same value) or inconsistent (if they have different values).

- Note that
$$\Phi(x) = \frac{1}{2}\left[\Phi(x+a) + \Phi(x-a)\right] \tag{186}$$
which implies that $\Phi(x)$ has no local maxima or minima. (See Figure 29.)

- In 2-dimensions, in Cartesian coordinates, Laplace's equation is
$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{187}$$

- One can show that
$$\Phi(x,y) = \frac{1}{2\pi R}\oint_C \Phi(x', y')\, ds' \tag{188}$$
where C is a circle centered at (x, y). (See Figure 30.) This result implies that $\Phi(x, y)$ has no local maxima or minima.

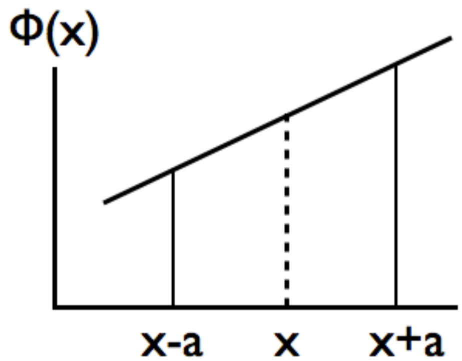

Figure 29: In 1-dimension, the most general solution to Laplace's equation $\nabla^2\Phi \equiv d^2\Phi/dx^2 = 0$ is a straight line. Note that $\Phi(x)$ is the average of its values at two equally-displaced points, $\Phi(x-a)$ and $\Phi(x+a)$.

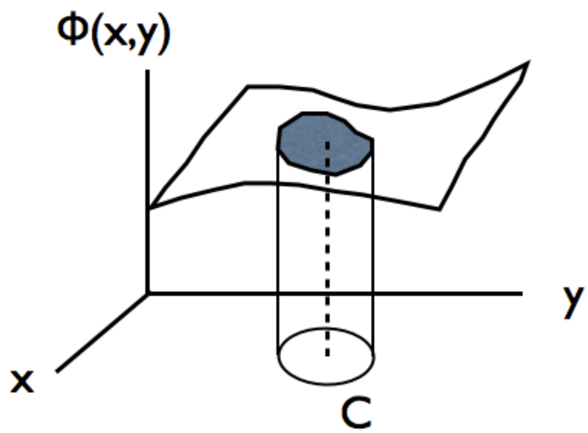

Figure 30: If Φ is a solution to Laplace's equation in 2-dimensions, then the value of Φ at any point (x, y) is the average of its values on a circle C of any radius centered at (x, y).

- Exercise: Prove the above statement using Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz \tag{189}$$

Here $f(z)$ is a complex-valued function that is *analytic* (i.e., complex differentiable) inside and on some closed curve C, and a is any point inside C. Recall that if $f(z)$ is analytic, then the real-valued functions $u(x,y)$, $v(x,y)$ defined by $f = u + i\,v$ satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{190}$$

which in turn imply that they individually satisfy Laplace's equation

$$\nabla^2 u = 0\,, \quad \nabla^2 v = 0 \tag{191}$$

Hint: Take C to be a circle of radius R centered at the point $a = (x,y)$ so that $z - a = R\,e^{i\phi}$ and $dz = iR\,e^{i\phi}\,d\phi$ for points on C. Then split Cauchy's integral formula into its real and imaginary parts.

- In 3-dimensions, one has

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_S \Phi(\mathbf{r}')\, da' \tag{192}$$

where S is a 2-sphere centered at $\mathbf{r}$. Again this implies that $\Phi(\mathbf{r})$ has no local maxima or minima.

- Exercise: Prove this last expression for $\Phi(\mathbf{r})$, assuming a point source located anywhere outside the sphere. For simplicity, you can take $\mathbf{r} = 0$ and put the point source on the z-axis, a distance d from the center of the sphere of radius R ($d > R$). (By the superposition principle, the result is then valid *any* distribution of charge outside the sphere.)

4.2 Boundary conditions and uniqueness theorems

- Theorem: A solution to Poisson's equation $\nabla^2\Phi = -\rho/\epsilon_0$ inside a volume V is uniquely determined (up to an additive constant) by specifying the charge density $\rho(\mathbf{r})$ in V and *either* the potential Φ *or* its normal derivative $\partial\Phi/\partial n$ on the closed boundary surface S.

- Specifying the potential on the boundary is called *Dirichlet* boundary conditions. The potential is uniquely determined for this case—i.e., the additive constant is zero.

- Specifying the normal derivative of the potential on the boundary is called *Neumann* boundary conditions. The additive constant may be non-zero for this case.

- For both Dirichlet and Neumann boundary conditions, the electric field—being the gradient of the potential—is uniquely determined. (The unspecified additive constant for Neumann BCs vanishes when taking the gradient.)

- One can also specify *mixed* boundary conditions, corresponding to specifying Φ on parts of S and $\partial\Phi/\partial n$ on the remaining parts of S.

- NOTE: Specifying both the potential and its normal derivative on the boundary surface *over-specifies* the problem—i.e., the potential may not be able to satisfy both of these conditions.

- Exercise: Prove the uniqueness theorem using Green's 1st identity

$$\int_V (\boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} U + T\nabla^2 U)\, dV = \oint_S (T\boldsymbol{\nabla} U)\cdot \hat{\mathbf{n}}\, da \tag{193}$$

by assuming that Φ_1 and Φ_2 are two solutions to Poisson's equation, and then showing that Φ_1 and Φ_2 differ at most by an additive constant. (Hint: Set $T = U = \Phi_1 - \Phi_2$, noting that $\nabla^2 U = 0$ and $\boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} U \geq 0$.)

- A related uniqueness theorem (see Griffiths, 3rd edition, page 118) is the following: In a volume V surrounded by *conductors* and containing a specified charge density, the electric field is uniquely determined if the total charge on each conductor is given.

4.3 Green's functions: Introduction

- Definition: A Green's function is a solution to a differential equation with a delta-function source. For Poisson's equation,

$$\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\mathbf{r}-\mathbf{r}') \tag{194}$$

  Note the prime on the Laplacian, meaning differentiation wrt $\mathbf{r}'$. (The factor of -4π is chosen for convenience to simplify some of the expressions for Green's functions.)

- Later we will show that $G(\mathbf{r},\mathbf{r}')$ is *symmetric* under interchange of $\mathbf{r}$ and $\mathbf{r}'$—i.e.,

$$G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r}) \tag{195}$$

  Since the Dirac delta function is also symmetric, then it actually doesn't matter whether we are differentiating $G(\mathbf{r},\mathbf{r}')$ with respect to $\mathbf{r}$ or $\mathbf{r}'$.

- Typically one thinks of $\mathbf{r}$ as the observation or field point and $\mathbf{r}'$ as the source point. The symmetry of the Green's function is related to the physical interchangeability of the source and observation points.

- Recall that

$$\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\,\delta(\mathbf{r}-\mathbf{r}') \tag{196}$$

  so $1/|\mathbf{r}-\mathbf{r}'|$ is an example of a Green's function.

- The most general solution is

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} + F(\mathbf{r},\mathbf{r}') \tag{197}$$

  where $F(\mathbf{r},\mathbf{r}')$ is a solution of Laplace's equation $\nabla^2 F(\mathbf{r},\mathbf{r}') = 0$.

- Exercise: Using Green's theorem

$$\int_V (T\nabla'^2 U - U\nabla'^2 T)\,dV' = \oint_S (T\boldsymbol{\nabla}' U - U\boldsymbol{\nabla}' T)\cdot\hat{\mathbf{n}}'\,da' \tag{198}$$

  show that

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,dV' + \frac{1}{4\pi}\oint_S \left[G(\mathbf{r},\mathbf{r}')\frac{\partial\Phi(\mathbf{r}')}{\partial n'} - \Phi(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\right] da' \tag{199}$$

  (Hint: Set $T = \Phi(\mathbf{r}')$, $U = G(\mathbf{r},\mathbf{r}')$.)

- The above equation for $\Phi(\mathbf{r})$ is an *integral equation* for $\Phi(\mathbf{r})$. The RHS cannot be used to calculate $\Phi(\mathbf{r})$ given *arbitrary* values for Φ and $\partial\Phi/\partial n$ on the boundary S, since that would overspecify $\Phi(\mathbf{r})$.

- However, by using the freedom in $F(\mathbf{r},\mathbf{r}')$, one can choose $G(\mathbf{r},\mathbf{r}')$ so that $\Phi(\mathbf{r})$ has the appropriate form for either Dirichlet or Neumann boundary conditions.

- For Dirichlet BCs, one chooses

$$G_D(\mathbf{r},\mathbf{r}')\Big|_S = 0 \tag{200}$$

  Then

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V G_D(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,dV' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{r}')\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\,da' \tag{201}$$

- For Neumann BCs, one chooses

$$\left.\frac{\partial G_N(\mathbf{r},\mathbf{r}')}{\partial n}\right|_S = -\frac{4\pi}{A} \tag{202}$$

where A is the area of S. Then

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V G_N(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,dV' + \frac{1}{4\pi}\oint_S G_N(\mathbf{r},\mathbf{r}')\frac{\partial\Phi(\mathbf{r}')}{\partial n'}\,da' + \langle\Phi\rangle_S \tag{203}$$

where $\langle\Phi\rangle_S$ is the average of Φ over the boundary surface S.

- Note that one cannot simply choose $\partial G_N/\partial n' = 0$ on S, since the divergence theorem implies

$$-4\pi = \int_V \nabla'^2 G_N(\mathbf{r},\mathbf{r}')\,dV' = \int_V \boldsymbol{\nabla}'\cdot\boldsymbol{\nabla}' G_N(\mathbf{r},\mathbf{r}')\,dV' = \oint_S \frac{\partial G_N(\mathbf{r},\mathbf{r}')}{\partial n'}\,da' \tag{204}$$

- Example: If S is the 'boundary' surface at $r\to\infty$, then the associated Dirichlet Green's function is:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} \qquad \text{(for } S \text{ boundary surface at } r\to\infty) \tag{205}$$

This is because the only solution to $\nabla^2 F = 0$ which vanishes as $r\to\infty$ is $F=0$.

- Exercise: Using Green's theorem, prove that a Dirichlet Green's function $G_D(\mathbf{r},\mathbf{r}')$ is symmetric under interchange of $\mathbf{r}$ and $\mathbf{r}'$. Hint: Take $U(\mathbf{r}'') = G_D(\mathbf{r},\mathbf{r}'')$ and $T(\mathbf{r}'') = G_D(\mathbf{r}',\mathbf{r}'')$, where $\mathbf{r}''$ is the integration variable. (Note: Symmetry for a Neumann Green's function $G_N(\mathbf{r},\mathbf{r}')$ is *not* automatic, but can imposed as a separate requirement.)

4.4 Method of images

- Basic idea: Solve Poisson's equation for Φ in some region having non-trivial boundary conditions by enlarging the region to include additional ('image') charges but *no* boundaries.
- The choice of image charges is such that the potential (or its normal derivative) originally specified on the boundary is reproduced by the original charges together with the image charges.
- The method of images is an *indirect* method of solving Poisson's equation. The potential for the problem of multiple charges *without boundary* was actually solved *first*; the physically equivalent problem that is obtained by replacing one of the equipotential surfaces with a conducting surface came afterward.
- Note that the solutions we obtain using the method of images for grounded conducting surfaces (i.e., $\Phi|_S = 0$) will give us the Dirichlet Green's functions associated with those boundary surfaces.

4.4.1 Example 1: Point charge above an infinite, grounded conducting plane

- Consider a point charge q located at a distance d above an infinite, grounded conducting plane. Find the potential in the region above the plane. (See Figure 31, panel (a).)
- Choose coordinates so that the conducting plane is given by $z=0$ and the charge q is located at $z=d$. (See Figure 31, panel (b).)
- An image charge $q_I = -q$ placed at $z=-d$ together with q produces a potential

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right) \tag{206}$$

which vanishes when $z=0$.

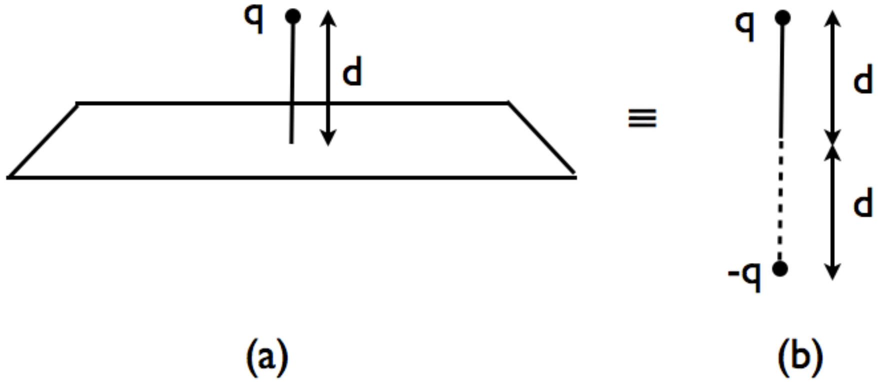

Figure 31: Panel (a): Point charge q a distance d above an infinite, grounded conducting plane. Panel (b): Equivalent problem with image charge $-q$ located a distance $2d$ from q. Note there is no conducting plane for the equivalent image problem.

- By the uniqueness theorems, the above expression is the *unique* solution to Poisson's equation in the region $z > 0$, satisfying $\Phi = 0$ on the conducting plane ($z = 0$).
- The induced surface charge on the conducting plane is given by

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial z}\bigg|_{z=0} = -\frac{qd}{2\pi} \frac{1}{(x^2 + y^2 + d^2)^{3/2}} \tag{207}$$

- The total induced charge, obtained by integrating σ over the surface, is

$$Q \equiv \int_S \sigma \, da = -q \tag{208}$$

  Note that this is the value of the image charge.
- The total force on the point charge q due to the induced charge on the conducting plane is obtained by integrating (minus) the force-per-unit-area

$$\mathbf{f} = \frac{1}{2} \frac{\sigma^2}{\epsilon_0} \, \hat{\mathbf{n}} \tag{209}$$

  over the surface. (Minus since $\mathbf{f}$ is the force-per-unit area *acting on a patch* of the conducting surface.)
- The result is

$$\mathbf{F}_q = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2} \, \hat{\mathbf{z}} \tag{210}$$

- This is the same as the force exerted on q by the image charge $q_I = -q$ at $z = -d$.
- The work required to bring the charge q in from infinity to its location a distance d above the infinite, grounded conducting plane, is given by

$$W = -\int_\infty^d \mathbf{F}_q \cdot d\mathbf{s} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{4d} \tag{211}$$

- Exercise: Prove this.

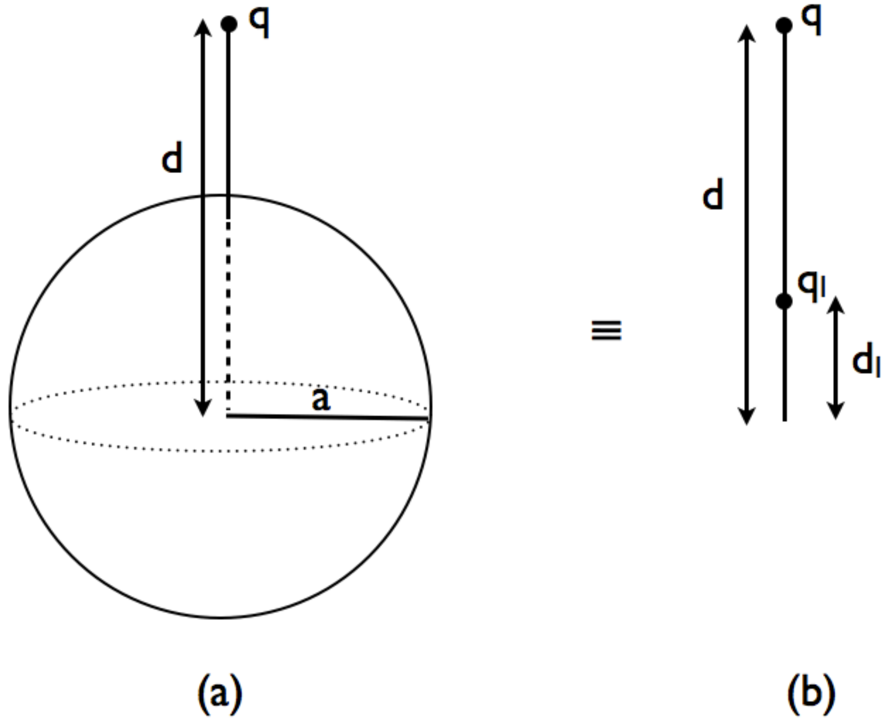

Figure 32: Panel (a): Point charge q a distance d from the center of a grounded conducting sphere of radius a. Panel (b): Equivalent problem with image charge q_I located a distance d_I from the center of the sphere. Note there is no conducting sphere for the equivalent image problem.

- This is *half* the work required to assemble the configuration consisting of the point charge q and image charge $-q$, *without* the conducting surface. The factor of 1/2 arises since no work is done on the induced charge as it moves around on the conducting plane (an equipotential surface) in response to the point charge q being brought in from infinity.

4.4.2 Example 2: Point charge exterior to a grounded conducting sphere

- Consider a point charge q located at a distance $d > a$ from the center of grounded conducting sphere of radius a. Find the potential outside the sphere—i.e., for $r > a$. (See Figure 32, panel (a).)
- Choose spherical polar coordinates so that the center of the sphere is at the origin of coordinates and the point charge is located at $z = d$.
- By symmetry, the image charge q_I will also lie on the z axis at a distance d_I from the center. (See Figure 32, panel (b).)
- To find q_I and d_I, we note that the potential due to q and q_I can be written as

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} + \frac{q_I}{\sqrt{r^2+d_I^2-2rd_I\cos\theta}}\right) \tag{212}$$

- When $|\mathbf{r}| = a$, this can be written as

$$\Phi(|\mathbf{r}| = a) = \frac{1}{4\pi\epsilon_0}\left(\frac{q/a}{\sqrt{1+(d/a)^2 - 2(d/a)\cos\theta}} + \frac{q_I/d_I}{\sqrt{1+(a/d_I)^2 - 2(a/d_I)\cos\theta}}\right) \tag{213}$$

- Exercise: Prove the last equality.
- From this expression, one sees that if

$$\frac{d}{a} = \frac{a}{d_I}\,, \qquad \frac{q_I}{d_I} = -\frac{q}{a} \tag{214}$$

then $\Phi(|\mathbf{r}| = a) = 0$, as desired.

- Thus,

$$d_I = \left(\frac{a}{d}\right) a = \left(\frac{a}{d}\right)^2 d\,, \qquad q_I = -q\,\frac{a}{d} \tag{215}$$

are the location and value of the image charge.

- Note that $d_I < a$ (since $a < d$), so the image charge is outside the region of interest ($r > a$) as it should be.
- Substituting for q_I and d_I, the potential becomes

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} - \frac{1}{\sqrt{a^2 + d^2r^2/a^2 - 2rd\cos\theta}}\right) \tag{216}$$

- The induced charge density on the spherical surface is

$$\sigma = -\epsilon_0 \frac{\partial\Phi}{\partial r}\bigg|_{r=a} = -\frac{q}{4\pi a}\frac{d^2 - a^2}{(a^2 + d^2 - 2ad\cos\theta)^{3/2}} \tag{217}$$

- The total induced charge is

$$Q \equiv \int_S \sigma\, da = -q\frac{a}{d} \tag{218}$$

As we saw for the other example, this is the value of the image charge.

