Math Methods in Physics II

DRAFT

Spring 2016

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1 Fourier series and Fourier transforms

1.1 Fourier series

- The *Fourier series* representation of a function $f(x)$ defined for $x \in [-L, L]$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{1.1}$$

where

$$a_0 = \frac{1}{L} \int_{-L}^{L} dx\, f(x) \tag{1.2}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} dx\, f(x) \cos\left(\frac{n\pi x}{L}\right) \quad \text{where} \quad n = 1, 2, \ldots \tag{1.3}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} dx\, f(x) \sin\left(\frac{n\pi x}{L}\right) \quad \text{where} \quad n = 1, 2, \ldots \tag{1.4}$$

- Exercise: Verify the above equations for the expansion coefficients by showing that

$$\frac{1}{L} \int_{-L}^{L} dx\, \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \delta_{nm} \tag{1.5}$$

$$\frac{1}{L} \int_{-L}^{L} dx\, \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) = \delta_{nm} \tag{1.6}$$

$$\frac{1}{L} \int_{-L}^{L} dx\, \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) = 0 \tag{1.7}$$

where n and m are positive integers $1, 2, \cdots$. Hint: Use the trigonometric identities

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)] \tag{1.8}$$

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)] \tag{1.9}$$

$$\sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)] \tag{1.10}$$

- The RHS of the Fourier expansion (1.1) is defined for $-\infty < x < \infty$ and is *periodic* with period $2L$. Thus, Fourier series can also be defined for periodic functions defined for all x.
- $f(x)$ is an *even* function iff $f(-x) = f(x)$.
- $f(x)$ is an *odd* function iff $f(-x) = -f(x)$.
- An even function $f(x)$ can be written as a sum containing only cosines—i.e., $b_n = 0$.
- An odd function $f(x)$ can be written as a sum containing only sines—i.e., $a_n = 0$.
- Exercise: Show that the Fourier expansion of the *step function*:

$$f(x) = \begin{cases} -1 & -L \le x < 0 \\ 1 & 0 \le x \le L \end{cases} \tag{1.11}$$

is given by

$$f(x) = \frac{4}{\pi} \left[\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3} \sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5} \sin\left(\frac{5\pi x}{L}\right) + \cdots \right] \tag{1.12}$$

- Exercise: Show that the Fourier expansion of the *top-hat function*:

$$f(x) = \begin{cases} 1 & -L/2 \le x \le L/2 \\ 0 & \text{otherwise} \end{cases} \tag{1.13}$$

for $x \in [-L, L]$ is given by

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos\left(\frac{\pi x}{L}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{L}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{L}\right) - \cdots\right] \tag{1.14}$$

- If one is given a function $f(x)$ defined for $x \in [0, L]$, one can sometimes extend it to $x \in [-L, L]$ as either an odd or even function. (The extension must be continuous at 0 and L for the Fourier series to converge to the values of $f(x)$ at these points.) If possible, this simplifies the subsequent calculation of the expansion coefficients, eliminating either the a_n or b_n coefficients, respectively.

- Exercise: Suppose that

$$f(x) = \begin{cases} x & 0 \le x \le L/2 \\ L - x & L/2 \le x \le L \end{cases} \tag{1.15}$$

This corresponds to a guitar string plucked in the middle. By extending $f(x)$ to $x \in [-L, L]$ as an odd function, show that

$$f(x) = \frac{4L}{\pi^2}\left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{25}\sin\left(\frac{5\pi x}{L}\right) - \cdots\right] \tag{1.16}$$

1.2 Fourier series in terms of complex exponentials

- The Fourier series expansion (1.1) can also be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/L} \tag{1.17}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} dx\, f(x)\, e^{-in\pi x/L} \tag{1.18}$$

- Exercise: Verify the above equations starting from the original Fourier expansion (1.1), substituting for the sines and cosines using

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \tag{1.19}$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \tag{1.20}$$

In the process show that

$$c_0 = \frac{a_0}{2} \tag{1.21}$$

$$c_n = \frac{1}{2}(a_n - ib_n) \quad \text{for } n = 1, 2, \cdots \tag{1.22}$$

$$c_n = \frac{1}{2}(a_{-n} + ib_{-n}) \quad \text{for } n = -1, -2, \cdots \tag{1.23}$$

Thus, the expansion coefficients c_n are in general *complex*, with $c_{-n} = c_n^*$ for a real function $f(x)$.

- Exercise: Verify Eq. (1.18) directly by showing that

$$\frac{1}{2L}\int_{-L}^{L} dx\, e^{in\pi x/L} e^{-im\pi x/L} = \delta_{nm} \tag{1.24}$$

where n and m are any integers $0, \pm 1, \pm 2, \cdots$.

- It is usually easier to work with the Fourier expansion in terms of complex exponentials (1.17), since exponential functions are typically easier to integrate.

- *Parseval's Theorem*:
$$\frac{1}{2L}\int_{-L}^{L} dx\, |f(x)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{1.25}$$
relates the the average value of the modulus-squared of $f(x)$ to the sum of the modulus-squared of the expansion coefficients c_n.

- Exercise: Prove Parseval's theorem.

1.3 Fourier series for different intervals

- Suppose the interval $[-L, L]$ is changed to $[-\pi, \pi]$ for example. Then one can show that
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{1.26}$$
where
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x) \tag{1.27}$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x) \cos(nx) \quad \text{where} \quad n = 1, 2, \ldots \tag{1.28}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x) \sin(nx) \quad \text{where} \quad n = 1, 2, \ldots \tag{1.29}$$

- In terms of complex exponentials
$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx} \tag{1.30}$$
where
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, f(x)\, e^{-inx} \tag{1.31}$$

- Exercise: Prove the above by making a change of variables from x to $y \equiv \pi x/L$.

1.4 Fourier transforms

- For a non-periodic function $f(x)$ defined for $-\infty < x < \infty$, one can extend the Fourier series expansion to the following *Fourier transform* pair:
$$f(x) = \int_{-\infty}^{\infty} dk\, \tilde{f}(k) e^{ikx} \tag{1.32}$$
where
$$\mathcal{F}[f] \equiv \tilde{f}(k) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} \tag{1.33}$$

- Compared to
$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/L} \tag{1.34}$$
where
$$c_n = \frac{1}{2L}\int_{-L}^{L} dx\, f(x)\, e^{-in\pi x/L} \tag{1.35}$$

we see that the discrete index n has been replaced by a continuous (real) variable k, the summation $\sum_{n=-\infty}^{\infty}$ has been replaced by an integral $\int_{-\infty}^{\infty} dk$, and the discrete set of expansion coefficients c_n have been replaced by the function $\widetilde{f}(k)$.

- $\widetilde{f}(k) \equiv \mathcal{F}[f]$ is called the *Fourier transform* of $f(x)$.
- $\widetilde{f}(k)$ is in general a complex function of k. If $f(x)$ is real, $\widetilde{f}(-k) = \widetilde{f}(k)^*$.
- $f(x) \equiv \mathcal{F}^{-1}[\widetilde{f}]$ is called the *inverse* Fourier transform of $\widetilde{f}(k)$.
- In different contexts (e.g., signal processing, quantum mechanics, ...) and in different textbooks, the definition of the Fourier transform and its inverse may have different powers of 2π in front of the integrals and different sign conventions and factors of 2π in the complex exponentials.
- One can give a heuristic "proof" of Eqs. (1.32) and (1.33) starting from Eqs. (1.34) and (1.35), and taking the limit as $L \to \infty$:

  (i) Substituting Eq. (1.35) for c_n back into Eq. (1.34) and defining $k_n \equiv n\Delta k \equiv n\pi/L$, one obtains

$$f(x) = \sum_{n=-\infty}^{\infty} \Delta k\, g(k_n, x) \tag{1.36}$$

  where

$$g(k_n, x) \equiv \left(\frac{1}{2\pi} \int_{-L}^{L} dy\, f(y) e^{-ik_n y} \right) e^{ik_n x} \tag{1.37}$$

  (ii) Taking the limit $L \to \infty$, we have

$$\Delta k \quad \to \quad dk \tag{1.38}$$

$$\sum_{n=-\infty}^{\infty} \Delta k \quad \to \quad \int_{-\infty}^{\infty} dk \tag{1.39}$$

$$g(k_n, x) \quad \to \quad g(k, x) \tag{1.40}$$

  where

$$g(k, x) = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dy\, f(y) e^{-iky} \right) e^{ikx} \equiv \widetilde{f}(k) e^{ikx} \tag{1.41}$$

  (iii) Thus,

$$f(x) \to \int_{-\infty}^{\infty} dk\, g(k, x) = \int_{-\infty}^{\infty} dk\, \widetilde{f}(k) e^{ikx} \tag{1.42}$$

  which is Eq. (1.32).