- The total force on the point charge q due to the induced charge on the conducting sphere is obtained by integrating the z-component (extra factor of $\cos\theta$ in the integral) of the force-per-unit area over the sphere:

$$\mathbf{F}_q = -\int_{r=a} \frac{1}{2}\frac{\sigma^2}{\epsilon_0}\,\hat{\mathbf{r}}\,\cos\theta\, da = -\frac{1}{4\pi\epsilon_0}\frac{a}{d}\frac{q^2d^2}{(d^2 - a^2)^2}\,\hat{\mathbf{z}} \tag{219}$$

- This is the same as the force exerted on q by the image charge $q_I = -q(a/d)$ at $z = d_I = a^2/d$.
- The work required to bring the charge q in from infinity to its location a distance d from the center of the sphere, is given by

$$W = -\int_\infty^d \mathbf{F}_q \cdot d\mathbf{s} = -\frac{1}{4\pi\epsilon_0}\frac{q^2 R}{2(d^2 - a^2)} \tag{220}$$

- Exercise: Prove this.
- As for the previous example, this is *half* the work required to assemble the configuration consisting of the point charge q and image charge $q_I = -q(a/d)$, *without* the conducting surface.

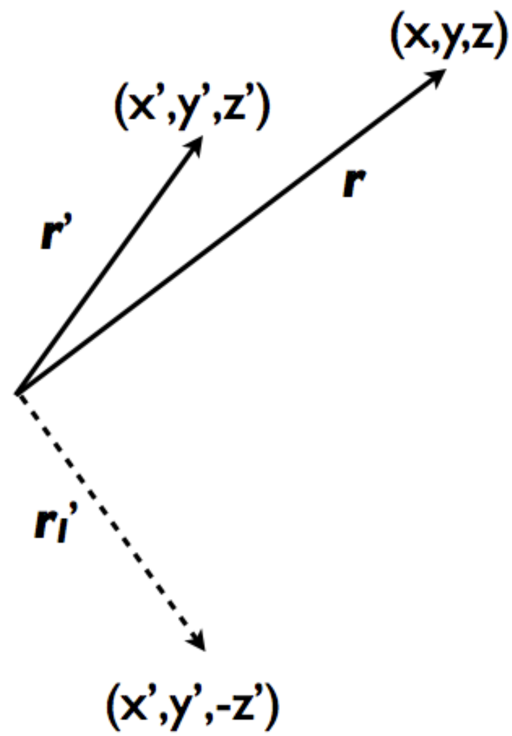

Figure 33: Relationship between the vectors $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}'_I$ for the Dirichlet Green's function for an infinite plane. These three vectors label label the field location, source location, and image charge location, respectively.

4.5 Green's function for an infinite plane

- The Dirichlet Green's function $G_D(\mathbf{r}, \mathbf{r}')$ for an infinite plane is effectively the potential for a point charge in the presence of an infinite, grounded conducting plane. We just need to write the potential in a form that is symmetric with respect to the field point $\mathbf{r}$ and point source location $\mathbf{r}'$.

- Choose coordinates so the plane is at $z = 0$. Then

$$\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}} \quad \text{and} \quad \mathbf{r}'_I = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} - z'\hat{\mathbf{z}} \tag{221}$$

  denote the point source and image charge locations. (See Figure 33.)

- In terms of these quantities, the potential $\Phi(\mathbf{r})$ for the method of images problem is

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'_I|}\right) \tag{222}$$

- Expressed in terms of Cartesian coordinates,

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right) \tag{223}$$

  The RHS is manifestly symmetric under interchange $\mathbf{r} \leftrightarrow \mathbf{r}'$, and vanishes on the plane $z = 0$.

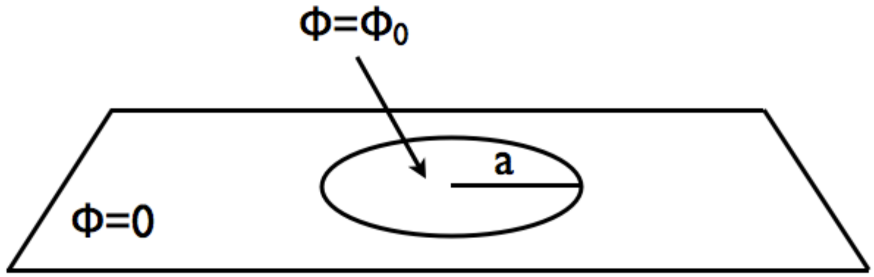

Figure 34: The potential on an infinite plane is specified to equal Φ_0 inside a circular disc of radius a, and to equal zero outside.

- Thus, the Dirichlet Green's function is

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \tag{224}$$

- In terms of $G_D(\mathbf{r},\mathbf{r}')$, the potential $\Phi(\mathbf{r})$ for $z>0$ with arbitrarily prescribed values on the plane $z=0$ is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V G_D(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,dV' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{r}')\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\,da' \tag{225}$$

- For the surface integral, we need

$$\left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\right|_S = -\left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial z'}\right|_{z'=0} = -\frac{2z}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}} \tag{226}$$

  The minus sign is because the outward pointing normal (away from the volume) is in the direction of decreasing z—i.e., $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$.

- If the charge distribution is zero (e.g, if we are interested in a solution of Laplace's equation in the region $z>0$), then the volume integral vanishes and $\Phi(\mathbf{r})$ is given simply by the surface integral.

- Exercise: (Jackson, Prob 2.7) Find the solution to Laplace's equation for $z>0$, where the potential on the plane $z=0$ is prescribed to have the value Φ_0 for a circular disc $x^2+y^2 \leq a^2$, and $\Phi = \Phi_0$ otherwise. Expand the integral in a power series of a/r, where $r^2 = x^2+y^2+z^2$, keeping the first few terms. (See Figure 34.)

- Answer:

$$\Phi(\mathbf{r}) = \frac{\Phi_0}{2}\left(\frac{a}{r}\right)^2\left(\frac{z}{r}\right)\left[1 - \frac{3}{4}\left(\frac{a}{r}\right)^2 + \frac{5}{8}\left(\frac{a}{r}\right)^4\left(1 + 3\,\frac{x^2+y^2}{a^2}\right) + \cdots\right] \tag{227}$$

4.6 Green's function exterior to a sphere

- Just as we saw for the infinite plane, the Dirichlet Green's function $G_D(\mathbf{r},\mathbf{r}')$ exterior to a sphere is given (up to an overall multiplicative constant) by the potential for a point charge exterior to a grounded conducting sphere. We just need to write the potential in a form that is symmetric with respect to the field point $\mathbf{r}$ and point source location $\mathbf{r}'$.

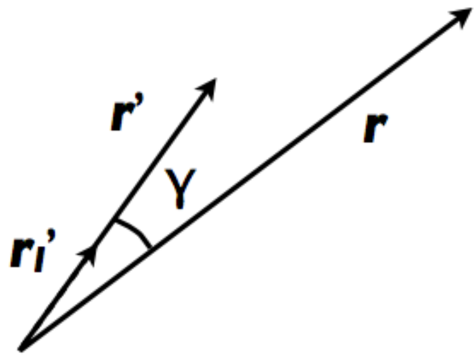

Figure 35: Relationship between the vectors $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}'_I$ for the Dirichlet Green's function exterior to a sphere of radius a. These three vectors label label the field location, source location, and image charge location, respectively. Note that both $r, r' > a$ while $r'_I < a$.

- Since $\mathbf{r}'$ now denotes the location of the point source, the value of the image charge and its location are given by

$$q_I = -q\,\frac{a}{r'}\,, \qquad \mathbf{r}'_I = \left(\frac{a}{r'}\right)^2 \mathbf{r}' \tag{228}$$

(See Figure 35.)

- In terms of these quantities, the potential $\Phi(\mathbf{r})$ for the method of images problem is

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{(a/r')}{|\mathbf{r} - \left(\frac{a}{r'}\right)^2 \mathbf{r}'|}\right) \tag{229}$$

- Expressed in terms of spherical polar coordinates,

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} - \frac{1}{\sqrt{a^2 + \frac{r^2 r'^2}{a^2} - 2rr'\cos\gamma}}\right) \tag{230}$$

where

$$\cos\gamma = \cos\theta\,\cos\theta' + \sin\theta\,\sin\theta'\,\cos(\phi - \phi') \tag{231}$$

is the angle between $\mathbf{r}$ and $\mathbf{r}'$.

- Note that the RHS of the potential is manifestly symmetric under interchange $\mathbf{r} \leftrightarrow \mathbf{r}'$, and vanishes on the sphere $r = a$.

- Thus, the Dirichlet Green's function is

$$G_D(\mathbf{r}, \mathbf{r}') = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} - \frac{1}{\sqrt{a^2 + \frac{r^2 r'^2}{a^2} - 2rr'\cos\gamma}} \tag{232}$$

- In terms of $G_D(\mathbf{r}, \mathbf{r}')$, the potential $\Phi(\mathbf{r})$ exterior to the sphere with arbitrarily prescribed values on the surface of the sphere is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V G_D(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')\,dV' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{r}')\frac{\partial G_D(\mathbf{r}, \mathbf{r}')}{\partial n'}\,da' \tag{233}$$

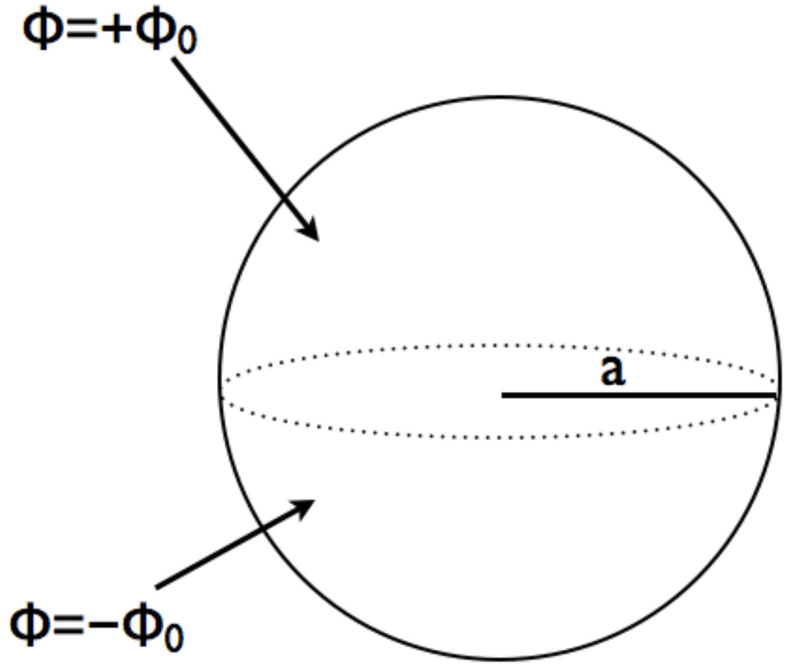

Figure 36: The potential on a sphere of radius a is specified to equal $+\Phi_0$ on the northern hemisphere and $-\Phi_0$ on the southern hemisphere, respectively.

- For the surface integral, we need

$$\left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\right|_S = -\left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial r'}\right|_{r'=a} = -\frac{(r^2-a^2)}{a(r^2+a^2-2ra\cos\gamma)^{3/2}} \tag{234}$$

The minus sign is because the outward pointing normal (away from the volume) is in the direction of decreasing r—i.e., $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$.

- If the charge distribution is zero (e.g, if we are interested in a solution of Laplace's equation exterior to the sphere), then the volume integral vanishes and $\Phi(\mathbf{r})$ is given simply by the surface integral.

- Exercise: Find the solution to Laplace's equation outside a sphere of radius a with prescribed potential $\pm\Phi_0$ in the upper and lower hemispheres, respectively. Expand the integral in a power series of a/r, keeping the first few terms. (See Figure 36.)

- Answer:

$$\Phi(\mathbf{r}) = \frac{3\Phi_0}{2}\left(\frac{a}{r}\right)^2\left[\cos\theta - \frac{7}{12}\left(\frac{a}{r}\right)^2\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \cdots\right] \tag{235}$$

- Note that the terms in the square brackets are proportional to the Legendre polynomials

$$P_1(x) = x\,, \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{236}$$

with $x = \cos\theta$.

- Legendre polynomials will appear again when using separation of variables in spherical coordinates to solve Laplace's equation.

4.7 Expansions in terms of orthonormal functions

- Basic idea: Expand a square-integrable function in terms of a set of orthonormal basis functions, similar to the decomposition $\mathbf{v} = \sum_i v_i \hat{\mathbf{e}}_i$ for vectors in a finite-dimensional vector space.

- Notation: Let $\xi \in [a, b]$ and $f(\xi)$ denote any square-integrable function.

- If a discrete set of functions $\{U_n(\xi) | n = 1, 2, \cdots\}$ satisfies

$$\int_a^b d\xi \, U_n^*(\xi) U_m(\xi) = \delta_{nm} \tag{237}$$

then the functions are said to be *orthonormal*.

- The functions $\{U_n(\xi) | n = 1, 2, \cdots\}$ form a *basis* (and are said to be *complete*) if any square-integrable function can be expanded as

$$f(\xi) = \sum_{n=1}^{\infty} A_n \, U_n(\xi) \tag{238}$$

- It follows from the orthonormality property of the $U_n(\xi)$ that

$$A_n = \int_a^b d\xi \, U_n^*(\xi) f(\xi) \tag{239}$$

- Substituting this expression for A_n back into the expansion for $f(\xi)$, one finds

$$\sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) = \delta(\xi - \xi') \tag{240}$$

This is another way of expressing the completeness of the functions $U_n(\xi)$.

- Exercise: Prove the last two statements.

4.7.1 Fourier series

- Let $x \in [-a/2, a/2]$. Then

$$\left\{ \sqrt{\frac{2}{a}} \sin\left(\frac{n2\pi x}{a}\right) \, , \, \sqrt{\frac{2}{a}} \cos\left(\frac{n2\pi x}{a}\right) \middle| \, n = 1, 2, \cdots \right\} \tag{241}$$

form an orthonormal basis for functions defined on the interval $[-a/2, a/2]$, or, equivalently, for *periodic functions* defined for $x \in (-\infty, \infty)$ with period a.

- Orthonormality:

$$\frac{2}{a} \int_{-a/2}^{a/2} dx \, \sin\left(\frac{n2\pi x}{a}\right) \sin\left(\frac{m2\pi x}{a}\right) = \delta_{nm} \tag{242}$$

$$\frac{2}{a} \int_{-a/2}^{a/2} dx \, \cos\left(\frac{n2\pi x}{a}\right) \cos\left(\frac{m2\pi x}{a}\right) = \delta_{nm} \tag{243}$$

$$\frac{2}{a} \int_{-a/2}^{a/2} dx \, \sin\left(\frac{n2\pi x}{a}\right) \cos\left(\frac{m2\pi x}{a}\right) = 0 \tag{244}$$

- Exercise: Prove the above using

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) \tag{245}$$

$$\cos A \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right) \tag{246}$$

$$\sin A \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right) \tag{247}$$

- Completeness:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n2\pi x}{a}\right) + B_n \sin\left(\frac{n2\pi x}{a}\right)\right] \tag{248}$$

where

$$A_0 = \frac{2}{a}\int_{-a/2}^{a/2} dx\, f(x) \tag{249}$$

$$A_n = \frac{2}{a}\int_{-a/2}^{a/2} dx\, f(x)\cos\left(\frac{n2\pi x}{a}\right) \tag{250}$$

$$B_n = \frac{2}{a}\int_{-a/2}^{a/2} dx\, f(x)\sin\left(\frac{n2\pi x}{a}\right) \tag{251}$$

- In terms of complex exponentials the equations simplify:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{in2\pi x/a} \tag{252}$$

where

$$C_n = \frac{1}{a}\int_{-a/2}^{a/2} dx\, f(x) e^{-in2\pi x/a} \tag{253}$$

- The orthonormal basis functions are now:

$$\left\{ \frac{1}{\sqrt{a}} e^{\frac{in2\pi x}{a}} \,\middle|\, n = 0, \pm 1, \pm 2, \cdots \right\} \tag{254}$$

- Orthonormality:

$$\frac{1}{a}\int_{-a/2}^{a/2} dx\, e^{i(n-m)2\pi x/a} = \delta_{nm} \tag{255}$$

- Completeness:

$$\frac{1}{a}\sum_{n=-\infty}^{\infty} e^{in2\pi(x-x')/a} = \delta(x-x') \tag{256}$$

- Parseval's theorem:

$$\frac{1}{a}\int_{-a/2}^{a/2} dx\, |f(x)|^2 = \sum_{n=-\infty}^{\infty} |C_n|^2 \tag{257}$$

4.7.2 Fourier transform

- For square-integrable *non-periodic* functions defined over $x \in (-\infty, \infty)$, the Fourier series expansion generalizes to the *Fourier transform*.
- The orthonormal basis functions are now labeled by a *continuous* index $k \in (-\infty, \infty)$:

$$U_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} \tag{258}$$

- For any square-integrable function $f(x)$, we have

$$f(x) = \int_{-\infty}^{\infty} dk\, C(k)\, \frac{1}{\sqrt{2\pi}} e^{ikx} \tag{259}$$

where

$$C(k) = \int_{-\infty}^{\infty} dx\, f(x)\, \frac{1}{\sqrt{2\pi}} e^{-ikx} \tag{260}$$

- $f(x)$ and $C(k)$ are said to be a *Fourier transform pair*.
- Some authors define the expansion of $f(x)$ without the factor of $1/\sqrt{2\pi}$, but then need a factor of $1/2\pi$ in the expression for $C(k)$.
- One sometimes writes $\tilde{f}(k)$ instead of $C(k)$.
- Orthonormality:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = \delta(k - k') \tag{261}$$

- Note that the basis functions themselves are *not* square-integrable, as they are normalised in the sense of equaling a Dirac delta function.
- Completeness:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = \delta(x - x') \tag{262}$$

- Note the symmetry between the orthonormality and completeness relations.
- Parseval's theorem:

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} dk\, |C(k)|^2 \tag{263}$$

4.8 Separation of variables (rectangular coords)

- Separation of variables is an attempt to solve Laplace's equation in some region by writing $\Phi(\mathbf{r})$ as a product of functions, each of a single variable, reducing the partial differential equation to a set of ordinary differential equations, which are easier to solve. (Note that Laplace's equation is separable in 11 different coordinate systems!)
- In rectangular (i.e., Cartesian) coordinates one writes

$$\Phi(x, y, z) = X(x)Y(y)Z(z) \tag{264}$$

- In terms of X, Y, and Z, Laplace's equation

$$0 = \nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \tag{265}$$

becomes

$$X''YZ + XY''Z + XYZ'' = 0 \tag{266}$$

where $'$ denotes ordinary derivative with respect to the (single) argument of the function—e.g., $X'(x) = dX/dx$.

- Dividing by $\Phi = XYZ$ yields
$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0 \tag{267}$$

- Note that this is a sum of three terms, which are functions of only x, y, and z, respectively. The only way that such a sum can equal zero is for each term is equal to a *constant* (called a *separation constant*), with the sum of constants equal to zero:
$$\frac{X''}{X} = C_1\,, \quad \frac{Y''}{Y} = C_2\,, \quad \frac{Z''}{Z} = C_3 \tag{268}$$
with
$$C_1 + C_2 + C_3 = 0 \tag{269}$$

- Whether the constants are positive or negative or zero depend on the particular BCs.

- For example, suppose that we are interested in solving Laplace's equation inside a rectangular region $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, where the potential is set to zero on all faces except $z = c$, where it equals some prescribed function, $\Phi(x, y, c) = f(x, y)$.

- Then the appropriate choice of separation constants is
$$C_1 \equiv -\alpha^2 \le 0\,, \quad C_2 \equiv -\beta^2 \le 0\,, \quad C_3 \equiv \gamma^2 = \alpha^2 + \beta^2 \ge 0 \tag{270}$$

- The solutions of the individual equations for non-zero α, β, and γ are
$$X(x) = A\,\sin(\alpha x) + B\,\cos(\alpha x)\,, \tag{271}$$
$$Y(y) = C\,\sin(\beta y) + D\,\cos(\beta y)\,, \tag{272}$$
$$Z(z) = E\,\sinh(\gamma z) + F\,\cosh(\gamma z) \tag{273}$$

- The solutions of the individual equations for α, β, and γ equal to zero are
$$X(x) = A_0\,x + B_0\,, \tag{274}$$
$$Y(y) = C_0\,y + D_0\,, \tag{275}$$
$$Z(z) = E_0\,z + F_0 \tag{276}$$

- The most general solution of Laplace's equation is then a linear combination of the product solutions XYZ for the different allowed values of α and β.