- Example: Let

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{x^2}{2\sigma_x^2} \right] \tag{1.43}$$

  be a (normalised) Gaussian with zero mean and variance σ_x^2. Then its Fourier transform is also a Gaussian:

$$\widetilde{f}(k) = \frac{1}{2\pi} \exp\left[-\frac{k^2}{2\sigma_k^2} \right] \tag{1.44}$$

  with $\sigma_k^2 = 1/\sigma_x^2$.

- Exercise: Prove the above expression for $\widetilde{f}(k)$. Hint: Complete the square in the exponential

$$-\frac{1}{2\sigma_x^2}(x^2 + 2ik\sigma_x^2 x) = -\frac{1}{2\sigma_x^2}(x + ik\sigma_x^2)^2 - \frac{1}{2}k^2\sigma_x^2 \tag{1.45}$$

and use the fact that

$$\int_{-\infty}^{\infty} dx\, \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{(x-\mu)^2}{2\sigma_x^2}\right] = 1 \tag{1.46}$$

for any μ (even complex).

- Note that the Gaussians $f(x)$ and $\widetilde{f}(k)$ obey an *uncertainty* relation:

$$\sigma_x \sigma_k = 1 \tag{1.47}$$

Thus, the more concentrated $f(x)$ is (i.e., the smaller σ_x) the more dispersed $\widetilde{f}(k)$ is (i.e., the larger σ_y).

- Example: Let

$$f(x) = \begin{cases} 1 & -L \le x \le L \\ 0 & \text{otherwise} \end{cases} \tag{1.48}$$

be a (non-periodic) top-hat function, which is non-zero between $-L$ and L. Then its Fourier transform is

$$\widetilde{f}(k) = \frac{L}{\pi}\,\mathrm{sinc}(kL) \tag{1.49}$$

where $\mathrm{sinc}(x) \equiv \sin(x)/x$.

- Exercise: Prove the above expression for $\widetilde{f}(k)$.
- Note that as $L \to \infty$ (i.e., as the top-hat function $f(x)$ becomes wider), the width of the main peak of $\widetilde{f}(k)$ goes to zero ($2\pi/L \to 0$), while the height of the peak goes to infinity ($L/\pi \to \infty$). This is another illustration of the uncertainty principle obeyed by $f(x)$ and $\widetilde{f}(k)$.
- Exercise: Prove the above statements.

1.5 Some properties of Fourier transform pairs

- Parseval's theorem for Fourier transform pairs is:

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = 2\pi \int_{-\infty}^{\infty} dk\, |\widetilde{f}(k)|^2 \tag{1.50}$$

Compare with Eq. (1.25) for Fourier series.

- Exercise: Prove Parseval's theorem for Fourier transform pairs.
- The *convolution theorem* for Fourier transform pairs is:

$$\mathcal{F}[f * g] = 2\pi \mathcal{F}[f]\mathcal{F}[g] \tag{1.51}$$

where the *convolution* of two functions $f(x)$ and $g(x)$ is defined by

$$(f * g)(x) \equiv \int_{-\infty}^{\infty} dy\, f(x-y)g(y) \tag{1.52}$$

- Thus the operation of convolution in x-space is simply multiplication of Fourier transforms in k-space.
- Exercise: Prove that convolution is symmetric—i.e., that $f * g = g * f$.
- Convolutions play a key role in signal processing applications since a linear filter $h(t)$ acting on an input $x(t)$ produces an output $y(t)$ given by the convolution $y = h * x$.
- Exercise: Prove the convolution theorem.

1.6 Fourier transforms in different contexts

- For signal processing applications, one often works with function of time $x(t)$ and their Fourier transforms $\widetilde{x}(f)$ related by:

$$x(t) = \int_{-\infty}^{\infty} df\, \widetilde{x}(f) e^{i2\pi f t} \tag{1.53}$$

where

$$\widetilde{x}(f) = \int_{-\infty}^{\infty} dx\, x(t) e^{-i2\pi f t} \tag{1.54}$$

- $\widetilde{x}(f)$ is in general a complex function of frequency f. Its absolute square $|\widetilde{x}(f)|^2$ is a measure of the energy content in the signal at frequency f.

- In quantum mechanics, one often works with *wave functions*, $\psi(x)$ or $\widetilde{\psi}(p)$, defined in either position or momentum space. The wave functions are related by:

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\, \widetilde{\psi}(p) e^{ipx/\hbar} \tag{1.55}$$

where

$$\widetilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx\, \psi(x) e^{-ipx/\hbar} \tag{1.56}$$

- The momentum p is related to the wavenumber $k = 2\pi/\lambda$ via $p = \hbar k$, where $\hbar$ is Planck's constant h divided by 2π.

2 Ordinary differential equations

2.1 General remarks

- Theorem: Any *linear* ordinary differential equation of order n has a solution containing n independent arbitrary constants, from which all solutions can be obtained. This solution is called the *general solution* of the linear differential equation.

- A *particular* solution satisfies the differential equation plus some other requirements–e.g., *initial conditions* or *boundary conditions*.

- NOTE: The above theorem is not true, in general, for *non-linear* equations. There may exist *singular* solutions, which are not obtainable from the general solution for any choice of integration constant.

- Example:

$$y' = \sqrt{y} \tag{2.1}$$

- Solution:

$$y = \frac{1}{4}(x+C)^2\,, \qquad y = 0 \tag{2.2}$$

- Example:

$$y' = \sqrt{1-y^2} \tag{2.3}$$

- Solution:

$$y = \sin(x+C)\,, \qquad y = \pm 1 \tag{2.4}$$

- Some useful integrals:

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + C \tag{2.5}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1} x + C \tag{2.6}$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a) + C = \ln(x+\sqrt{x^2-a^2}) + C \tag{2.7}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}(x/a) + C = \ln(x+\sqrt{x^2+a^2}) + C \tag{2.8}$$

2.2 First-order ordinary differential equations

- The most general first-order ODE for $y = y(x)$ can be written as

$$y' \equiv \frac{dy}{dx} = F(x,y) \quad \text{or} \quad A(x,y)dx + B(x,y)dy = 0 \tag{2.9}$$

- When the function $F(x,y)$, or $A(x,y)$ and $B(x,y)$, satisfy certain properties, the differential equation simplifies and we can use one of the following methods to solve it.

2.2.1 Separable

- ODE:

$$y' = F(x, y) = f(x)g(y) \tag{2.10}$$

- Method: Rewrite the equation so that the LHS is a function of y and the RHS is a function of x, and then integrate both sides:

$$\int \frac{dy}{g(y)} = \int f(x)dx + C \tag{2.11}$$

- Example:

$$y' = x + xy \tag{2.12}$$

- Solution:

$$\ln(1 + y) = \frac{1}{2}x^2 + C \tag{2.13}$$

2.2.2 Exact

- ODE:

$$A(x, y)dx + B(x, y)dy = 0\,, \quad \text{with} \quad \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \tag{2.14}$$

- Method: There exists a function $U(x, y)$ such that

$$dU = A(x, y)dx + B(x, y)dy = 0\,, \quad \text{with} \quad A = \frac{\partial U}{\partial x}\,, \quad B = \frac{\partial U}{\partial y} \tag{2.15}$$

The solution is $U(x, y) = C$. One method of finding $U(x, y)$ is to first integrate in the x-direction

$$U(x, y) = \int A(x, y)dx + g(y) \tag{2.16}$$

and then use $\partial U/\partial y = B$ to determine the function $g(y)$.

- Example:

$$(3x + y)\,dx + x\,dy = 0 \tag{2.17}$$

- Solution:

$$U(x, y) = \frac{3}{2}x^2 + yx = C \tag{2.18}$$

2.2.3 Inexact

- If the differential $Adx + Bdy$ is not exact to begin with, it is always possible to find a function $\mu(x, y)$, called an *intergrating factor*, for which

$$\mu(x, y)\,[A(x, y)dx + B(x, y)dy] \quad \text{is exact} \tag{2.19}$$

Unfortunately, for general A and B, there is no general method for finding $\mu(x, y)$.

- We will give an explicit construction of μ for the case of *linear* first-order ODEs below.