- The BCs that the potential vanishes when $x = 0$, $y = 0$, and $z = 0$ imply
$$B = 0\,, \quad D = 0\,, \quad F = 0\,, \quad B_0 = 0\,, \quad D_0 = 0\,, \quad F_0 = 0\,, \tag{277}$$

- The BCs that the potential vanishes when $x = a$ and $y = b$ imply
$$\alpha = \frac{n\pi}{a}\,, \quad \beta = \frac{m\pi}{b}\,, \quad A_0 = 0\,, \quad C_0 = 0 \tag{278}$$
where n, m are positive integers.

- We need only consider n and m positive, since $n = 0$ and $m = 0$ leads to the trivial $\Phi = 0$ solution of Laplace's equation; while n and m negative introduce only an overall sign change from n and m positive, which can be absorbed in the multiplicative constants.

- Although E_0 is not constrained to vanish by the BCs, it will not enter into the final expression for Φ since $\gamma = 0$ iff $\alpha = \beta = 0$, and the X and Y solutions for $\alpha = \beta = 0$ are identically zero (we saw above that A_0, B_0, C_0, and D_0 are all constrained to vanish).

- Thus, the most general solution of Laplace's equation satisfying all of the boundary conditions except $\Phi(x, y, c) = f(x, y)$ can be written as

$$\Phi(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh\left(\gamma_{nm} z\right) \tag{279}$$

where

$$\gamma_{nm} = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}} \tag{280}$$

- Imposing the final BC at $z = c$ yields

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh\left(\gamma_{nm} c\right) \tag{281}$$

- Orthonomality of the sine functions on the intervals $x \in [0, a]$ and $y \in [0, b]$, lead to the solution

$$A_{nm} \sinh\left(\gamma_{nm} c\right) = \frac{4}{ab} \int_0^a dx \int_0^b dy\, f(x, y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \tag{282}$$

- To do the integration one needs to specify the explicit form of $f(x, y)$.

- To solve Laplace's equation inside the same rectangular region for more complicated BCs (e.g., where more than one face has non-zero values), we can simply superimpose the 'single-face' solutions, which all have a form similar to the above solution.

- Exercise: Solve the 2-dimensional Laplace equation in the region $0 \leq x \leq a$, $0 \leq y < \infty$ subject to the BCs that the potential vanishes on the 'sides' (i.e., at $x = 0$ and $x = a$) and at the 'top' (i.e., $y \to \infty$), and is equal to a constant Φ_0 when $y = 0$.

- Answer:

$$\Phi(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{-n\pi y/a} \tag{283}$$

where

$$A_n = \frac{2}{a} \int_0^a dx\, \Phi_0 \sin\left(\frac{n\pi x}{a}\right) = \begin{cases} \frac{4\Phi_0}{n\pi} & n = 1, 3, \cdots \\ 0 & n = 2, 4, \cdots \end{cases} \tag{284}$$

- Exercise: Using

$$\frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) = \sum_{n=\text{odd}}^{\infty} \frac{z^n}{n} \tag{285}$$

show that one can explicitly evaluate the summation for $\Phi(x, y)$ yielding the analytical expression

$$\Phi(x, y) = \frac{2\Phi_0}{\pi} \tan^{-1}\left(\frac{\sin(\pi x/a)}{\sinh(\pi y/a)}\right) \tag{286}$$

4.9 Separation of variables (spherical polar coords)

- In spherical polar coordinates, Laplace's equation is

$$0 = \nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Phi}{\partial \phi^2} \tag{287}$$

- If one assumes the product form

$$\Phi(r,\theta,\phi) \equiv R(r)P(\theta)Q(\phi) \tag{288}$$

Laplace's equation reduces to the following ordinary differential equations:

$$Q''(\phi) = -m^2\, Q(\phi) \tag{289}$$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = l(l+1)\, R(r) \tag{290}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\, \frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right] P(\theta) = 0 \tag{291}$$

where l and m are (at this stage) arbitrary real separation constants.

- Exercise: Prove the above.
- The solutions of the ϕ-equation are

$$Q(\phi) = A_0 + B_0\phi\,, \qquad \text{for } m = 0 \tag{292}$$

$$Q(\phi) = A\, e^{im\phi} + B\, e^{-im\phi}\,, \qquad \text{for } m \neq 0 \tag{293}$$

- If ϕ can take on the full range of values $\phi \in [0, 2\pi]$, then the requirement that $Q(\phi)$ be single-valued (i.e., $Q(\phi+2\pi n) = Q(\phi)$ for integer n) implies $B_0 = 0$ and m equal an integer. (This will normally be the case for the examples that we consider.)
- The radial equation has the general solution

$$R(r) = Ar^l + Br^{-(l+1)} \tag{294}$$

- Exercise: Prove this.
- Note that if $l \geq 0$, finiteness of $\Phi(r,\theta,\phi)$ at the origin ($r = 0$) implies $B = 0$. Similarly, requiring $\Phi(r,\theta,\phi) \to 0$ as $r \to \infty$ implies $A = 0$.
- The θ-equation can be put into more standard form by making a change of variables from θ to $x = \cos\theta$:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P(x) = 0 \tag{295}$$

and then expanding the derivative,

$$(1-x^2)P''(x) - 2x\, P'(x) + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P(x) = 0 \tag{296}$$

The above equation is called the *associated Legendre's equation.*

- If $m = 0$, the above equation is called *Legendre's equation*:

$$(1-x^2)P''(x) - 2x\, P'(x) + l(l+1)\, P(x) = 0 \tag{297}$$

4.9.1 Legendre polynomials

- To find a power series solution to Legendre's equation, we first note that $x = 0$ is a regular point of the differential equation.
- Substituting

$$P(x) = \sum_{n=0}^{\infty} a_n x^n \tag{298}$$

into Legendre's equation and differentiating term by term, we obtain the recurrence relation

$$a_{n+2} = \frac{n(n+1) - l(l+1)}{(n+1)(n+2)}\, a_n \tag{299}$$

- Exercise: Prove this.

- Since the recurrence relation relates a_{n+2} to a_n, the two independent solutions to Legendre's equation are given by setting $a_0 = 1$, $a_1 = 0$ and $a_1 = 1$, $a_0 = 0$. These solutions will be *even* and *odd* functions of x, respectively.

- One can show that the power series solutions diverge at $x = \pm 1$ (corresponding to the North and South poles of the sphere) unless the series terminates after some finite value of n.

- From the recurrence relation, we see that if l is a non-negative integer, $l = 0, 1, \cdots$, one of the power series solutions terminates (the even solution if l is even, and the odd solution if l is odd). The other solution can be set to zero (by hand) by setting $a_1 = 0$ (or $a_0 = 0$).

- The finite solutions are polynomials of order l. When appropriately normalised, they are called *Legendre polynomials*, denoted $P_l(x)$.

- NOTE: If l is a negative integer, $l = -1, -2, \cdots$, one also obtains a polynomial solution. But these solutions are the *same* as those for l non-negative (e.g., $l = -2$ yields the same solution as $l = 1$) so there is no loss of generality in restricting attention to $l = 0, 1, \cdots$.

- Legendre polynomials are normalized by the condition that $P_l(1) = 1$.

- Exercise: Show that the first four Legendre polynomials are given by

$$P_0(x) = 1 \tag{300}$$

$$P_1(x) = x \tag{301}$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{302}$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{303}$$

  See Figures 37-40 for various graphical representations of these functions.

- Rodrigues' formula:

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2 - 1)^l \tag{304}$$

- Note that

$$P_l(-x) = (-1)^l P_l(x) \tag{305}$$

- Orthonormality:

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'} \tag{306}$$

  Thus, Legendre polynomials form a set of *orthogonal* polynomials.

- Exercise: Prove the above. (Hint: The proof of orthogonality is simple if you integrate Legendre's equation times $P_{l'}(x)$. The derivation of the normalization constant is harder, but can proved using mathematical induction and Rodrigues's formula for $P_l(x)$.)

- Completeness: Any square-integrable function $f(x)$ defined on the interval $x \in [-1, 1]$ can be expanded in terms of Legendre polynomials:

$$f(x) = \sum_{l=0}^{\infty} A_l\, P_l(x) \tag{307}$$

  where

$$A_l = \frac{2l+1}{2}\int_{-1}^{1} dx\, f(x)\, P_l(x) \tag{308}$$

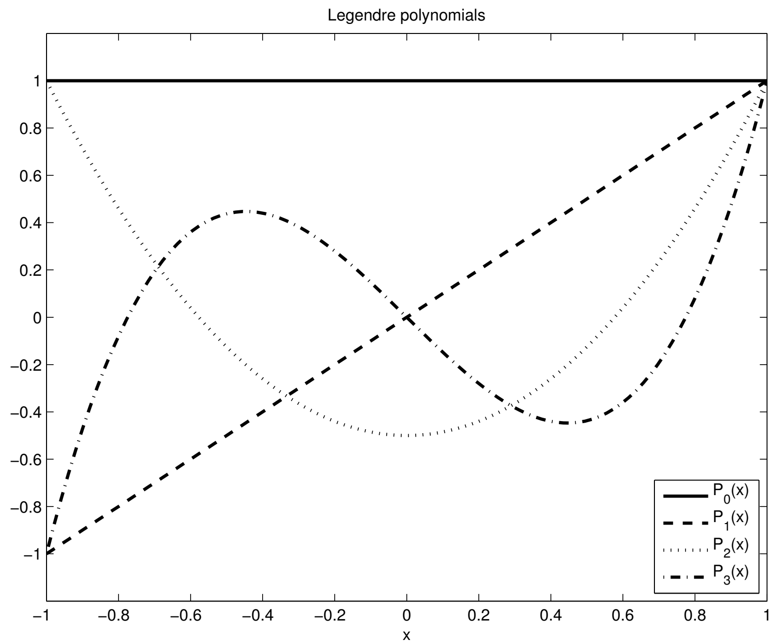

Figure 37: First few Legendre polynomials $P_l(x)$ plotted as functions of $x \in [-1, 1]$.

- Example: The function

$$f(x) = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases} \tag{309}$$

can be expanded as

$$f(x) = \frac{3}{2} P_1(x) - \frac{7}{8} P_3(x) + \frac{11}{16} P_5(x) + \cdots \tag{310}$$

See Figure 41.

- Exercise: Prove this.

- Thus, the general solution to Laplace's equation for problems with *azimuthal symmetry* (i.e., no ϕ-dependence so $m = 0$) is

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(A_l \, r^l + B_l \, r^{-(l+1)} \right) P_l(\cos\theta) \tag{311}$$

- Exercise: Find the solution to Laplace's equation outside a sphere of radius a with specified potential

$$\Phi(r = a, \theta) = \begin{cases} +\Phi_0 & \text{for } 0 \le \theta < \pi/2 \\ -\Phi_0 & \text{for } \pi/2 < \theta \le \pi \end{cases} \tag{312}$$

- Answer:

$$\Phi(r, \theta) = \Phi_0 \left[\frac{3}{2} \left(\frac{a}{r} \right)^2 P_1(\cos\theta) - \frac{7}{8} \left(\frac{a}{r} \right)^4 P_3(\cos\theta) + \frac{11}{16} \left(\frac{a}{r} \right)^6 P_5(\cos\theta) + \cdots \right] \tag{313}$$

NOTE: We obtained this result earlier using the Dirichlet Green's function exterior to the sphere.

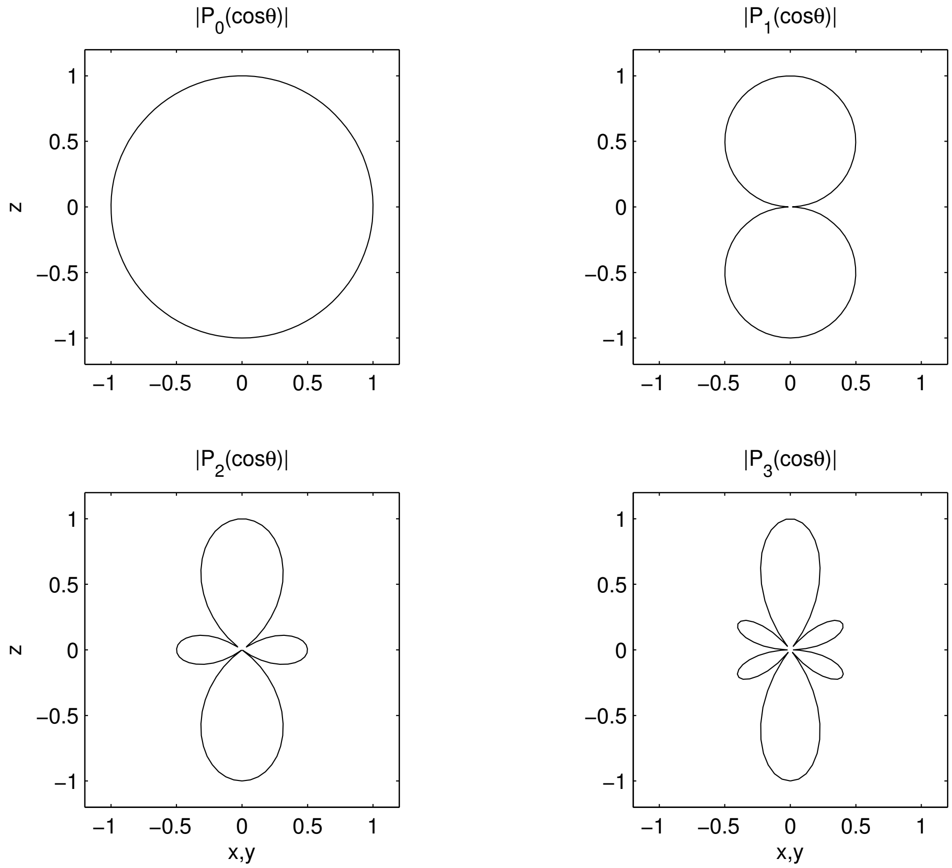

Figure 38: The *magnitude* $|P_l(\cos\theta)|$ of the first few Legendre polynomials plotted as functions of $\cos\theta$ in the x-z (or y-z) plane. The angle θ is measured wrt the positive z-axis. Note that by plotting the magnitude, information about the *sign* (i.e., $\pm$) of the Legendre polynomials $P_l(\cos\theta)$ is lost in this graphical representation.

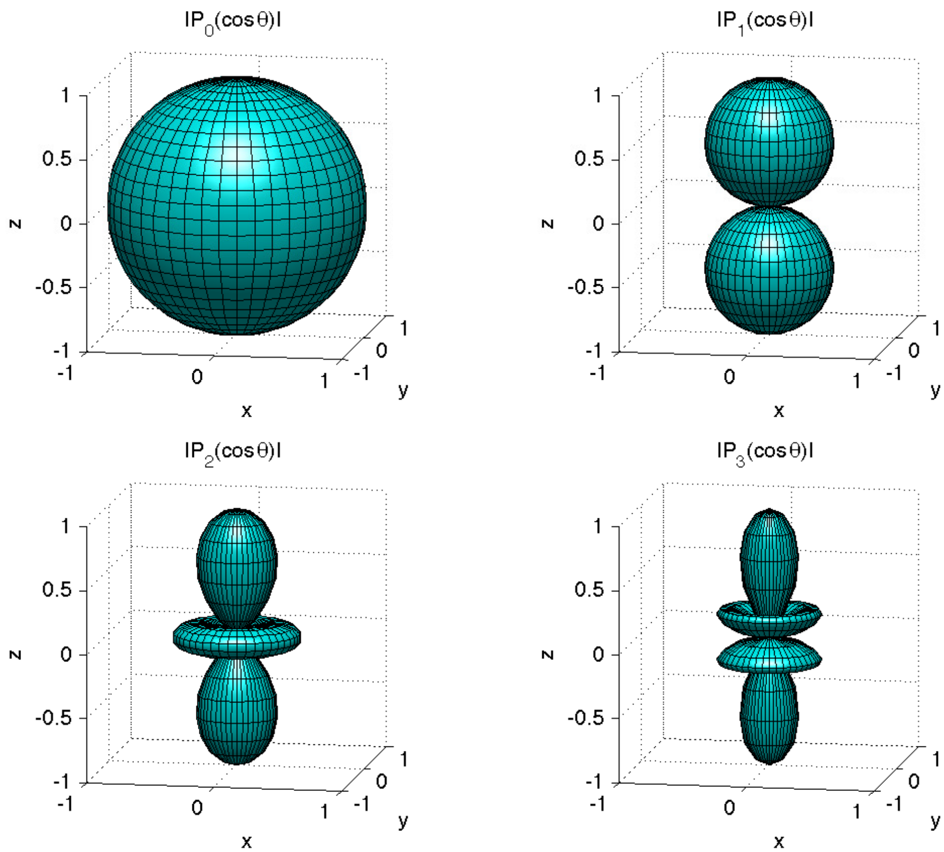

Figure 39: Same as Figure 38 but illustrated as surfaces of revolution (since there is no ϕ-dependence for the Legendre polynomials).

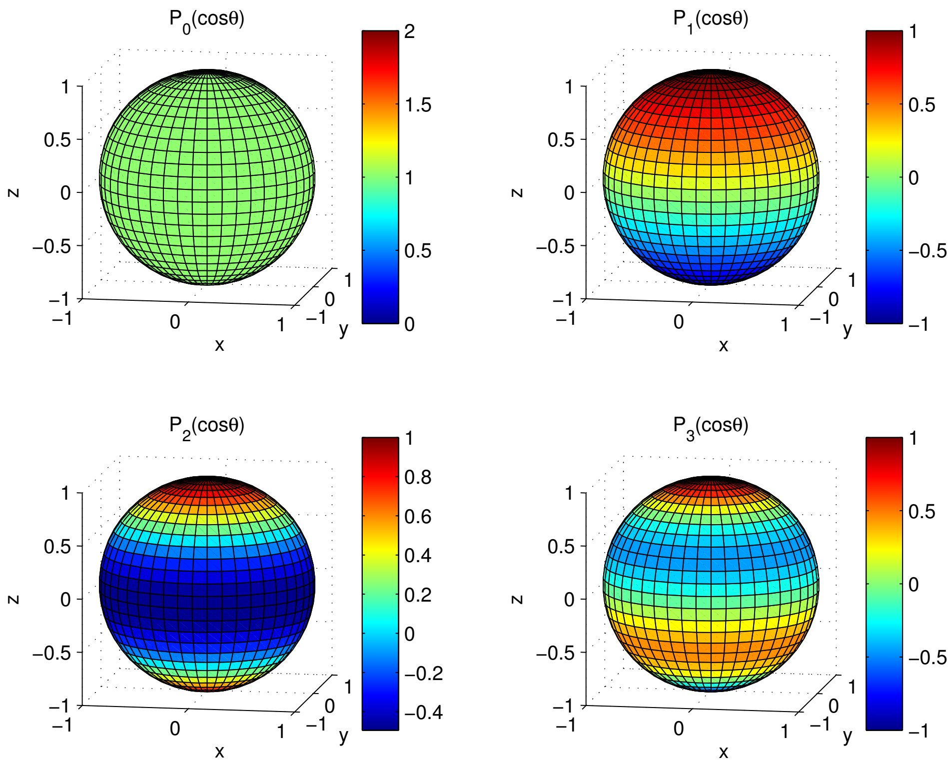

Figure 40: First few Legendre polynomials $P_l(\cos\theta)$ represented as functions on the unit 2-sphere. The color associated with each point on the sphere is the value of $P_l(\cos\theta)$ for that point (θ, ϕ). Note that in contrast to Figures 38 and 39, information about the sign of the Legendre polynomials is preserved in this graphical representation.