2.2.4 Linear

- ODE:

$$y' + P(x)\, y = Q(x) \tag{2.20}$$

- Method: Multiply by the integrating factor

$$\mu(x) = e^{I(x)}\,, \qquad I(x) = \int^x P(x')dx' \tag{2.21}$$

obtaining the solution

$$e^{I(x)} y(x) = \int^x e^{I(x')} Q(x') dx' + C \tag{2.22}$$

The first term on the RHS corresponds to a *particular solution* of the equation, and the second term to the *complementary* function, which is the general solution of the differential equation with $Q(x) = 0$.

- Example:

$$(1 + x^2)y' + 6xy = 2x \tag{2.23}$$

- Solution:

$$y = \frac{C}{(1+x^2)^3} + \frac{1}{3} \tag{2.24}$$

2.2.5 Bernoulli equation

- ODE:

$$y' + P(x)\, y = Q(x)\, y^n \tag{2.25}$$

- Method: Divide the whole equation by y^n and make a change of variables $v = y^{1-n}$, which yields

$$v' + (1-n)P(x)\, v = (1-n)Q(x) \tag{2.26}$$

The ODE for $v(x)$ is linear, which can be solved using the previous method.

- Example:

$$y' + y = xy^{2/3} \tag{2.27}$$

- Solution:

$$y^{1/3} = x - 3 + Ce^{-x/3} \tag{2.28}$$

2.2.6 Homogeneous

- ODE:

$$y' = F(x, y) = F(y/x) \tag{2.29}$$

- Method: Make a change of variables $v = y/x$, which yields

$$v'\, x + v = F(v) \tag{2.30}$$

The ODE for $v(x)$ is separable and has solution

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x} + C \tag{2.31}$$

- Example:

$$y' = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \tag{2.32}$$

- Solution:

$$y = x\, \sin^{-1}(Cx) \tag{2.33}$$

2.3 Second-order ordinary differential equations

2.3.1 Constant coefficients, zero RHS

- ODE:

$$ay'' + by' + cy = 0 \tag{2.34}$$

  where a, b, c are constants.

- Method: Solve the *auxiliary* equation

$$aD^2 + bD + c = 0 \tag{2.35}$$

  for the differential operator $D \equiv \frac{d}{dx}$. The solution is

$$(D - \lambda_1)(D - \lambda_2) = 0\,, \qquad \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.36}$$

  If $\lambda_1 \neq \lambda_2$, then the solution to the differential equation is

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \tag{2.37}$$

  where C_1 and C_2 are constants.

  If $\lambda_1 = \lambda_2 \equiv \lambda$, then the solution is

$$y = C_1 e^{\lambda x} + C_2 x e^{\lambda x} \tag{2.38}$$

- Example:

$$y'' - 6y' + 9y = 0 \tag{2.39}$$

- Solution:

$$y = C_1 e^{3x} + C_2 x e^{3x} \tag{2.40}$$

2.3.2 Constant coefficients, non-zero RHS

- ODE:

$$ay'' + by' + cy = F(x) \tag{2.41}$$

  where a, b, c are constants. $F(x)$ is called the *source term* or *forcing function.*

- Method: The general solution to the above differential equation is

$$y = y_p + y_c \tag{2.42}$$

  where y_p is a particular solution of the differential equation and y_c (the complementary function) is the general solution of the differential equation with the RHS set equal to zero.

  Since y_c can be found using the previous method, we concentrate here only on methods for finding y_p.

- Exponential RHS:

$$F(x) = ke^{\alpha x} \tag{2.43}$$

- Solution:

$$y_p(x) = \begin{cases} Ce^{\alpha x} & \alpha \neq \lambda_1 \text{ and } \alpha \neq \lambda_2 \\ Cxe^{\alpha x} & \alpha = \lambda_1 \text{ or } \lambda_2 \text{ and } \lambda_1 \neq \lambda_2 \\ Cx^2 e^{\alpha x} & \alpha = \lambda_1 = \lambda_2 \end{cases} \tag{2.44}$$

  where the constant C is determined by substituting y_p into the differential equation.

- Sine or cosine RHS:

$$F(x) = k\sin(\alpha x) \text{ or } k\cos(\alpha x) \tag{2.45}$$

- Solution: First solve the differential equation for the complex exponential $F(x) = ke^{i\alpha x}$, and then take the imaginary (for sine) or real part (for cosine) of the solution.