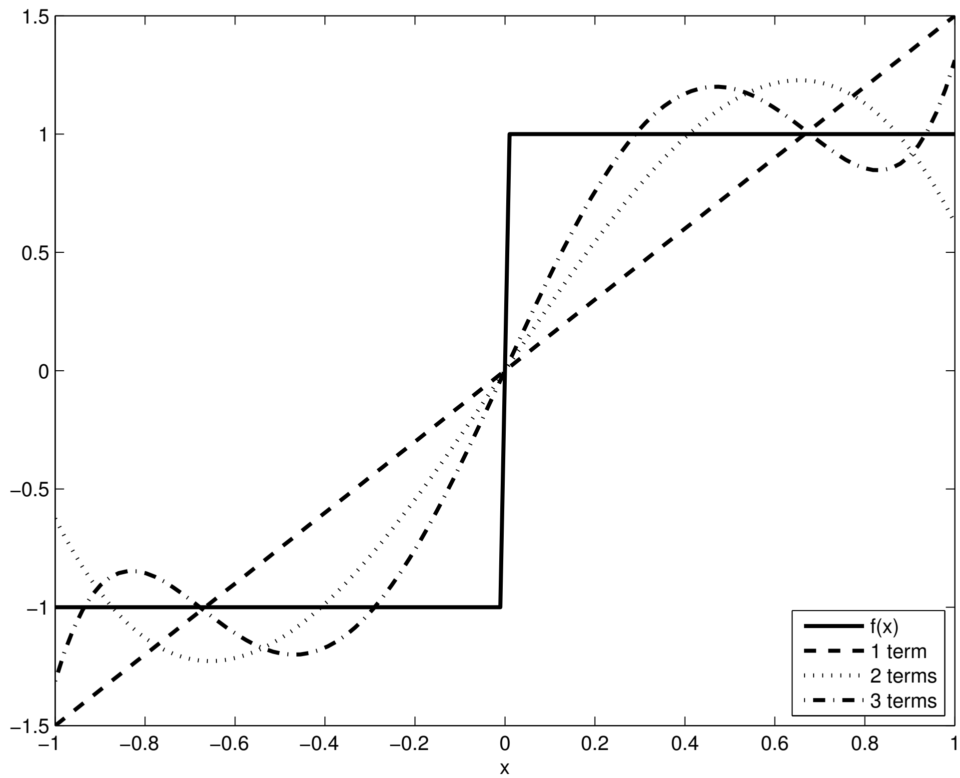

Figure 41: Expansion of the function $f(x) = \pm 1$ for $x \gtrless 0$, in terms Legendre polynomials. This plot shows how the approximation to $f(x)$ improves as more terms in the expansion are used.

- Generating function:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n \tag{314}$$

- Using the generating function, one can derive the following *recurrence relations*:

$$(n+1)\, P_{n+1} = (2n+1)x\, P_n - n\, P_{n-1} \tag{315}$$
$$P_n = P'_{n+1} - 2x\, P'_n + P'_{n-1} \tag{316}$$
$$n\, P_n = x\, P'_n - P'_{n-1} \tag{317}$$
$$(n+1)\, P_n = P'_{n+1} - x\, P'_n \tag{318}$$
$$(2n+1)\, P_n = P'_{n+1} - P'_{n-1} \tag{319}$$
$$(1-x^2)\, P'_n = n(P_{n-1} - x\, P_n) \tag{320}$$

- Note that Legendre's equation

$$(1-x^2)\, P''_n - 2x\, P'_n + n(n+1)\, P_n = 0 \tag{321}$$

can be obtained by differentiating (320) wrt x and then using (317). In addition, the normalization $P_n(1) = 1$ also follows simply from the generating function.

- Exercise: Prove the above relations by differentiating the generating function wrt t and x separately, and then combining the various expressions.

- Another important result that follows trivially from the generating function expression is an expansion of the potential of a point charge in terms of Legendre polynomials:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}}\, P_l(\cos\gamma) \tag{322}$$

where $r_<$ ($r_>$) is the smaller (larger) of r and r', and γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$:

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' \equiv \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\, \cos(\phi - \phi') \tag{323}$$

(See Figure 42.)

- Exercise: Prove the expansion for $1/|\mathbf{r}-\mathbf{r}'|$.

4.9.2 Associated Legendre functions

- When $m \neq 0$, we need to solve the associated Legendre's equation

$$(1-x^2)\, P'' - 2x\, P' + \left[l(l+1) - \frac{m^2}{(1-x^2)}\right] P = 0 \tag{324}$$

- It turns out that power series solutions of this differential equation also diverge at the poles ($x = \pm 1$) unless $l = 0, 1, \cdots$ (as before) and $m = -l, -l+1, \cdots, l$.

- The finite solutions are called *associated Legendre functions* are are given by derivatives of the Legendre polynomials:

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \tag{325}$$

and

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!}\, P_l^m(x) \tag{326}$$

for $m > 0$.

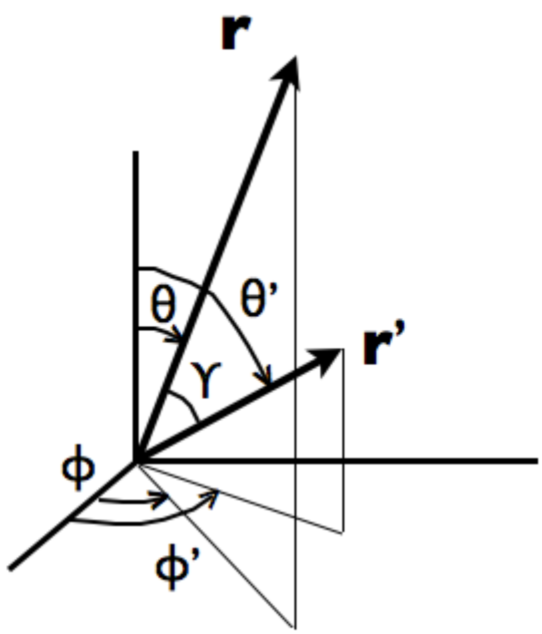

Figure 42: Position vectors $\mathbf{r}$ and $\mathbf{r}'$, and the spherical coordinates (θ, ϕ) and (θ', ϕ') specifying their directions. γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. For this example $r_> = r$ and $r_< = r'$.

- The above phase convention is that of Condon and Shortley.
- The associated Legendre functions are *not* polynomials in x on account of the square root factor $(1-x^2)^{m/2}$ for odd m. But since we are ultimately interested in the replacement $x = \cos\theta$, these non-polynomial factors are just proportional to $\sin^m\theta$. Thus, the associated Legendre functions can be written as polynomials in $\cos\theta$ if m is even, and polynomials in $\cos\theta$ multiplied by $\sin\theta$ if m is odd.
- Exercise: Show that the first few associated Legendre functions are given by:

  $l = 0$:

$$P_0^0(\cos\theta) = 1 \tag{327}$$

  $l = 1$:

$$P_1^0(\cos\theta) = \cos\theta \tag{328}$$

$$P_1^1(\cos\theta) = -\sin\theta \tag{329}$$

  $l = 2$:

$$P_2^0(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right) \tag{330}$$

$$P_2^1(\cos\theta) = -3\sin\theta\,\cos\theta \tag{331}$$

$$P_2^2(\cos\theta) = 3(1 - \cos^2\theta) \tag{332}$$

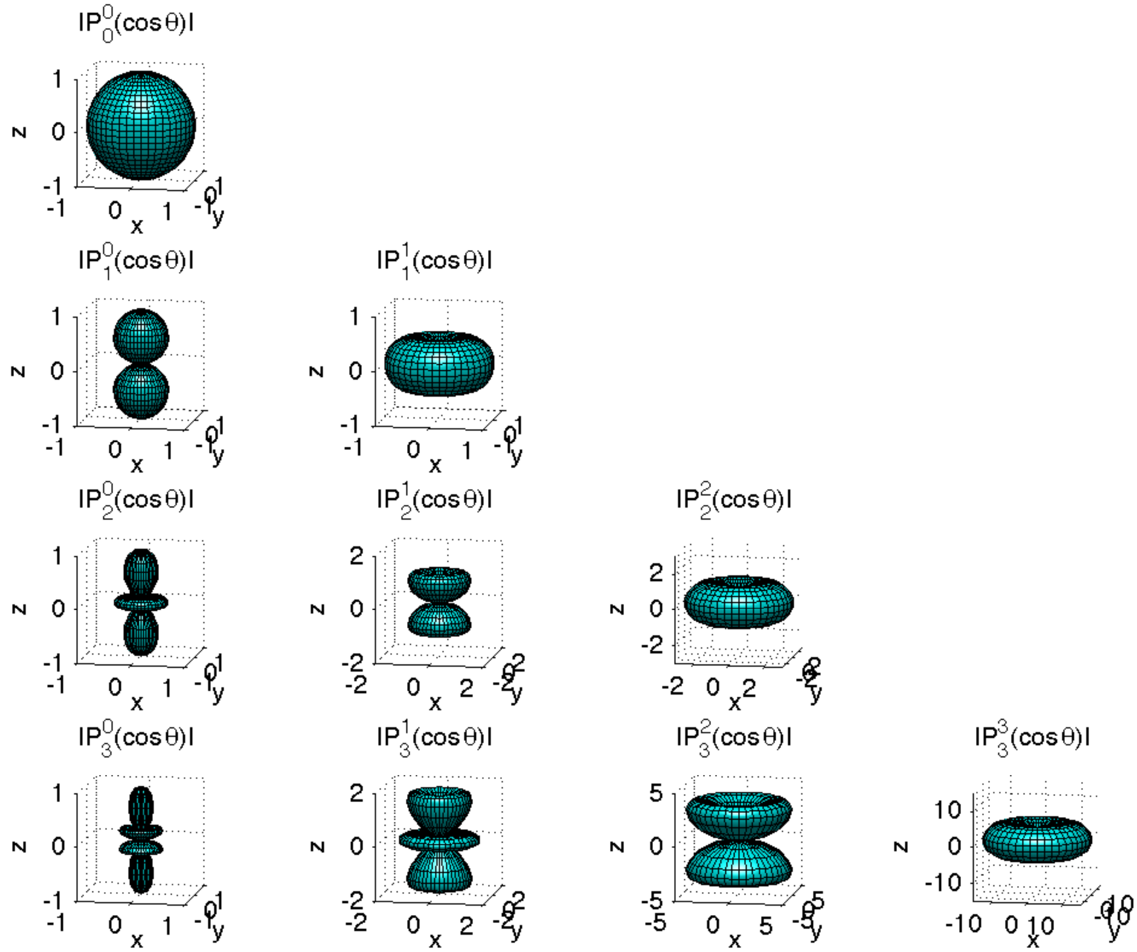

Figure 43: The *magnitude* $|P_l^m(\cos\theta)|$ of the first few associated Legendre functions plotted as surfaces of revolution. Similar to the plots in Figures 38 and 39, the sign (i.e., $\pm$) of the associated Legendre functions $P_l^m(\cos\theta)$ is lost in this graphical representation.

$\underline{l=3}$:

$$P_3^0(\cos\theta) = \frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right) \tag{333}$$

$$P_3^1(\cos\theta) = -\frac{3}{2}\sin\theta\,\left(5\cos^2\theta - 1\right) \tag{334}$$

$$P_3^2(\cos\theta) = 15\left(\cos\theta - \cos^3\theta\right) \tag{335}$$

$$P_3^3(\cos\theta) = -15\sin\theta\,\left(1 - \cos^2\theta\right) \tag{336}$$

See Figure 43 for plots of the magnitude of the first few of these functions.

- Using Rodrigues' formula, we can write down a formula valid for both positive and negative values of m:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\,\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \tag{337}$$

- Orthonormality: For each m

$$\int_{-1}^{1} dx\, P_l^m(x) P_{l'}^m(x) = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\,\delta_{ll'} \tag{338}$$

- Completeness: For each m, the associated Legendre functions form a complete set (in the index l) for square-integrable functions on $x \in [-1, 1]$:

$$f(x) = \sum_{l=0}^{\infty} A_l \, P_l^m(x) \tag{339}$$

where

$$A_l = \frac{2l+1}{2} \frac{(l-m)!}{(l+m)!} \int_{-1}^{1} dx \, f(x) \, P_l^m(x) \tag{340}$$

4.9.3 Spherical harmonics

- *Spherical harmonics* are proportional to the product of the solutions of the angular equations for Laplace's equation in spherical polar coordinates:

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi} \tag{341}$$

They are *complex* functions on the unit 2-sphere with spherical coordinates (θ, ϕ).

- The proportionality constants have been chosen so that

$$\int_{S^2} d\Omega \, Y^*_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'} \tag{342}$$

where

$$d\Omega \equiv d(\cos\theta) \, d\phi = \sin\theta \, d\theta \, d\phi \tag{343}$$

- Note that

$$Y_{l,-m}(\theta, \phi) = (-1)^m Y^*_{lm}(\theta, \phi) \tag{344}$$

- For the antipodal point on the sphere

$$Y_{lm}(\pi - \theta, \phi + \pi) = (-1)^l Y_{lm}(\theta, \phi) \tag{345}$$

- For $m = 0$

$$Y_{l0} = \sqrt{\frac{2l+1}{4\pi}} \, P_l(\cos\theta) \tag{346}$$

- Expressions for the first few spherical harmonics:

  $\underline{l = 0}$:

$$Y_{00}(\theta, \phi) = \sqrt{\frac{1}{4\pi}} \tag{347}$$

  $\underline{l = 1}$:

$$Y_{11}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi} \tag{348}$$

$$Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta \tag{349}$$

$$Y_{1,-1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi} \tag{350}$$

$\underline{l=2}$:

$$Y_{22}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi} \tag{351}$$

$$Y_{21}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi} \tag{352}$$

$$Y_{20}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \tag{353}$$

$$Y_{2,-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\phi} \tag{354}$$

$$Y_{2,-2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{-2i\phi} \tag{355}$$

- Since $Y_{lm}(\theta,\phi)$ differs from $P_l^m(\theta)$ by only a constant multiplicative factor and phase $e^{im\phi}$, the magnitude $|Y_l^m(\theta,\phi)|$ has the same shape as $|P_l^m(\theta)|$ (see Figure 43).

- Completeness: Any square-integrable function $f(\theta,\phi)$ on the unit 2-sphere can be expanded in terms of spherical harmonics:

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}\, Y_{lm}(\theta,\phi) \tag{356}$$

where

$$A_{lm} = \int_{S^2} d\Omega\, f(\theta,\phi)\, Y^*_{lm}(\theta,\phi) \tag{357}$$

- Equivalently, the completeness property can be written as

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) = \delta(\hat{\mathbf{\Omega}},\hat{\mathbf{\Omega}}') \tag{358}$$

where $\delta(\hat{\mathbf{\Omega}},\hat{\mathbf{\Omega}}')$ is the Dirac delta function on the 2-sphere:

$$\delta(\hat{\mathbf{\Omega}},\hat{\mathbf{\Omega}}') = \delta(\cos\theta - \cos\theta')\delta(\phi-\phi') = \frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi') \tag{359}$$

- Thus, the general solution to Laplace's equation in spherical polar coordinates is

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}\, r^l + B_{lm}\, r^{-(l+1)}\right] Y_{lm}(\theta,\phi) \tag{360}$$

- *Addition theorem*:

$$\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) = \frac{2l+1}{4\pi}\, P_l(\cos\gamma) \tag{361}$$

where

$$\cos\gamma = \hat{\mathbf{\Omega}}\cdot\hat{\mathbf{\Omega}}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') \tag{362}$$

- Completeness of the spherical harmonics and the addition theorem imply

$$\delta(\hat{\mathbf{\Omega}},\hat{\mathbf{\Omega}}') = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}P_l(\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{\Omega}}')\,, \tag{363}$$

which is an expansion of the Dirac delta function on the 2-sphere in terms of the Legendre polynomials.

- *Transformation under a rotation*:

$$Y_{lm}(R\hat{\mathbf{\Omega}}) = \sum_{m'=-l}^{l} D_{lm,m'} Y_{lm'}(\hat{\mathbf{\Omega}}) \,, \tag{364}$$

  where R denotes an arbitrary rotation.

- The fact that $Y_{lm}(R\hat{\mathbf{\Omega}})$ can be written as a linear combination of the $Y_{lm'}(\hat{\mathbf{\Omega}})$ with the *same* l is a consequence of the spherical harmonics being eigenfunctions of the (rotationally-invariant) Laplacian on the unit 2-sphere with eigenvalues depending only on l:

$$^{(2)}\nabla^2 Y_{lm}(\hat{\mathbf{\Omega}}) = -l(l+1) Y_{lm}(\hat{\mathbf{\Omega}}) \,. \tag{365}$$

- The coefficients $D^l_{mm'}$ are closely related to the Clebsch-Gordan coefficients. They satisfy

$$\sum_{m''=-l}^{l} D_{lm,m''} D^*_{lm',m''} = \delta_{mm'} \tag{366}$$

  as a consequence of

$$\int_{S^2} d\hat{\mathbf{\Omega}}\, Y^*_{lm}(R\hat{\mathbf{\Omega}}) Y_{l'm'}(R\hat{\mathbf{\Omega}}) = \delta_{ll'}\delta_{mm'} \tag{367}$$

- Using the addition theorem, it follows that the potential for a point source can be written as:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi) \tag{368}$$

  where $r_<$ ($r_>$) is the smaller (larger) of r and r'.

- This expression is fully-factorized into a product of functions of the unprimed and primed coordinates.

4.9.4 Proof of the addition theorem for spherical harmonics

- Goal: To prove the addition theorem

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi) \tag{369}$$

  where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') \tag{370}$$

- Definitions and coordinate systems:

  1) Let $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}'$ be two unit vectors, with dot product $\hat{\mathbf{\Omega}} \cdot \hat{\mathbf{\Omega}}' = \cos\gamma$. We will keep $\hat{\mathbf{\Omega}}'$ fixed, but allow $\hat{\mathbf{\Omega}}$ to vary.

  2) Choose a coordinate system on the 2-sphere so that $\hat{\mathbf{\Omega}}$ has coordinates (θ,ϕ) and $\hat{\mathbf{\Omega}}'$ has coordinates (θ',ϕ'). (See Figure 44, panel (a).) Since $\hat{\mathbf{\Omega}}'$ is fixed, (θ',ϕ') are constants. (For example, they are *not* integrated over in any of the following expressions.) If $\gamma = 0$, then $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}'$ correspond to the same point on the 2-sphere, so that $(\theta,\phi) = (\theta',\phi')$.

  3) We can also consider a rotated coordinate system on the 2-sphere with the North Pole given by $\hat{\mathbf{\Omega}}'$. Then $\hat{\mathbf{\Omega}}$ has spherical coordinates (γ,ψ) wrt this rotated coordinate system, where ψ is an arbitrary azimuthal coordinate, since choosing $\hat{\mathbf{\Omega}}'$ as the North Pole of the rotated coordinates doesn't uniquely determine the zero of the azimuthal angle. (See Figure 44, panel (b).)

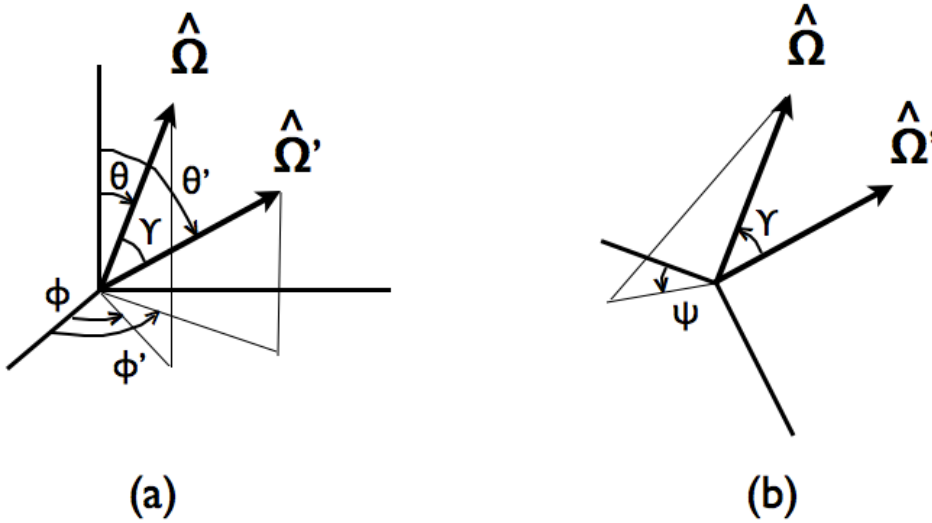

Figure 44: Panel (a): Coordinate system in which $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}'$ have coordinates (θ, ϕ) and (θ', ϕ'), respectively. γ is the angle between $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}'$. The vector $\hat{\mathbf{\Omega}}'$ is kept fixed, while $\hat{\mathbf{\Omega}}$ is allowed to vary. Panel (b): Same two unit vectors $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}'$ as in panel (a), but with respect to a coordinate system in which $\hat{\mathbf{\Omega}}'$ is the North Pole. In this coordinate system, $\hat{\mathbf{\Omega}}$ has coordinates (γ, ψ).

4) At times, we will think of γ as a function of (θ, ϕ). At other times, we will think of (θ, ϕ) as functions of (γ, ψ).

5) The Laplacian on the 2-sphere is invariant under rotations. Thus, if $f_l(\theta, \phi)$ is a eigenfunction of the (θ, ϕ)-Laplacian on the 2-sphere with eigenvalue l, then $f_l(\gamma, \psi) \equiv f_l(\theta(\gamma, \psi), \phi(\gamma, \psi))$ is an eigenfunction of the (γ, ψ)-Laplacian on the 2-sphere *with the same eigenvalue* l.

- Consider $P_l(\cos\gamma)$, and view it as a function of (θ, ϕ). Then we can write

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} A_{lm}\, Y_{lm}(\theta, \phi) \tag{371}$$

- The fact that there is no sum over an l' index is a consequence of item 5 above as $P_l(\cos\gamma)$ is an eigenfunction of the (γ, ψ)-Laplacian on the 2-sphere (and hence also of the (θ, ϕ)-Laplacian) with eigenvalue l.

- Using the orthonormality of the spherical harmonics it follows that

$$A_{lm} = \int_{S^2} d\Omega_{\theta,\phi}\, P_l(\cos\gamma) Y^*_{lm}(\theta, \phi) \tag{372}$$

where

$$d\Omega_{\theta,\phi} \equiv d(\cos\theta)\, d\phi = \sin\theta\, d\theta\, d\phi \tag{373}$$

- Since

$$Y_{l0}(\gamma, \psi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\gamma) \tag{374}$$

we can also write

$$A_{lm} = \sqrt{\frac{4\pi}{2l+1}} \int_{S^2} d\Omega_{\theta,\phi}\, Y_{l0}(\gamma,\psi) Y^*_{lm}(\theta,\phi) \tag{375}$$

- By a similar argument (or by appealing to the transformation properties of the spherical harmonics under a rotation), we can write

$$Y_{lm}(\theta,\phi) = \sum_{m'=-l}^{l} B_{lm,m'} Y_{lm'}(\gamma,\psi) \tag{376}$$

where the (θ,ϕ) variables on the LHS are to be thought of a functions of (γ,ψ).

- The expansion coefficients are given by

$$B_{lm,m'} = \int_{S^2} d\Omega_{\gamma,\psi}\, Y_{lm}(\theta,\phi) Y^*_{lm'}(\gamma,\psi) \tag{377}$$

where

$$d\Omega_{\gamma,\psi} \equiv d(\cos\gamma)\, d\psi = \sin\gamma\, d\gamma\, d\psi \tag{378}$$

- If we consider $\hat{\mathbf{\Omega}}$ to point in the same direction as $\hat{\mathbf{\Omega}}'$, then $\gamma = 0$, which implies

$$Y_{lm}(\theta',\phi') = Y_{lm}(\theta,\phi)\big|_{\gamma=0} = \sum_{m'=-l}^{l} B_{lm,m'} Y_{lm'}(0,\psi) = B_{lm,0}\sqrt{\frac{2l+1}{4\pi}} \tag{379}$$

where the last equality follows from

$$Y_{lm}(0,\psi) = \begin{cases} \sqrt{\frac{2l+1}{4\pi}} & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \tag{380}$$

- The integral expression for the $B_{lm,0}$ expansion coefficient is given by equation (377):

$$B_{lm,0} = \int_{S^2} d\Omega_{\gamma,\psi}\, Y_{lm}(\theta,\phi) Y^*_{l0}(\gamma,\psi) \tag{381}$$

- Comparing this with the integral expression for the A_{lm} expansion coefficient, we see that

$$A^*_{lm} = \sqrt{\frac{4\pi}{2l+1}} \int_{S^2} d\Omega_{\theta,\phi}\, Y^*_{l0}(\gamma,\psi) Y_{lm}(\theta,\phi) \tag{382}$$

$$= \sqrt{\frac{4\pi}{2l+1}} \int_{S^2} d\Omega_{\gamma,\psi}\, Y^*_{l0}(\gamma,\psi) Y_{lm}(\theta,\phi) \tag{383}$$

$$= \sqrt{\frac{4\pi}{2l+1}}\, B_{lm,0} \tag{384}$$

$$= \frac{4\pi}{2l+1} Y_{lm}(\theta',\phi') \tag{385}$$

where we used the rotational invariance of the area element on the 2-sphere, $d\Omega_{\theta,\phi} = d\Omega_{\gamma,\psi}$, to get the second equality.

- Thus,

$$A_{lm} = \frac{4\pi}{2l+1} Y^*_{lm}(\theta',\phi') \tag{386}$$

which gives us

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi) \tag{387}$$

as desired.

4.10 Separation of variables (cylindrical coords)

- In cylindrical polar coordinates (ρ, ϕ, z), Laplace's equation is

$$0 = \nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} \tag{388}$$

- If one assumes the product form

$$\Phi(\rho, \phi, z) \equiv R(\rho)Q(\phi)Z(z) \tag{389}$$

then Laplace's equation reduces to

$$0 = \frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2}\frac{1}{Q}\frac{d^2Q}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} \tag{390}$$

where we've divided the whole equation by $\Phi = RQZ$.

- If the separation constants are chosen so that

$$Z''(z) = k^2 Z(z)\,, \tag{391}$$

$$Q''(\phi) = -\nu^2 Q(\phi) \tag{392}$$

then

$$R''(\rho) + \frac{1}{\rho}R'(\rho) + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R(\rho) = 0 \tag{393}$$

- If we choose the separation constants with the opposite sign for the z-equation:

$$Z''(z) = -k^2 Z(z)\,, \tag{394}$$

$$Q''(\phi) = -\nu^2 Q(\phi) \tag{395}$$

then

$$R''(\rho) + \frac{1}{\rho}R'(\rho) - \left(k^2 + \frac{\nu^2}{\rho^2}\right)R(\rho) = 0 \tag{396}$$

- Exercise: Prove the above statements.
- The solutions to the z-equation with positive separation constant $+k^2$ are

$$Z(z) = A\sinh(kz) + B\cosh(kz) \tag{397}$$

or, equivalently,

$$Z(z) = A'\,e^{kz} + B'\,e^{-kz} \tag{398}$$

This choice of separation constant is needed when solving Laplace's equation with boundary condition $\Phi \to 0$ as $z \to \infty$, or if the potential is specified to have some non-zero value on the 2-d boundary surface $z =$ constant.

- The solutions to the z-equation with negative separation constant $-k^2$ are

$$Z(z) = A\sin(kz) + B\cos(kz) \tag{399}$$

This choice of separation constant is needed when solving Laplace's equation with $\Phi = 0$ on the 2-d boundary surfaces $z = a$ and $z = b$, where a, b are finite constants.

- The solutions to the ϕ-equation are

$$Q(\phi) = C_0 + B_0\phi\,, \quad \text{for } \nu = 0 \tag{400}$$

$$Q(\phi) = C\sin(\nu\phi) + D\cos(\nu\phi)\,, \quad \text{for } \nu \neq 0 \tag{401}$$

- If ϕ can take on the full range of values $\phi \in [0, 2\pi]$, then the requirement that $Q(\phi)$ be single-valued implies $B_0 = 0$ and ν equal an integer.

- The ρ-equations can be put into more standard form by making a change of variables $\rho \to x = k\rho$, with

$$y(x)\big|_{x=k\rho} \equiv R(\rho) \tag{402}$$

The two different equations corresponding to the different choice of sign for the separation constant $\pm k^2$ become

$$y''(x) + \frac{1}{x}\, y'(x) + \left(1 - \frac{\nu^2}{x^2}\right) y(x) = 0 \tag{403}$$

or

$$y''(x) + \frac{1}{x}\, y'(x) - \left(1 + \frac{\nu^2}{x^2}\right) y(x) = 0 \tag{404}$$

- Equation (403) is called *Bessel's equation of order* ν; equation (404) is called the *modified Bessel's equation of order* ν.

- Note that if $y(x)$ is a solution of Bessel's equation, then $\bar{y}(x) \equiv y(ix)$ is a solution of the modified Bessel's equation.

- Exercise: Prove the above.

4.10.1 Bessel functions

- To solve Bessel's equation, we note that $x = 0$ is a regular singular point of the differential equation. (The functions $p(x) \equiv 1/x$ and $q(x) \equiv (1 - \nu^2/x^2)$, which multiply $y'(x)$ and $y(x)$, respectively, are singular at $x = 0$, but $x\,p(x)$ and $x^2\,q(x)$ are both finite at $x = 0$.)

- The method of Frobenius says that such a differential equation will admit a power series solution of the form

$$y(x) = x^\sigma \sum_{n=0}^{\infty} a_n x^n \tag{405}$$

- Substituting this expansion into Bessel's equation and equating the coefficients multiplying like powers of x leads to

$$a_0(\sigma^2 - \nu^2) = 0 \tag{406}$$

$$a_1(1 + 2\sigma + \sigma^2 - \nu^2) = 0 \tag{407}$$

$$a_{n+2} = -\frac{a_n}{(n + 2 + \sigma)^2 - \nu^2} \tag{408}$$

- The first of the above equations is called the *indicial equation.* For $a_0 \neq 0$ it has the solutions:

$$\sigma = \pm\nu \tag{409}$$

- Substituting these solutions for σ into the second equation leads to

$$a_1(1 \pm 2\nu) = 0 \tag{410}$$

- For $\nu \neq \pm 1/2$, this equation implies $a_1 = 0$. But even for $\nu = \pm 1/2$, we can set $a_1 = 0$.

- Thus, $a_1 = 0$ together with the recurrence relation implies $a_n = 0$ for all *odd* values of n.

- For the even expansion coefficients, we can rewrite the recurrence relation as

$$a_{2n} = a_0\, \frac{(-1)^n \Gamma(1 + \nu)}{2^{2n} n! \Gamma(n + 1 + \nu)} \tag{411}$$

for $n = 0, 1, 2, \cdots$.

- Recall that the gamma function is defined by

$$\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x} \tag{412}$$

for $\mathrm{Re}(z) > 0$. The gamma function generalizes the factorial function to non-integer arguments in the sense that

$$\Gamma(n+1) = n! \quad \text{for } n = 0, 1, \cdots \tag{413}$$
$$\Gamma(z+1) = z\,\Gamma(z) \quad \text{for } \mathrm{Re}(z) > 0 \tag{414}$$

- If the normalization constant a_0 is chosen to be

$$a_0 = \frac{1}{2^\nu \Gamma(1+\nu)} \tag{415}$$

then

$$a_{2n} = \frac{(-1)^n}{2^{2n+\nu} n! \Gamma(n+1+\nu)} \tag{416}$$

- The power series solution is thus

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1+\nu)} \left(\frac{x}{2}\right)^{2n+\nu} \tag{417}$$

$J_\nu(x)$ is called *Bessel's function of the 1st kind.*

- Asymptotic form:

$$x \ll 1: \qquad J_\nu(x) \to \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \tag{418}$$
$$x \gg 1, \nu: \qquad J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{419}$$

- Thus, $J_0(0) = 1$, $J_\nu(0) = 0$ for all $\nu \neq 0$; while for large x, $J_\nu(x)$ behaves like a damped sinusoid, and has infinitely many zeros $x_{\nu n}$:

$$J(x_{\nu n}) = 0\,, \qquad \text{for } n = 1, 2, \cdots \tag{420}$$

See Figure 45.

- Exercise: Show that the zeros of $J_\nu(x)$ are given by

$$x_{\nu n} \simeq n\pi + \left(\nu - \frac{1}{2}\right)\frac{\pi}{2} \tag{421}$$

- If ν is not an integer, then $J_{-\nu}(x)$ is the second independent solution to Bessel's equation.
- If $\nu = m$ is an integer, then one can show that $J_{-m}(x)$ is proportional to $J_m(x)$:

$$J_{-m}(x) = (-1)^m J_m(x) \tag{422}$$

so $J_{-m}(x)$ is not an independent solution for this case.

- Exercise: Prove the above.
- A second solution, which is independent of $J_\nu(x)$ for all ν (integer or not), is

$$N_\nu(x) := \frac{J_\nu(x)\, \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \tag{423}$$

$N_\nu(x)$ is called a *Neumann function* or *Bessel's function of the 2nd kind*, and is sometimes denoted by $Y_\nu(x)$.

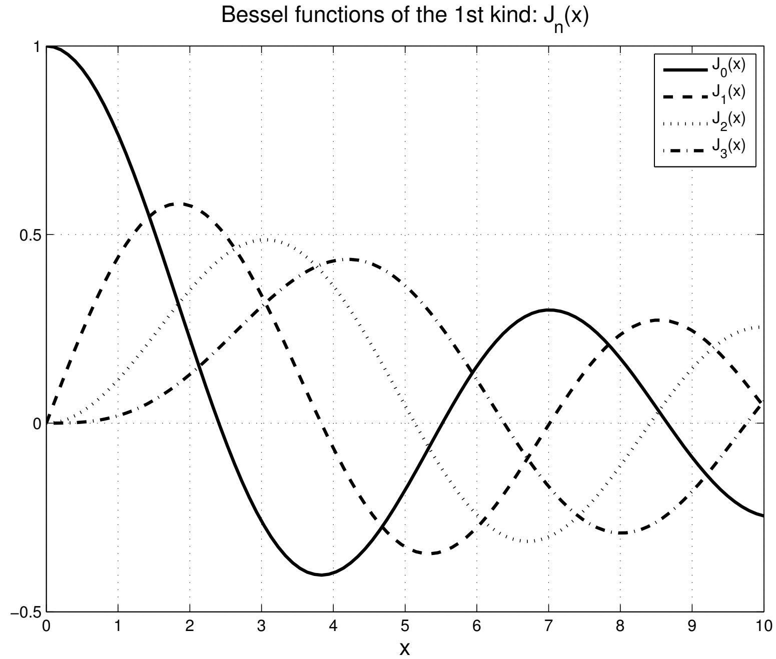

Figure 45: First few Bessel functions of the 1st kind for integer ν.

- For $\nu = m$ an integer, one needs to use L'Hopital's rule to show that the RHS of the expression defining $N_m(x)$ is well-defined.
- Asymptotic form:

$$x \ll 1: \qquad N_\nu(x) \to \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + 0.5772\cdots\right], & \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu, & \nu \neq 0 \end{cases} \tag{424}$$

$$x \gg 1, \nu: \qquad N_\nu(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{425}$$

- Note that for all ν, $N_\nu(x) \to -\infty$ as $x \to 0$.
- As we saw for $J_\nu(x)$, for large x, $N_\nu(x)$ behaves like a damped sinusoid, 90° out of phase with $J_\nu(x)$. See Figure 46.
- Thus, the most general solution to the radial part of Laplace's equation is

$$R(\rho) = A\, J_\nu(k\rho) + B\, N_\nu(k\rho) \tag{426}$$

- Since $N_\nu(x)$ blows up at $x = 0$, if $\rho = 0$ is in the region of interest, then all of the B coefficients must vanish to yield a finite value of the potential on the axis.
- *Hankel functions* (or *Bessel functions of the 3rd kind*) are defined by

$$H_\nu^{(1)}(x) := J_\nu(x) + iN_\nu(x) \tag{427}$$

$$H_\nu^{(2)}(x) := J_\nu(x) - iN_\nu(x) \tag{428}$$

- *Modified* (or *hyperbolic*) *Bessel functions* are defined by

$$I_\nu(x) := i^{-\nu} J_\nu(ix) \tag{429}$$

$$K_\nu(x) := \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \tag{430}$$

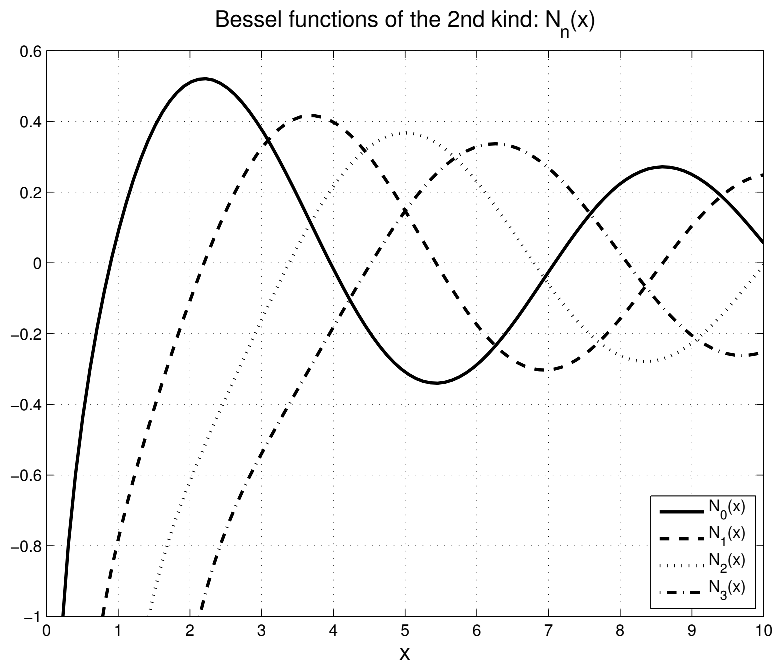

Figure 46: First few Bessel functions of the 2nd kind for integer ν.

Note the pure imaginary arguments on the RHS.

- These are two linearly independent solutions of the modified Bessel's equation (404). See Figures 47 and 48.
- Asymptotic form:

$$x \ll 1: \qquad I_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \tag{431}$$

$$K_\nu(x) \to \begin{cases} -\left[\ln\left(\frac{x}{2}\right) + 0.5772\cdots\right], & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu, & \nu \neq 0 \end{cases} \tag{432}$$

$$x \gg 1, \nu: \qquad I_\nu(x) \to \frac{1}{\sqrt{2\pi x}}\, e^x \left[1 + O\left(\frac{1}{x}\right)\right] \tag{433}$$

$$K_\nu(x) \to \sqrt{\frac{\pi}{2x}}\, e^{-x}\left[1 + O\left(\frac{1}{x}\right)\right] \tag{434}$$

- Thus, $I_0(0) = 1$, $I_\nu(0) = 0$ for all $\nu \neq 0$, while $K_\nu(x) \to \infty$ as $x \to 0$ for all ν.
- For large x, $I_\nu(x) \to \infty$ while $K_\nu(x) \to 0$ for all ν.
- Thus, the most general solution to the radial part of Laplace's equation for the choice of negative separation constant $-k^2$ is

$$R(\rho) = A\, I_\nu(k\rho) + B\, K_\nu(k\rho) \tag{435}$$

- Since $K_\nu(x)$ blows up at $x = 0$, if $\rho = 0$ is in the region of interest, then all of the B coefficients must vanish to yield a finite value of the potential on the axis.
- Similarly, since $I_\nu(x)$ blows up as $x \to \infty$, if the potential is to vanish as $\rho \to \infty$, then all of the A coefficients must vanish.