- Method of undetermined coefficients:
$$F(x) = e^{\alpha x} P_n(x) \tag{2.46}$$
where $P_n(x)$ is a polynomial of degree n.

- Solution:
$$y_p(x) = \begin{cases} e^{\alpha x} Q_n(x) & \alpha \neq \lambda_1 \text{ and } \alpha \neq \lambda_2 \\ x e^{\alpha x} Q_n(x) & \alpha = \lambda_1 \text{ or } \lambda_2 \text{ and } \ \lambda_1 \neq \lambda_2 \\ x^2 e^{\alpha x} Q_n(x) & \alpha = \lambda_1 = \lambda_2 \end{cases} \tag{2.47}$$
where $Q_n(x)$ is a polynomial of the same degree as $P_n(x)$, whose coefficients are determined by substituting y_p into the differential equation.

- Several terms on the RHS:
$$F(x) = e^x + 4\sin(2x) + (x^2 - x) \tag{2.48}$$

- Solution: Solve for y_p for each source term separately, then add the solutions together. This works because the differential equation is *linear*.

  NOTE: For an arbitrary periodic forcing function of period $2L$, we can expand
$$F(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \tag{2.49}$$
and then solve for y_p for each complex exponential term in the expansion.

- When we don't know what to guess:

- Solution: First solve the auxiliary equation for D so that the differential equation becomes
$$(D - \lambda_1)(D - \lambda_2)y = F(x) \tag{2.50}$$
Then define
$$u = (D - \lambda_2)y \tag{2.51}$$
and solve the resulting first-order equation
$$(D - \lambda_1)u = F(x) \tag{2.52}$$
for $u(x)$. Finally, knowing $u(x)$, solve Eq. (2.51) for $y(x)$.

  NOTE: We are basically solving the second-order ODE as two first-order ODEs.

- Example: Forced vibrations and resonance:
$$\ddot{y} + 2b\dot{y} + \omega^2 y = F\sin(\omega' t) \tag{2.53}$$
where F is a constant.

- Solution: Use the method of complex exponentials to find the particular solution $y_p(t) = \mathrm{Im}(Ce^{i\omega' t})$, where
$$C = \frac{F}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}} e^{i\phi}, \quad \tan\phi = -\frac{2b\omega'}{\omega^2 - \omega'^2} \tag{2.54}$$

- NOTE: There are two definitions of *resonance*, depending on which frequency is held fixed and which is varied.

  (i) For fixed forcing frequency ω', the amplitude is maximum when the natural frequency ω equals ω'.

  (ii) For fixed natural frequency ω, the amplitude is maximum when the forcing frequency ω' is given by $\omega' = \sqrt{\omega^2 - 2b^2}$ (assuming small damping, $b \ll \omega$).

2.3.3 Dependent variable y is absent

- Method: Make the substitutions

$$y' = p\,, \quad y'' = p' \tag{2.55}$$

obtaining a first-order equation for $p(x)$ which might be solvable. Then integrate $y' = p(x)$ to find $y(x)$.

- Example:

$$y'' + 2y' = 4x \tag{2.56}$$

- Solution:

$$y = x^2 - x - \frac{1}{2}C_1 e^{-2x} + C_2 \tag{2.57}$$

2.3.4 Independent variable x is absent

- Method: Make the substitutions

$$y' = p\,, \qquad y'' = p' = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy} \tag{2.58}$$

obtaining a first-order equation for $p(y)$ which might be solvable. Then integrate $y' = p(y)$ to find $y(x)$.

- Example:

$$1 + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \tag{2.59}$$

- Solution:

$$(x - C_2)^2 + y^2 = C_1^2 \tag{2.60}$$

2.3.5 First derivative y' and independent variable x are absent

- ODE:

$$y'' + f(y) = 0 \tag{2.61}$$

- Method: Solve as for the previous case, obtaining a separable equation for $p(y)$ which can be integrated:

$$\frac{1}{2}p^2 + \int f(y)\,dy = C_1 \tag{2.62}$$

Then rewrite in terms of $dy/dx = p(y)$, obtaining a separable equation for $y(x)$ which can be integrated:

$$\int \frac{dy}{\sqrt{2\left(C_1 - \int f(y)\,dy\right)}} = x + C_2 \tag{2.63}$$