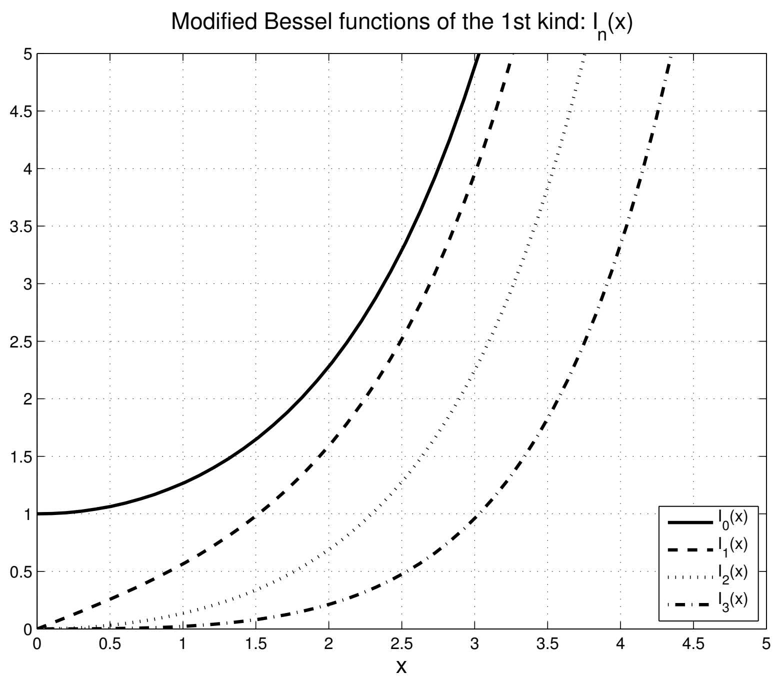

Figure 47: First few modified Bessel functions of the 1st kind for integer ν.

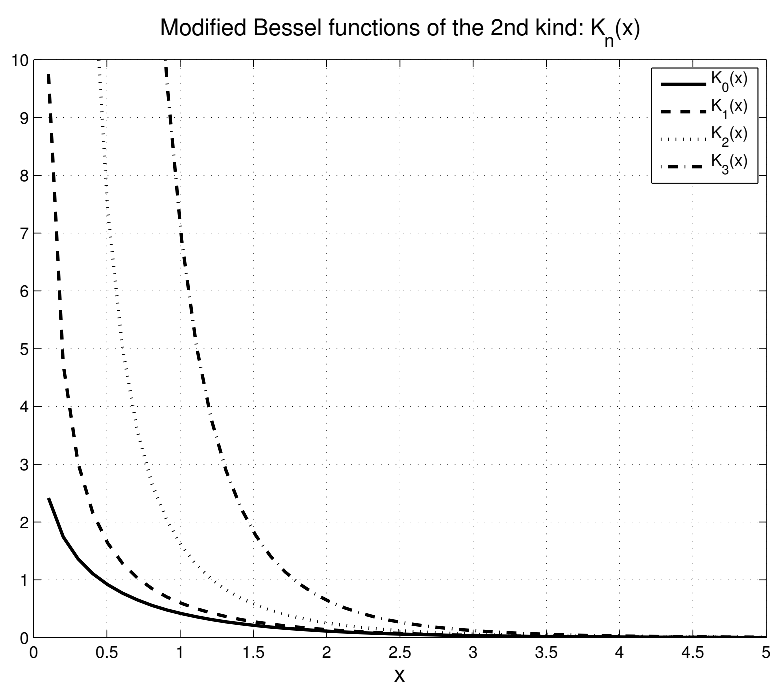

Figure 48: First few modified Bessel functions of the 2nd kind for integer ν.

- Recurrence relations:

$$\frac{d}{dx}\left(x^\nu J_\nu(x)\right) = x^\nu J_{\nu-1}(x) \tag{436}$$

$$\frac{d}{dx}\left(x^{-\nu} J_\nu(x)\right) = -x^{-\nu} J_{\nu+1}(x) \tag{437}$$

$$J'_\nu(x) = -\frac{\nu}{x} J_\nu(x) + J_{\nu-1}(x) \tag{438}$$

$$J'_\nu(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x) \tag{439}$$

$$2J'_\nu(x) = J_{\nu-1}(x) - J_{\nu+1}(x) \tag{440}$$

$$\frac{2\nu}{x} J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x) \tag{441}$$

- <u>Exercise</u>: Prove the above.

- Note that the recurrence relations also hold for $N_\nu(x)$, $H_\nu^{(1)}(x)$, $H_\nu^{(2)}(x)$, since they are relatively simple linear combinations of $J_\nu(x)$ and $J_{-\nu}(x)$.

- Orthogonality:

$$\int_a^b d\rho\, \rho J_\nu(k\rho) J_\nu(k'\rho) = 0 \quad \text{for } k \neq k' \tag{442}$$

  where

$$\rho \left[J_\nu(k\rho) \frac{dJ_\nu}{d\rho}(k'\rho) - J_\nu(k'\rho) \frac{dJ_\nu}{d\rho}(k\rho) \right] \Bigg|_{\rho=a}^{b} = 0 \tag{443}$$

- <u>Exercise</u>: Prove this. (Hint: Let $f(\rho) = J_\nu(k\rho)$, $g(\rho) = J_\nu(k'\rho)$, write down Bessel's equation for f and g, multiply these equations by g and f, subtract, and then integrate.)

- An explicit example satifying the above boundary condition is to choose $a = 0$, rename $b = a$, and then choose k and k' so that $J_\nu(ka) = 0 = J_\nu(k'a)$. For this case k and k' take on discrete values

$$k \equiv k_{\nu n} = \frac{x_{\nu n}}{a}\,, \qquad k' \equiv k_{\nu n'} = \frac{x_{\nu n'}}{a}\,, \qquad n, n' = 1, 2, \cdots \tag{444}$$

  where $x_{\nu n}$ and $x_{\nu n'}$ are the nth and n'th zeroes of $J_\nu(x)$.

- Note that the orthogonality of Bessel functions is wrt to different arguments $x = k\rho$ and $x' = k'\rho$ of a *single* function $J_\nu(x)$, and not wrt *different* functions $J_\nu(x)$ and $J_{\nu'}(x)$ of the same argument $x = k\rho$. (This latter case held for the Legendre polynomials $P_l(x)$ and $P_{l'}(x)$.)

- The orthogonality of Bessel functions is similar to the orthogonality of the sine functions $\sin(n2\pi x/a)$ for different values of n.

- Normalization:

$$\int_a^b d\rho\, \rho J_\nu(k\rho) J_\nu(k\rho) = \frac{1}{2} \left[\left(\rho^2 - \frac{\nu^2}{k^2} \right) J_\nu^2(k\rho) + \rho^2 [J'_\nu(k\rho)]^2 \right] \Bigg|_{\rho=a}^{b} \tag{445}$$

  where $J'_\nu(k\rho)$ denotes derivative wrt to its argument $x = k\rho$.

- <u>Exercise</u>: Prove the normalization condition. (Note: This is a rather tricky proof, requiring some clever integration by parts and the use of Bessel's equation to substitute for $x^2 J_\nu(x)$ in one of the integrals.)

- Again the RHS can be simplified for the case described above where we set $a = 0$, rename $b = a$, and take $k = x_{\nu n}/a$ for some integer n. Then

$$\int_0^a d\rho\, \rho J_\nu(x_{\nu n}\rho/a) J_\nu(x_{\nu n}\rho/a) = \frac{1}{2}a^2 [J'_\nu(x_{\nu n})]^2 = \frac{1}{2}a^2 J^2_{\nu+1}(x_{\nu n}) \tag{446}$$

where a recurrence relation was used to get the last equality.

- Exercise: Prove this.

- We can put the orthogonality and normalisation equations together as a single equation:

$$\int_0^a d\rho\, \rho J_\nu(x_{\nu n}\rho/a) J_\nu(x_{\nu n'}\rho/a) = \frac{1}{2}a^2 J^2_{\nu+1}(x_{\nu n})\, \delta_{nn'} \tag{447}$$

where we have explicitly indicated the zeroes $x_{\nu n}$ and $x_{\nu n'}$ of $J_\nu(x)$.

- If the interval $[0, a]$ becomes infinite $[0, \infty)$, then the orthogonality and normalisation conditions actually become simpler

$$\int_0^\infty d\rho\, \rho J_\nu(k\rho) J_\nu(k'\rho) = \frac{1}{k}\delta(k - k') \tag{448}$$

where k now takes on a continuous range of values.

- This is similar to the transition from Fourier series (basis functions $e^{ik_n x}$ with $k_n = n2\pi/a$) to Fourier transforms (basis functions e^{ikx} with k a real variable):

$$\int_{-a/2}^{a/2} dx\, e^{i2\pi(n-n')x/a} = a\, \delta_{nn'} \longrightarrow \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = 2\pi\, \delta(k - k') \tag{449}$$

4.10.2 Spherical Bessel functions

- *Spherical Bessel functions* are defined by

$$j_n(x) := \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x) \tag{450}$$

$$n_n(x) := \sqrt{\frac{\pi}{2x}}\, N_{n+\frac{1}{2}}(x) \tag{451}$$

where $n = 0, 1, 2, \cdots$.

- One can also define

$$h_n^{(1)}(x) := j_n(x) + i n_n(x) \tag{452}$$

$$h_n^{(2)}(x) := j_n(x) - i n_n(x) \tag{453}$$

- Given the explicit form of $J_{n+\frac{1}{2}}(x)$ one can show that

$$j_n(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right) \tag{454}$$

$$n_n(x) = -x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right) \tag{455}$$

- In particular, it follows that

$$j_0(x) = \frac{\sin x}{x}\,, \qquad n_0(x) = -\frac{\cos x}{x} \tag{456}$$

See Figures 49 and 50 for plots of the first few spherical Bessel functions.

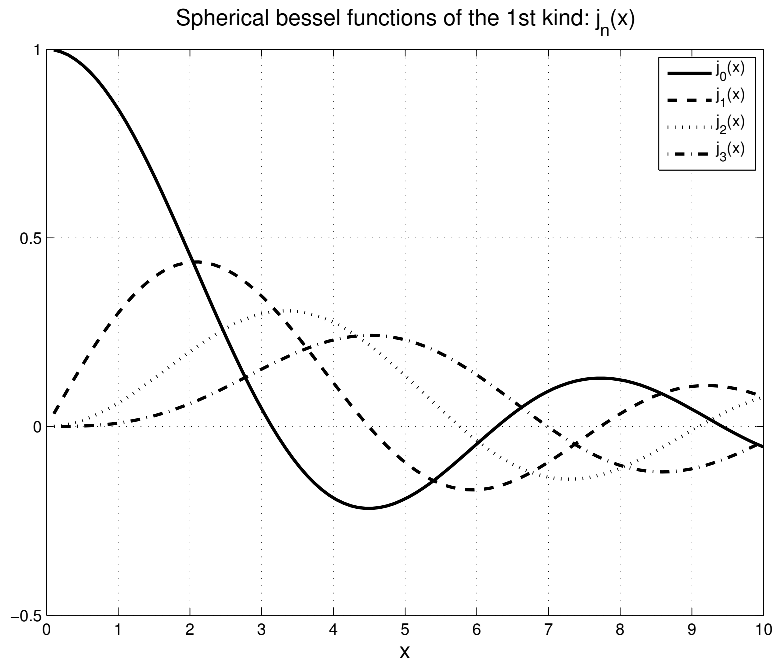

Figure 49: First few spherical Bessel functions of the 1st kind.

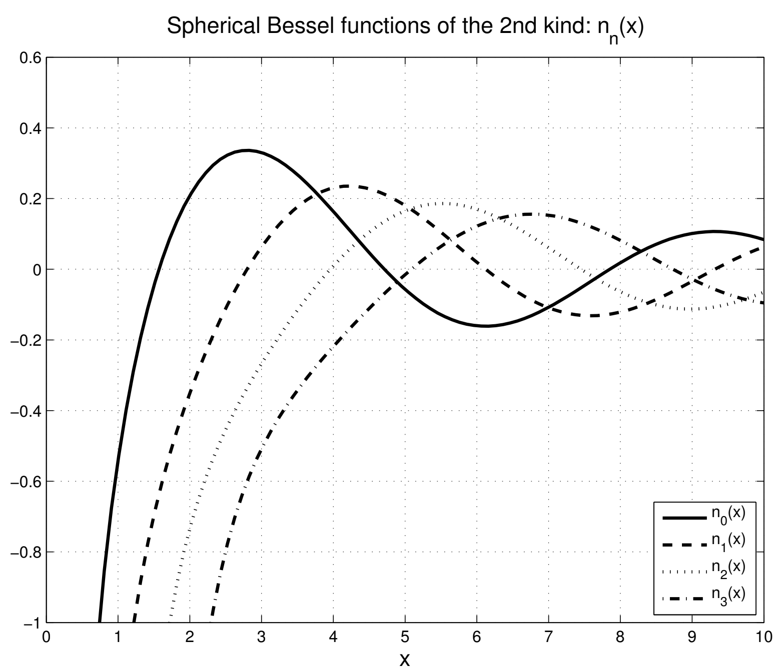

Figure 50: First few spherical Bessel functions of the 2nd kind.

- Exercise: Prove the above expression for $j_0(x)$ directly from its definition in terms of the ordinary Bessel function $J_{\frac{1}{2}}(x)$.

- Given the relationship between $j_n(x)$ and $J_{n+\frac{1}{2}}(x)$, one can show that the spherical Bessel functions satisfy the differential equation

$$j_n''(x) + \frac{2}{x} j_n'(x) + \left[1 - \frac{n(n+1)}{x^2}\right] j_n(x) = 0 \tag{457}$$

- Exercise: Prove this.

- Alternatively, one arrives at the same differential equation by using separation of variables in *spherical polar coordinates* to solve the *Helmholtz equation*:

$$\nabla^2 \Phi(r, \theta, \phi) + k^2 \Phi(r, \theta, \phi) = 0 \tag{458}$$

- The ϕ equation is the standard harmonic oscillator equation with separation constant $-m^2$; the θ equation is the associated Legendre's equation with separation constants l and m; and the radial equation is

$$R''(r) + \frac{2}{r} R'(r) + \left[k^2 - \frac{l(l+1)}{r^2}\right] R(r) = 0 \tag{459}$$

- Making the change of variables $x = kr$ with $y(x)|_{x=kr} = R(r)$, leads to

$$y''(x) + \frac{2}{x} y'(x) + \left[1 - \frac{l(l+1)}{x^2}\right] y(x) = 0 \tag{460}$$

  which is the differential equation (457) we found earlier with solution $y(x) = j_l(x)$.

4.10.3 Examples

- Example 1: Solve Laplace's equation interior to a cylinder of radius a and height L, with zero potential on the bottom and sides of the cylinder, and specified potential $f(\rho, \phi)$ on the top. (See Figure 51.)

- Answer:

  Choose cylindrical polar coordinates so that the axis of the cylinder coincides with the $\hat{\mathbf{z}}$ axis, and that the bottom and top of the cylinder have $z = 0$ and $z = L$ respectively.

  Note:

  1) The fact the potential has a non-zero value on the top of the cylinder implies using a positive separation constant $+k^2$ for the z-equation.

  2) The BC that the potential vanish at $z = 0$ implies that there are no $\cosh(kz)$ terms.

  3) Single-valuedness of $\Phi(\phi)$ implies that $\nu = m$ is an integrer.

  4) Finiteness at the axis ($\rho = 0$) implies that there are no $N_m(x)$ terms in the general solution to Bessel's equation.

  5) The BC that the potential vanish when $\rho = a$ implies that k take on the discrete values $k_{mn} = x_{mn}/a$, where x_{mn} is the nth zero $J_m(x)$.

  Thus,

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(x_{mn}\rho/a) \sinh(x_{mn} z/a) \left[A_{mn} \sin(m\phi) + B_{mn} \cos(m\phi)\right] \tag{461}$$

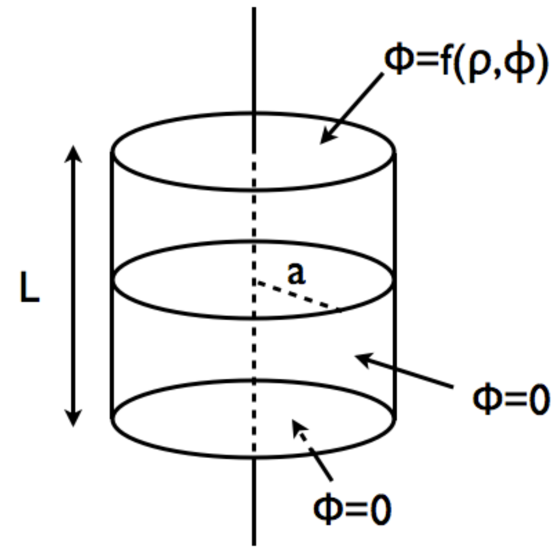

Figure 51: Cylinder of radius a and height L, with zero potential on the bottom and sides, and specified potential $\Phi = f(\rho, \phi)$ on the top.

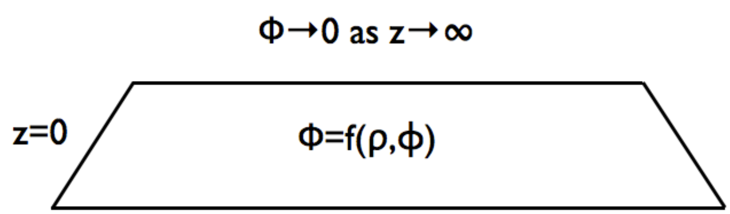

Figure 52: Infinite 2-d plane ($z = 0$) with specified potential $\Phi = f(\rho, \phi)$ on the plane and $\Phi \to 0$ as $z \to \infty$.

where the BC that $\Phi(\rho, \phi, L) = f(\rho, \phi)$ implies

$$A_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(x_{mn}) \sinh(x_{mn}L/a)} \int_0^{2\pi} d\phi \int_0^a d\rho \, \rho f(\rho, \phi) J_m(x_{mn}\rho/a) \sin(m\phi) \quad (462)$$

$$B_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(x_{mn}) \sinh(x_{mn}L/a)} \int_0^{2\pi} d\phi \int_0^a d\rho \, \rho f(\rho, \phi) J_m(x_{mn}\rho/a) \cos(m\phi) \quad (463)$$

Note that for $m = 0$, the term $B_{mn} \cos(m\phi)$ should be $B_{0n}/2$ where B_{0n} is calculated from the above integral.

- Example 2: Solve Laplace's equation above an infinite 2-d plane with specified potential $f(\rho, \phi)$ and vanishing potential infinitely far from the plane. (See Figure 52.)

- Answer:

  Choose cylindrical polar coordinates so that the 2-d plane corresponds to $z = 0$.

  Note:

1) The fact the range of z extends to ∞ implies using a positive separation constant $+k^2$ for the z-equation.

2) The BC that the potential vanishes as $z \to \infty$ implies that there are no e^{+kz} terms.

3) Single-valuedness of $\Phi(\phi)$ implies that $\nu = m$ is an integrer.

4) Finiteness at the axis ($\rho = 0$) implies that there are no $N_m(x)$ terms in the general solution to Bessel's equation.

Thus,

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, J_m(k\rho) e^{-kz} \left[A_m(k)\, \sin(m\phi) + B_m(k)\, \cos(m\phi)\right] \tag{464}$$

where the BC that $\Phi(\rho, \phi, 0) = f(\rho, \phi)$ implies

$$A_m(k) = \frac{k}{\pi} \int_0^{2\pi} d\phi \int_0^{\infty} d\rho\, \rho f(\rho, \phi) J_m(k\rho) \sin(m\phi) \tag{465}$$

$$B_m(k) = \frac{k}{\pi} \int_0^{2\pi} d\phi \int_0^{\infty} d\rho\, \rho f(\rho, \phi) J_m(k\rho) \cos(m\phi) \tag{466}$$

Note that for $m = 0$, the term $B_m(k)\cos(m\phi)$ should be $B_0(k)/2$ where $B_0(k)$ is calculated from the above integral.

- Such expansions are called *Fourier-Bessel expansions*.

4.11 Eigenfunction expansions of Green's functions

- The general form of a homogeneous, linear, second-order ordinary differential equation is

$$y''(x) + p(x)\, y'(x) + q(x)\, y(x) = 0 \tag{467}$$

where $p(x)$ and $q(x)$ are arbitrary functions of x.

- The differential equation is said to be in *self-adjoint* (or Hermitian) form if

$$\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] + g(x)\, y(x) = 0 \tag{468}$$

- <u>Exercise</u>: Prove that *any* homogeneous, linear, second-order ordinary differential equation can be put into self-adjoint form by multiplying the equation (467) by $\exp[\int^x p(x')dx']$. You should obtain equation (468) with

$$f(x) = \exp\left[\int^x p(x')\, dx'\right], \quad g(x) = \exp\left[\int^x p(x')\, dx'\right] q(x) \tag{469}$$

- All of the equations that we are interested in (e.g., Legendre's equation, Bessel's equation, the radial and angular parts of Laplace's equation in spherical polar coordinates, etc.) are already in self-adjoint form.

- Let $y_1(x)$ and $y_2(x)$ be two solutions of equation (467) or (468). Then $y_1(x)$ and $y_2(x)$ are linearly independent if and only if the *Wronskian*

$$W(x) \equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \tag{470}$$

is non-zero.

- The Wronskian itself satisfies a differential equation

$$W'(x) = -p(x)\,W(x) \tag{471}$$

which can be obtained by differentiating the equation that defines W and then using equation (467). to substitute for y_1'' and y_2''.

- Exercise: Prove the above.

- From the above equation for W it follows that

$$W(x) = C\exp\left[-\int^x p(x')\,dx'\right] = \frac{C}{f(x)} \tag{472}$$

(Note: We will need to use this result later on when expanding Green's functions in cylindrical polar coordinates.)

- Consider the region $x \in [a,b]$ (where a, b could be infinite), and define an *inner product*

$$\langle y_1|y_2\rangle \equiv \int_a^b y_1^*(x)y_2(x)\,w(x)dx \tag{473}$$

Here $*$ denotes complex conjugation and $w(x) > 0$ is a (real) *weight* function.

- For most cases $w = 1$, but for Legendre polynomials and Bessel functions, the weight functions are $w(\theta) = \sin\theta$ and $w(\rho) = \rho$, respectively.

- Orthogonality and normalization of functions is defined wrt the inner product.

- A linear operator $\mathcal{L}$ is said to be *self-adjoint* (or Hermitian) wrt the inner product if and only if

$$\langle \mathcal{L}y_1|y_2\rangle = \langle y_1|\mathcal{L}y_2\rangle \tag{474}$$

or, equivalently,

$$\int_a^b [\mathcal{L}y_1(x)]^*\,y_2(x)w(x)\,dx = \int_a^b y_1^*(x)\,[\mathcal{L}y_2(x)]\,w(x)\,dx \tag{475}$$

- Consider the linear differential operator $\mathcal{L}$ defined by

$$\mathcal{L}y(x) \equiv \frac{1}{w(x)}\left\{\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] + g(x)\,y(x)\right\} \tag{476}$$

which is proportional to the LHS of the self-adjoint form (468) of a homogeneous, linear, second-order ordinary differential equation.

- One can show that $\mathcal{L}$ is Hermitian wrt the above inner product iff

$$[y_1^* f y_2' - y_2 f y_1^{*\prime}]\,\Big|_a^b = 0 \tag{477}$$

- Exercise: Prove the above result assuming $f(x)$ and $g(x)$ are real.

- In particular, $\mathcal{L}$ will be Hermitian if the functions $y_1(x)$ and $y_2(x)$ vanish on the boundary $x = a, b$. Such solutions are said to satisfy *homogeneous* boundary conditions, and we will assume such BCs when we calculate Dirichlet Green's functions. (Other boundary conditions are possible, but we will not consider them here.)

- A *Sturm-Liouville* equation has the form

$$\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] + g(x)\,y(x) - \lambda\,w(x)y(x) = 0 \tag{478}$$

where λ is some fixed constant.

- In terms of the linear operator $\mathcal{L}$, the Sturm-Liouville equation can be written as

$$\mathcal{L}y(x) = \lambda\, y(x) \tag{479}$$

which is an eigenvalue equation.

- In order that $\mathcal{L}$ be Hermitian, the homogeneous boundary conditions on the eigenfunctions restrict the allowed values of λ:

$$\mathcal{L}\psi_n(x) = \lambda_n\, \psi_n(x) \tag{480}$$

- Recall that for a Hermitian operator:

  1) the eigenvalues λ_n are real.

  2) the eigenfunctions $\psi_n(x)$ and $\psi_{n'}(x)$ corresponding to distinct eigenvalues are orthogonal.

  3) the set of eigenfunctions $\{\psi_n(x)\}$ span the space of square-integrable functions satisfying the same BCs as the eigenfunctions.

- Several examples of differential equations, their eigenfunctions, and eigenvalues are given in the next subsection, Section 4.12.

- Suppose we want to solve the *inhomogeneous* equation

$$\mathcal{L}y(x) = \frac{F(x)}{w(x)} \tag{481}$$

where $F(x)$ is some source term.

- By expanding $y(x)$ in terms of the eigenfunctions

$$y(x) = \sum_n A_n\, \psi_n(x) \tag{482}$$

one can show that

$$y(x) = \int_a^b dx' F(x') \left(\sum_n \frac{\psi_n^*(x')\psi_n(x)}{N_n \lambda_n} \right) \tag{483}$$

where

$$N_n := \int_a^b |\psi_n(x)|^2\, w(x) dx \tag{484}$$

is the normalization of the eigenfunction $\psi_n(x)$.

- Exercise: Prove the above.

- Recalling that a Dirichlet Green's function satisfies

$$\mathcal{L}G_D(x, x') = \delta(x - x') \Longleftrightarrow y(x) = \int_a^b dx'\, G_D(x, x')F(x') \tag{485}$$

we can conclude that

$$G_D(x, x') = \sum_n \frac{\psi_n^*(x')\psi_n(x)}{N_n \lambda_n} \tag{486}$$

- NOTES:

  1) The summation $\sum_n$ should be replaced by an integration $\int dk$ if the eigenvalues are labeled by a continuous index k.

2) The above result extends to functions of many variables:

$$G_D(\mathbf{r},\mathbf{r}') = \sum_n \frac{\psi_n^*(\mathbf{r}')\psi_n(\mathbf{r})}{N_n\lambda_n} \tag{487}$$

3) It also extends to solutions of the more general differential equation

$$\mathcal{L}y(x) - \lambda\, y(x) = \frac{F(x)}{w(x)} \tag{488}$$

were λ is a constant, not equal to any of the eigenvalues λ_n. The expression for the Dirichlet Green's function for this case is

$$G_D(x,x') = \sum_n \frac{\psi_n^*(x')\psi_n(x)}{N_n(\lambda_n - \lambda)} \tag{489}$$

- Exercise: Prove this last statement.
- Example 1: The Dirichlet Green's function for Poisson's equation in 1-dimension

$$\frac{d^2}{dx^2}G_D(x,x') = \delta(x-x') \tag{490}$$

for the finite interval $0 \le x \le a$, can be expanded in terms of sinusoids

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \tag{491}$$

with eigenvalues

$$\lambda_n = -\left(\frac{n\pi}{a}\right)^2 \tag{492}$$

Thus,

$$G_D(x,x') = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)}{-\left(\frac{n\pi}{a}\right)^2} \tag{493}$$

- Example 2: The infinite space Dirichlet Green's function for Poisson's equation

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{494}$$

can be expanded in terms of the complex exponentials

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\, e^{i\mathbf{k}\cdot\mathbf{r}} \tag{495}$$

where $\mathbf{k}$ is a vector in 3-dimensional space, and the normalization factor has been chosen so that $N_{\mathbf{k}} = 1$. These are eigenfunctions of the 3-d Laplacian operator $\mathcal{L} = \nabla^2$ with eigenvalues

$$\lambda_{\mathbf{k}} = -k^2 \tag{496}$$

Thus,

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{4\pi}{(2\pi)^3}\int_{\text{all space}} dV_{\mathbf{k}}\, \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2} \tag{497}$$

Note that the extra factor of 4π and the absence of the minus sign in the denominator come from the definition (494) of the Green's function in 3-dimensions.

- Example 3: The Dirichlet Green's for Poisson's equation inside a rectangular box ($0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$) can be expanded in terms of a product of sinusoids:

$$\psi_{lmn}(\mathbf{r}) = \sqrt{\frac{8}{abc}} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi z}{c}\right) \tag{498}$$

These are eigenfunctions of the Laplacian with eigenvalues

$$\lambda_{lmn} = -\left[\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2\right] \tag{499}$$

Thus,

$$G_D(\mathbf{r},\mathbf{r}') = \frac{32\pi}{abc} \sum_{l,m,n=1}^{\infty} \frac{\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)}{\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2} \tag{500}$$

4.12 Summary of key eigenfunction formulas

- Cartesian coordinate:

$$-\infty < x < \infty \quad \text{(similar for } y, z) \tag{501}$$

Differential equation:

$$\frac{d^2\psi}{dx^2} = \lambda\, \psi(x) \tag{502}$$

Weight function:

$$w(x) = 1 \tag{503}$$

Eigenfunctions:

$$\psi_k(x) = e^{ikx}\,, \qquad -\infty < k < \infty \tag{504}$$

Eigenvalues:

$$\lambda_k = -k^2 \tag{505}$$

Orthonormality:

$$\int_{-\infty}^{\infty} dx\; e^{i(k-k')x} = 2\pi\, \delta(k-k') \tag{506}$$

Completeness:

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ik(x-x')} = \frac{1}{\pi}\int_0^{\infty} dk\; \cos[k(x-x')] \tag{507}$$

- Cartesian coordinate:

$$0 \le x \le a \quad \text{(similar for } y, z) \tag{508}$$

Differential equation:

$$\frac{d^2\psi}{dx^2} = \lambda\, \psi(x) \tag{509}$$

Weight function:

$$w(x) = 1 \tag{510}$$

Eigenfunctions:

$$\psi_n(x) = \sin\left(\frac{n\pi x}{a}\right)\,, \qquad n = 1, 2, \cdots \tag{511}$$

Eigenvalues:

$$\lambda_n = -\left(\frac{n\pi}{a}\right)^2 \tag{512}$$

Orthonormality:

$$\int_0^a dx\, \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n'\pi x}{a}\right) = \frac{a}{2}\,\delta_{nn'} \tag{513}$$

Completeness:

$$\delta(x - x') = \frac{2}{a}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right) \tag{514}$$

- Azimuthal angle:

$$0 \le \phi \le 2\pi \tag{515}$$

Differential equation:

$$\frac{d^2\psi}{d\phi^2} = \lambda\, \psi(\phi) \tag{516}$$

Weight function:

$$w(\phi) = 1 \tag{517}$$

Eigenfunctions:

$$\psi_m(\phi) = e^{im\phi}\,, \qquad m = 0, \pm 1, \pm 2, \cdots \tag{518}$$

Eigenvalues:

$$\lambda_m = -m^2 \tag{519}$$

Orthonormality:

$$\int_0^{2\pi} d\phi\; e^{i(m-m')\phi} = 2\pi\, \delta_{mm'} \tag{520}$$

Completeness:

$$\delta(\phi - \phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{m=1}^{\infty} \cos[m(\phi - \phi')] \tag{521}$$

- Polar angle:

$$0 \le \theta \le \pi \quad (\text{or } \; -1 \le x \le 1\,, \quad \text{where } x = \cos\theta) \tag{522}$$

Differential equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) = \lambda\, P(\theta) \tag{523}$$

Weight function:

$$w(\theta) = \sin\theta \tag{524}$$

Eigenfunctions:

$$P_l(\cos\theta)\,, \qquad l = 0, 1, 2, \cdots \tag{525}$$

Eigenvalues:

$$\lambda_l = -l(l+1) \tag{526}$$

Orthonormality:

$$\int_0^{\pi} d\theta\, \sin\theta P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{ll'} \tag{527}$$

Completeness:

$$\frac{1}{\sin\theta}\delta(\theta - \theta') = \frac{2l+1}{2}\sum_{l=0}^{\infty} P_l(\cos\theta) P_l(\cos\theta') \tag{528}$$

- 2-sphere coordinates:

$$0 \leq \theta \leq \pi\,, \quad 0 \leq \phi \leq 2\pi \tag{529}$$

Differential equation:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = \lambda\, Y(\theta,\phi) \tag{530}$$

Weight function:

$$w(\theta,\phi) = \sin\theta \tag{531}$$

Eigenfunctions:

$$Y_{lm}(\theta,\phi)\,, \qquad l = 0,1,2,\cdots\,, \qquad m = -l,-l+1,\cdots,l \tag{532}$$

Eigenvalues:

$$\lambda_{lm} = -l(l+1) \tag{533}$$

Orthonormality:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, Y_{lm}(\theta,\phi) Y^*_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \tag{534}$$

Completeness:

$$\frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \tag{535}$$

- Cylindrical radius:

$$0 \leq \rho \leq a \tag{536}$$

Differential equation:

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - \frac{\nu^2}{\rho^2}R(\rho) = \lambda\, R(\rho)\,, \quad \nu \text{ real (fixed)} \tag{537}$$

Weight function:

$$w(\rho) = \rho \tag{538}$$

Eigenfunctions:

$$R_n(\rho) = J_\nu(x_{\nu n}\rho/a)\,, \qquad n = 1,2,\cdots \tag{539}$$

Eigenvalues:

$$\lambda_n = -\frac{x_{\nu n}^2}{a^2} \tag{540}$$

Orthonormality:

$$\int_0^a d\rho\, \rho J_\nu(x_{\nu n}\rho/a)J_\nu(x_{\nu n'}\rho/a) = \frac{1}{2}a^2 J^2_{\nu+1}(x_{\nu n})\,\delta_{nn'} \tag{541}$$

Completeness:

$$\frac{1}{\rho}\,\delta(\rho-\rho') = \frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_\nu(x_{\nu n}\rho/a)J_\nu(x_{\nu n}\rho'/a)}{J^2_{\nu+1}(x_{\nu n})} \tag{542}$$

- Cylindrical radius:

$$0 \leq \rho < \infty \tag{543}$$

Differential equation:

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - \frac{\nu^2}{\rho^2}R(\rho) = \lambda\, R(\rho)\,, \quad \nu \text{ real (fixed)} \tag{544}$$

Weight function:

$$w(\rho) = \rho \tag{545}$$

Eigenfunctions:

$$R_k(\rho) = J_\nu(k\rho)\,, \qquad k \geq 0 \tag{546}$$

Eigenvalues:

$$\lambda_k = -k^2 \tag{547}$$

Orthonormality:

$$\int_0^\infty d\rho\, \rho J_\nu(k\rho) J_\nu(k'\rho) = \frac{1}{k}\, \delta(k-k') \tag{548}$$

Completeness:

$$\frac{1}{\rho}\, \delta(\rho-\rho') = \int_0^\infty dk\, k J_\nu(k\rho) J_\nu(k\rho') \tag{549}$$

4.13 Expanding Green's functions by solving 1-d δ-function equations

- An alternative method of calculating a Green's function is to expand the equation

$$\nabla^2 G_D(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{550}$$

with respect to two of the three coordinates (e.g., x, y, or θ, ϕ, etc.) and solve the resulting ODE with a Dirac delta function source in just the remaining coordinate (e.g., z or r, etc.).

- This method is best illustrated by two examples:

- Example 1: Find an expression for the Dirichlet Green's function for Poisson's equation in 1-dimension for $0 \leq x \leq a$. (Note: For $-\infty < x < \infty$, the Dirichlet Green's function is identically zero, as the only solution to $d^2G_D/dx^2 = \delta(x-x')$ which vanishes as $|x| \to \infty$ is $G_D(x, x') = 0$.)

- Solution:

1) The solution to

$$\frac{d^2}{dx^2} G_D(x, x') = \delta(x-x') \tag{551}$$

for $x < x'$ and $x > x'$ where the RHS is zero is:

$$G_D(x, x') = \begin{cases} A(x') + B(x')\, x & \text{for } x < x' \\ C(x') + D(x')\, x & \text{for } x > x' \end{cases} \tag{552}$$

2) Applying the homogeneous BCs at $x = 0$ and $x = a$ leads to

$$A(x') = 0\,, \qquad C(x') = -D(x')\, a \tag{553}$$

so that

$$G_D(x, x') = \begin{cases} B(x')\, x & \text{for } x < x' \\ D(x')\, (x-a) & \text{for } x > x' \end{cases} \tag{554}$$

3) Applying the symmetry of $G_D(x, x')$ further reduces the freedom of the integration constants to an overall multiplicative constant:

$$B(x') = E\,(x'-a)\,, \qquad D(x') = E\, x' \tag{555}$$

so that

$$G_D(x, x') = E\, x_<(x_> - a) \tag{556}$$

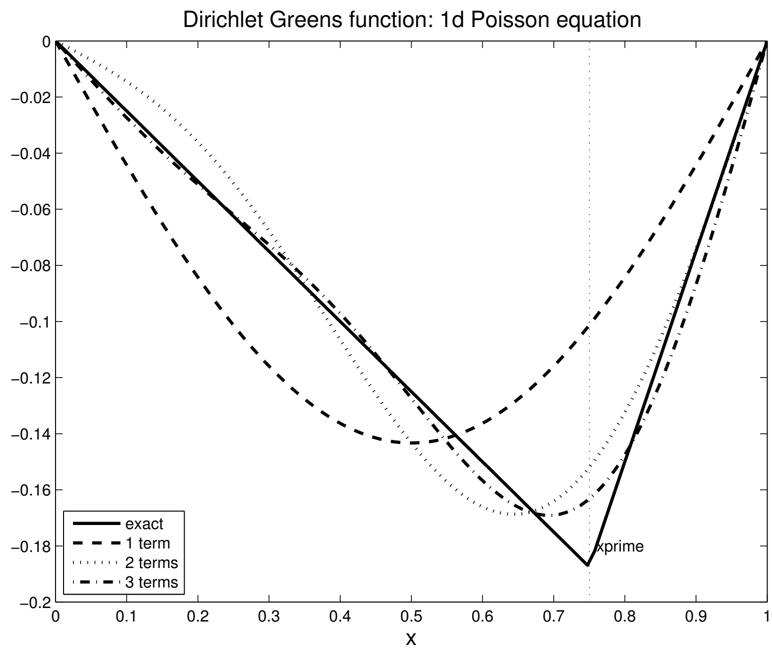

Figure 53: Dirichlet Green's function for the 1-dimensional Poisson's equation for the finite interval $0 \leq x \leq a$. The exact solution and Fourier series expansions containing one, two, and three terms are plotted, for the case $a = 1$, $x' = 0.75a$.

4) The constant E can be determined by integrating the differential equation for $G_D(x, x')$ across the delta function singularity from $x = x' - \epsilon$ to $x = x' + \epsilon$, then taking the limit $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} \left\{ \frac{d}{dx} G_D(x, x') \bigg|_{x=x'-\epsilon}^{x'+\epsilon} \right\} = 1 \Leftrightarrow E = \frac{1}{a} \tag{557}$$

5) Thus,

$$G_D(x, x') = -\frac{1}{a}\, x_<(a - x_>) \tag{558}$$

Note: When we used the eigenfunction method to determine the Green's function, we found (493):

$$G_D(x, x') = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right)}{-\left(\frac{n\pi}{a}\right)^2} \tag{559}$$

Thus,

$$-\frac{1}{a} x_<(a - x_>) = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right)}{-\left(\frac{n\pi}{a}\right)^2} \tag{560}$$

is the Fourier series expansion of the 1-dimensional Dirichlet Green's function. (See Figure 53.)