- Example:

$$m\ddot{x} = -kx \tag{2.64}$$

- Solution:

$$x = \sqrt{\frac{2E}{k}}\,\sin(\omega t + \phi_0)\,, \qquad \omega \equiv \sqrt{\frac{k}{m}} \tag{2.65}$$

2.3.6 Euler (or Cauchy) equation

- ODE:

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = f(x) \tag{2.66}$$

- Method: Make the change of variables

$$x = e^t \quad \Rightarrow \quad x\frac{dy}{dx} = \frac{dy}{dt}\,, \qquad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} \tag{2.67}$$

obtaining a second-order linear equation with constant coefficients, which can be solved using previous methods.

- Example:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0 \tag{2.68}$$

- Solution:

$$y = C_1 x^2 + C_2 \frac{1}{x^2} \tag{2.69}$$

2.3.7 Reduction of order

- ODE:

$$y'' + f(x)y' + g(x)y = 0\,, \quad \text{where} \quad u(x) \text{ is a solution} \tag{2.70}$$

- Method: Make the substitution

$$y(x) = u(x)v(x) \tag{2.71}$$

obtaining the differential equation

$$v'' + \left(2\frac{u'}{u} + f(x)\right) v' = 0 \tag{2.72}$$

which is a second-order ODE for $v(x)$ that has v absent. Make another substitution $p = v'$, $p' = v''$, obtaining a first-order, separable ODE for $p(x)$. Finally, integrate $p = v'$ to find $v(x)$, and then multiply by $u(x)$ to find $y(x) = u(x)v(x)$.

NOTE: If one sets the integration constants to zero, one obtains the second independent solution. Keeping the integration constants gives the general solution.

- Example:

$$x^2(2-x)y'' + 2xy' - 2y = 0\,, \qquad u = x \text{ is a solution} \tag{2.73}$$

- Solution:

$$y = 1 - \frac{1}{x} \tag{2.74}$$

2.4 Laplace transforms

- The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}[f] \equiv F(p) \equiv \int_0^\infty dt\, f(t)e^{-pt} \tag{2.75}$$

(Note the limits on the integral. This implies that $F(p)$ does not depend on the values of $f(t)$ for $t < 0$.)

- The Laplace transform is a *linear* transform, since it satisfies

$$\mathcal{L}[af] = a\mathcal{L}[f]\,, \qquad \mathcal{L}[f_1 + f_2] = \mathcal{L}[f_1] + \mathcal{L}[f_2] \tag{2.76}$$

where a is a constant.

- Examples of Laplace transform pairs:

$$f(t) = 1\,, \qquad F(p) = \frac{1}{p}\,, \quad \text{with } \mathrm{Re}(p) > 0 \tag{2.77}$$

$$f(t) = e^{-at}\,, \qquad F(p) = \frac{1}{a+p}\,, \quad \text{with } \mathrm{Re}(a+p) > 0 \tag{2.78}$$

$$f(t) = e^{iat}\,, \qquad F(p) = \frac{p+ia}{p^2+a^2}\,, \quad \text{with } \mathrm{Re}(p-ia) > 0 \tag{2.79}$$

$$f(t) = e^{-iat}\,, \qquad F(p) = \frac{p-ia}{p^2+a^2}\,, \quad \text{with } \mathrm{Re}(p+ia) > 0 \tag{2.80}$$

$$f(t) = \cos at\,, \qquad F(p) = \frac{p}{p^2+a^2}\,, \quad \text{with } \mathrm{Re}(p\pm ia) > 0 \tag{2.81}$$

$$f(t) = \sin at\,, \qquad F(p) = \frac{a}{p^2+a^2}\,, \quad \text{with } \mathrm{Re}(p\pm ia) > 0 \tag{2.82}$$

$$f(t) = \frac{e^{-at}-e^{-bt}}{b-a}\,, \qquad F(p) = \frac{1}{(p+a)(p+b)}\,, \quad \text{with } \mathrm{Re}(p+a),\ \mathrm{Re}(p+b) > 0 \tag{2.83}$$

- The Laplace transform is another method for solving second-order linear ODEs with constant coefficients and specified boundary (or initial) conditions y_0, y'_0. The Laplace transform converts such an ODE into an algebraic equation that can be solved for $Y(p)$. The solution $y(t)$ is obtained by finding the inverse Laplace transform of $Y(p)$, using (for now) a table of Laplace transform pairs.