- Exercise: Extend the above analysis to determine the Dirichlet Green's function for the 1-dimensional simple harmonic oscillator equation

$$\frac{d^2}{dx^2} G_D(x, x') + k^2\, G_D(x, x') = \delta(x - x') \tag{561}$$

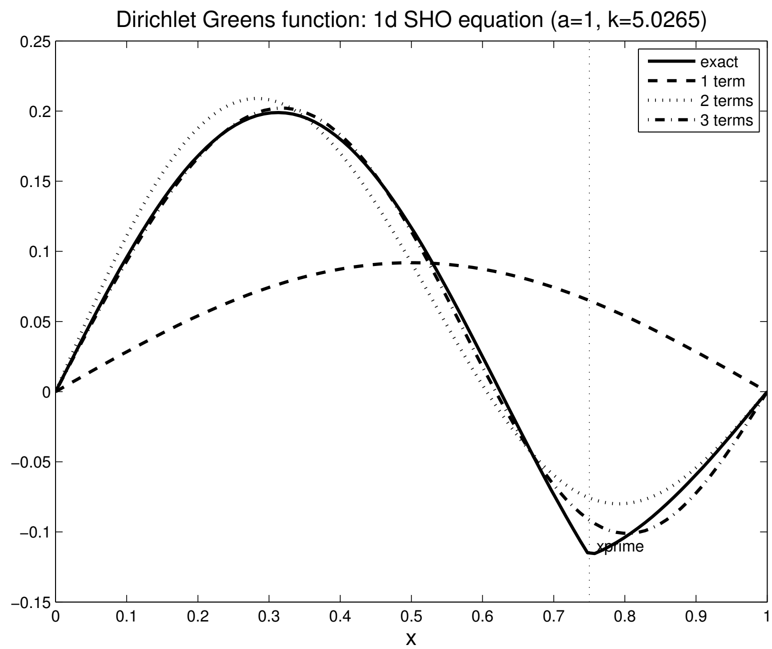

Figure 54: Dirichlet Green's function for the 1-dimensional simple harmonic oscillator equation for the finite interval $0 \le x \le a$. The exact solution and Fourier series expansions containing one, two, and three terms are plotted, for the case $a = 1$, $x' = 0.75a$, and $k = (4/5) \times 2\pi/a$.

for the finite interval $0 \le x \le a$. Assume that $k^2 \neq (n\pi/a)^2$ for any positive integer n. By solving the 1-d delta function equation, you should find

$$G_D(x, x') = -\frac{\sin(kx_<)\sin(k(a - x_>))}{k\sin(ka)} \tag{562}$$

Alternatively, in terms of an eigenfunction expansion

$$G_D(x, x') = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)}{k^2 - \left(\frac{n\pi}{a}\right)^2} \tag{563}$$

This last expression can be thought of as a Fourier series representation of the RHS of (562). (See Figure 54.) Note also that in the limit $k \to 0$, we recover the results (558) and (559) for the 1-d Poisson's equation, as we should.

- Example 2: Find an expression for the Dirichlet Green's function for a rectangular box ($0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$), singling out the z-coordinate for special treatment.

- Solution:

  1) Begin by expanding the Dirac delta function $\delta(\mathbf{r} - \mathbf{r}')$ in terms of the appropriate eigenfunctions for the rectangular box, leaving the $\delta(z - z')$ factor as is:

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(x - x')\delta(y - y')\delta(z - z') \tag{564}$$

$$= \delta(z - z')\frac{4}{ab}\sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right) \tag{565}$$

  2) Do the same for the Green's function, leaving the z, z' dependence to be determined:

$$G_D(\mathbf{r}, \mathbf{r}') = \frac{4}{ab}\sum_{l,m=1}^{\infty} g_{lm}(z, z')\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right) \tag{566}$$

3) Substitute the above expressions into $\nabla^2 G_D(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}')$, obtaining

$$\frac{d^2}{dz^2} g_{lm}(z,z') - k_{lm}^2 g_{lm}(z,z') = -4\pi\,\delta(z-z') \tag{567}$$

where

$$k_{lm}^2 := \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \tag{568}$$

4) Solve this equation for $z < z'$ and $z > z'$ where the RHS is zero:

$$g_{lm}(z,z') = \begin{cases} A(z')\,\sinh(k_{lm}z) + B(z')\,\cosh(k_{lm}z) & \text{for } z < z' \\ C(z')\,\sinh(k_{lm}z) + D(z')\,\cosh(k_{lm}z) & \text{for } z > z' \end{cases} \tag{569}$$

5) Apply the homogeneous BCs at $z = 0$ and $z = c$, yielding

$$B(z') = 0\,, \qquad C(z') = -D(z')\frac{\cosh(k_{lm}c)}{\sinh(k_{lm}c)} \tag{570}$$

so that

$$g_{lm}(z,z') = \begin{cases} A(z')\,\sinh(k_{lm}z) & \text{for } z < z' \\ D(z')\,\frac{\sinh[k_{lm}(c-z)]}{\sinh(k_{lm}c)} & \text{for } z > z' \end{cases} \tag{571}$$

6) Apply the symmetry of $G_D(\mathbf{r},\mathbf{r}')$ to further reduce the freedom of the integration constants to an overall multiplicative constant:

$$A(z') = E\,\frac{\sinh\left[k_{lm}(c-z')\right]}{\sinh(k_{lm}c)}\,, \qquad D(z') = E\,\sinh(k_{lm}z') \tag{572}$$

so that

$$g_{lm}(z,z') = E\,\frac{\sinh(k_{lm}z_<)\sinh\left[k_{lm}(c-z_>)\right]}{\sinh(k_{lm}c)} \tag{573}$$

7) Determine the constant E by integrating the differential equation for $g_{lm}(z,z')$ across the delta function singularity from $z = z' - \epsilon$ to $z = z' + \epsilon$, taking the limit $\epsilon \to 0$:

$$\lim_{\epsilon\to 0}\left\{ \frac{d}{dz} g_{lm}(z,z')\Big|_{z=z'-\epsilon}^{z'+\epsilon} \right\} = -4\pi \tag{574}$$

which leads to

$$E = \frac{4\pi}{k_{lm}} \tag{575}$$

8) Substitute this constant back into the formulas to obtain the solutions

$$g_{lm}(z,z') = 4\pi\frac{\sinh(k_{lm}z_<)\sinh\left[k_{lm}(c-z_>)\right]}{k_{lm}\sinh(k_{lm}c)} \tag{576}$$

and

$$\begin{aligned} G_D(\mathbf{r},\mathbf{r}') =& \frac{16\pi}{ab}\sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right) \\ &\times \frac{\sinh(k_{lm}z_<)\sinh\left[k_{lm}(c-z_>)\right]}{k_{lm}\sinh(k_{lm}c)} \end{aligned} \tag{577}$$

Note: If we compare this expression for $G_D(\mathbf{r},\mathbf{r}')$ with what we obtained earlier using the eigenfunction method:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{32\pi}{abc}\sum_{l,m,n=1}^{\infty} \frac{\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)}{\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2} \tag{578}$$

we can conclude that

$$\frac{\sinh(k_{lm}z_<)\sinh\left[k_{lm}(c-z_>)\right]}{k_{lm}\sinh(k_{lm}c)} = \frac{2}{c}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi z'}{c}\right)}{k_{lm}^2+\left(\frac{n\pi}{c}\right)^2}\sin\left(\frac{n\pi z}{c}\right) \tag{579}$$

Thus, the RHS is the Fourier series representation of the 1-dimensional Dirichlet Green's function satisfying

$$\frac{d^2}{dz^2}G_D(z,z') - k_{lm}^2\, G_D(z,z') = -\delta(z-z') \tag{580}$$

for $0 \le z \le c$. If we take the limit $k_{lm} \to 0$ of these last two equations, we recover the Fourier series expansion of the previous example with z and c replacing x and a. (One needs to apply L'Höpital's rule twice to take the limit $k_{lm} \to 0$ of the LHS of (579).)

4.14 Green's functions: More examples

- Example 1: Expand the infinite space Dirichlet Green's function $G_D(\mathbf{r},\mathbf{r}') = 1/|\mathbf{r}-\mathbf{r}'|$ in spherical polar coordinates, using the method of the previous section, singling out the r-coordinate for special treatment.

- Answer:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{581}$$

Note: Since

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma) \tag{582}$$

from the generating function for the Legendre polynomials, the above result may be thought of as an alternative derivation of the addition theorem for spherical harmonics:

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{583}$$

- Example 2: Proceeding as above, derive the Dirichlet Greens's function between two concentric spheres of radii a and b (with $a < b$).

- Answer:

$$G_D(\mathbf{r},\mathbf{r}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)}{\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \tag{584}$$

Note: From the above result we can obtain expressions for the Dirichlet Green's functions corresponding to the following special cases:

i) Infinite space Dirichlet Green's function in spherical polar coords (set $a \to 0$, $b \to \infty$):

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l-0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{585}$$

ii) Dirichlet Green's function exterior to a sphere of radius a (set $b \to \infty$):

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{a}{r'|\mathbf{r}-\frac{a^2}{r'^2}\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{a^{2l+1}}{(rr')^{l+1}}\right)Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{586}$$

iii) Dirichlet Green's function interior to a sphere of radius a (set $a \to 0$, then rename $b = a$):

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{a}{r'|\mathbf{r}-\frac{a^2}{r'^2}\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{(rr')^l}{a^{2l+1}}\right)Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{587}$$

- Example 3: Expand the infinite space Dirichlet Green's function $G_D(\mathbf{r},\mathbf{r}') = 1/|\mathbf{r}-\mathbf{r}'|$ in cylindrical polar coordinates, using the method of the previous section, singling out the ρ-coordinate for special treatment.

- Answer:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, \cos\left[k(z-z')\right] e^{im(\phi-\phi')} I_m(k\rho_<)K_m(k\rho_>) \tag{588}$$

To prove the above result we note the following:

1) We have the expansions

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{\delta(\rho-\rho')}{\rho}\frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, \cos\left[k(z-z')\right] e^{im(\phi-\phi')} \tag{589}$$

and

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, g_m(k,\rho,\rho')\cos\left[k(z-z')\right] e^{im(\phi-\phi')} \tag{590}$$

2) The differential equation satisfied by $g_m(k,\rho,\rho')$ is

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) - \left(k^2 + \frac{m^2}{\rho^2}\right) = -4\pi\,\frac{\delta(\rho-\rho')}{\rho} \tag{591}$$

For $\rho < \rho'$ and $\rho > \rho'$ this is the *modified* Bessel's equation.

3) BCs at $\rho \to 0$, $\rho \to \infty$, and symmetry of the solution imply

$$g_m(k,\rho,\rho') = E\, I_m(k\rho_<)K_m(k\rho_>) \tag{592}$$

4) Integrating the differential equation across the delta function singularity from $\rho = \rho' - \epsilon$ to $\rho = \rho' + \epsilon$ yields

$$-4\pi = E\,k\rho'\left[I_m(k\rho')K_m'(k\rho') - I_m'(k\rho')K_m(k\rho')\right] = E\,k\rho' W(k\rho') \tag{593}$$

where $W(x)$ is the Wronskian of $I_m(x)$ and $K_m(x)$, with $x = k\rho'$.

5) Since the modified Bessel's equation is in Sturm-Liouville form, we know that

$$W(x) = \frac{C}{x}\,, \quad \text{with } C = -1 \tag{594}$$

where C was evaluated using the asymptotic form of $I_m(x)$ and $K_m(x)$, as $x \to \infty$.

6) Thus, $E = 4\pi$ implying

$$g_m(k,\rho,\rho') = 4\pi\, I_m(k\rho_<)K_m(k\rho_>) \tag{595}$$

leading to the final expression for $G_D(\mathbf{r},\mathbf{r}')$.

- Example 4: Determine the Dirichlet Green's function between two infinite planes at $z = 0$ and $z = L$. (Jackson, Prob 3.17)

- Answers:
  Eigenfunction method:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\int_0^\infty dk\, k\frac{e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)J_m(k\rho)J_m(k\rho')}{\left(\frac{n\pi}{L}\right)^2+k^2} \tag{596}$$

  Singling out the z-coordinate for special treatment:

$$G_D(\mathbf{r},\mathbf{r}') = 2\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\frac{\sinh(kz_<)\sinh\left[k(L-z_>)\right]}{\sinh(kL)} \tag{597}$$

  Singling out the ρ-coordinate for special treatment:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)I_m\left(\frac{n\pi\rho_<}{L}\right)K_m\left(\frac{n\pi\rho_>}{L}\right) \tag{598}$$

- Example 5: Determine the Dirichlet Green's function inside a cylindrical can of height L and radius a. (Choose cylindrical polar coordinates so that the axis of the cylinder corresponds to $\rho=0$, and the bottom and top of the can to $z=0$ and $z=L$, respectively.) (Jackson, Prob 3.23)

- Answers:
  Eigenfunction method:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{8}{La^2}\sum_{m=-\infty}^{\infty}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}\sin\left(\frac{l\pi z}{a}\right)\sin\left(\frac{l\pi z'}{a}\right)J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{l\pi}{L}\right)^2\right]J_{m+1}^2(x_{mn})} \tag{599}$$

  Singling out the z-coordinate for special treatment:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{4}{a}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{x_{mn}J_{m+1}^2(x_{mn})}\frac{\sinh\left(\frac{x_{mn}z_<}{a}\right)\sinh\left[\frac{x_{mn}}{a}(L-z_>)\right]}{\sinh\left(\frac{x_{mn}L}{a}\right)} \tag{600}$$

  Singling out the ρ-coordinate for special treatment:

$$\begin{aligned}G_D(\mathbf{r},\mathbf{r}') =& \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{a}\right)\sin\left(\frac{n\pi z'}{a}\right)\frac{I_m\left(\frac{n\pi\rho_<}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)}\\ &\times\left[I_m\left(\frac{n\pi a}{L}\right)K_m\left(\frac{n\pi\rho_>}{L}\right)-K_m\left(\frac{n\pi a}{L}\right)I_m\left(\frac{n\pi\rho_>}{L}\right)\right]\end{aligned} \tag{601}$$

- Example 6: Determine a closed form expression for the infinite space Dirichlet Green's function in 2-dimensions, and an expansion for it in terms of the eigenfunctions of the azimuthal coordinate ϕ. (Jackson, Prob 2.17)

- Answers:
  Closed form expression:

$$G_D(\mathbf{r},\mathbf{r}') = -\ln\left(|\mathbf{r}-\mathbf{r}'|^2\right) = -\ln\left(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')\right) \tag{602}$$

  Expansion:

$$G_D(\mathbf{r},\mathbf{r}') = -\ln\left(\rho_>^2\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos\left[m(\phi-\phi')\right] \tag{603}$$

Note:

1) The closed form expression can be verified by showing that

$$\nabla^2 \ln\rho = 0 \quad \text{for } \rho \neq 0 \tag{604}$$

and

$$\int_D \nabla^2(\ln\rho)\, da = \oint_C (\boldsymbol{\nabla} \ln\rho)\cdot\hat{\mathbf{n}}\, dl = 2\pi \tag{605}$$

where D is any 2-d disk containing the origin with boundary circle C.

2) The second expression can be verified by expanding

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{\delta(\rho-\rho')}{\rho}\left\{\frac{1}{2\pi} + \frac{1}{\pi}\sum_{m=1}^{\infty}\cos\left[m(\phi-\phi')\right]\right\} \tag{606}$$

and

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi}\, g_0(\rho,\rho') + \frac{1}{\pi}\sum_{m=1}^{\infty} g_m(\rho,\rho')\,\cos\left[m(\phi-\phi')\right] \tag{607}$$

then substituting into $\nabla^2 G_D(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')$ to obtain

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_0}{d\rho}\right) = -4\pi\,\frac{\delta(\rho-\rho')}{\rho} \tag{608}$$

and

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) - \frac{m^2}{\rho^2}\, g_m = -4\pi\,\frac{\delta(\rho-\rho')}{\rho} \tag{609}$$

Solving these equations as we did for previous examples (with the BCs being finite at $\rho = 0$ and only logarithmic divergence as $\rho\to\infty$) leads to

$$g_0(\rho,\rho') = -4\pi\,\ln\rho_>\,, \qquad g_m(\rho,\rho') = \frac{2\pi}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \tag{610}$$

which yields the final result.

4.15 Using Green's functions to solve boundary value problems

- Given the Dirichlet Green's function for a particular geometry, the potential $\Phi(\mathbf{r})$ is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V G_D(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\, dV' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{r}')\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\, da' \tag{611}$$

- Note that

$$G_D(\mathbf{r},\mathbf{r}')\Big|_S = 0\,, \qquad \frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n}\Big|_S \equiv \hat{\mathbf{n}}'\cdot\boldsymbol{\nabla}' G_D(\mathbf{r},\mathbf{r}')\Big|_S \tag{612}$$

where $\hat{\mathbf{n}}'$ is the normal to the boundary S pointing *outward* from the volume V.

- In general, both integrals are needed to determine $\Phi(\mathbf{r})$. However, if $\rho(\mathbf{r}) = 0$ inside V, then only the surface integral is needed (i.e., solution to Laplace's equation). Also, if $\Phi(\mathbf{r})$ vanishes on the boundary (i.e., if the boundary surfaces are grounded conductors), then only the volume integral is needed.

- Example 1: Show that the following methods of solving Laplace's equation in the interior of a sphere of radius a with specified potential on the boundary ($\Phi(r=a) \equiv f(\theta,\phi)$) are equivalent: (Jackson, Prob 3.5)

i) Dirichlet Green's function method:

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi} \oint_S \Phi(\mathbf{r}') \frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\, da' \tag{613}$$

where

$$G_D(\mathbf{r},\mathbf{r}') = 4\pi \sum_{l-0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left(\frac{r_<^l}{r_>^{l+1}} - \frac{(rr')^l}{a^{2l+1}} \right) Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi) \tag{614}$$

ii) Separation of variables method:

$$\Phi(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} r^l\, Y_{lm}(\theta,\phi) \tag{615}$$

where

$$A_{lm} = \frac{1}{a^l} \int_{S^2} d\Omega\, f(\theta,\phi) Y_{lm}^*(\theta,\phi) \tag{616}$$

iii) Method of images:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi} \int_{S^2} d\Omega'\, f(\theta',\phi') \frac{a(a^2-r^2)}{(r^2+a^2-2ar\,\cos\gamma)^{3/2}} \tag{617}$$

- Notes:

  For (i), you should find

$$\left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\right|_S = \left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial r'}\right|_{r'=a} = -\frac{4\pi}{a^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi) \left(\frac{r}{a}\right)^l \tag{618}$$

  For (iii), note that

$$\frac{1}{|\mathbf{r} - \frac{a}{r'}\mathbf{r}'|} = \frac{1}{\sqrt{r^2+a^2-2ar\,\cos\gamma}} = \frac{1}{a} \sum_{l=o}^{\infty} \left(\frac{r}{a}\right)^l P_l(\cos\gamma) \tag{619}$$

  Then take the derivative of both sides wrt r to pull down a factor of l.

- Example 2: Use Dirichlet Green's function for the interior of a sphere to calculate the potential inside a grounded conducting sphere of radius b due to a ring of charge (radius $a < b$, total charge Q) in the xy-plane.

- Answer:

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n!} P_{2n}(\cos\theta) \left(\frac{r_<^{2n}}{r_>^{2n+1}} - \frac{(ra)^{2n}}{b^{4n+1}} \right) \tag{620}$$

  where $r_<$ ($r_>$) is the smaller (larger) of r and a.

- Hints:

$$\rho(\mathbf{r}') = \frac{Q\delta(r'-a)\delta(\cos\theta')}{2\pi a^2} \tag{621}$$

  and

$$P_{2n+1}(0) = 0\,, \qquad P_{2n}(0) = \frac{(-1)^n(2n-1)!!}{2^n n!} \tag{622}$$