- Note that

$$\mathcal{L}[y'] = pY(p) - y_0 \tag{2.84}$$

$$\mathcal{L}[y''] = p^2Y(p) - py_0 - y'_0 \tag{2.85}$$

- <u>Example</u>:

$$y'' + y = \sin t\,, \quad y_0 = 1\,, \quad y'_0 = 0 \tag{2.86}$$

- <u>Solution</u>:

$$y(t) = \cos t + \frac{1}{2}(\sin t - t\cos t) \tag{2.87}$$

2.4.1 Convolution

- The Laplace transform of the differential equation

$$Ay'' + By' + Cy = f(t)\,, \quad y_0 = 0\,, \quad y'_0 = 0 \tag{2.88}$$

has solution

$$Y(p) = H(p)F(p)\,, \quad H(p) = \frac{1}{Ap^2+Bp+C} = \frac{1}{A(p+a)(p+b)} \tag{2.89}$$

- $H(p)$ is called the *transfer* function; it relates the 'output' $Y(p)$ to the 'input' $F(p)$, which is the Laplace transform of the forcing function.

- Thus, to find $y(t)$, one needs to take the inverse Laplace transform of the product, $H(p)F(p)$, of two Laplace transforms. This turns out to equal the *convolution* of the functions $h(t)$ and $f(t)$ defined by

$$(h * f)(t) \equiv \int_0^t d\tau\, h(t-\tau)f(\tau) \tag{2.90}$$

NOTE: If $h(t) = 0$ and $f(t) = 0$ for $t < 0$, then the limits on the convolution integral can be replaced by $-\infty$ and ∞.

- The *convolution theorem* is:

$$\mathcal{L}[h * f] = H(p)F(p) = \mathcal{L}[h]\,\mathcal{L}[f] \tag{2.91}$$

- Exercise: Prove this. (Hint: You will need to change the order of integration when taking the Laplace transform of $h * f$.)

- Convolution is commutative: $(h * f)(t) = (f * h)(t)$.

- It is sometimes simpler to use convolution to solve an ODE.

- Example:

$$y'' + 5y' + 6y = e^{-2t}\,, \quad y_0 = 0\,, \quad y_0' = 0 \tag{2.92}$$

- Solution:

$$y(t) = (t-1)e^{-2t} + e^{-3t} \tag{2.93}$$

2.5 Dirac delta function

- The Dirac delta function is a mathematical quantity that can be used to represent the charge density $\rho(\mathbf{r})$ of a point charge q located at $\mathbf{r}_0$:

$$\rho(\mathbf{r}) = q\,\delta(\mathbf{r} - \mathbf{r}_0) \tag{2.94}$$

- The charge density is zero at all points away from the charge, and is infinite at the location of the charge in such a way that its integral over any volume V containing the charge is q:

$$\int_V dV\,\rho(\mathbf{r}) = q \tag{2.95}$$

- NOTE: Similar statements can be made for the mass density $\mu(\mathbf{r})$ of a point charge m.

- The Dirac delta function is not an ordinary mathematical function. Mathematicians call a Dirac delta function a *generalized function* or a *distribution*.

- In 1-dimension, the Dirac delta function $\delta(x - x_0)$ can be thought of as the limit of a sequence of functions $f_n(x)$ all of which have unit area, but which get narrower and higher as $n \to \infty$. Some examples:

  i) A sequence of top-hat functions centered at x_0 with width $2/n$:

$$f_n(x) = \begin{cases} n/2\,, & x_0 - 1/n < x < x_0 + 1/n \\ 0 & \text{otherwise} \end{cases} \tag{2.96}$$

  ii) A set of Gaussian probability distributions with mean $\mu = x_0$ and width $\sigma = 1/n$:

$$f_n(x) = \frac{n}{\sqrt{2\pi}} e^{-n^2(x-x_0)^2/2} \tag{2.97}$$

  iii) A set of sinc functions centered at x_0 of the form:

$$f_n(x) = \frac{n}{\pi}\,\mathrm{sinc}\,(n(x - x_0)) \tag{2.98}$$

- Alternatively, a Dirac delta function can be defined in terms of its action on a set of *test functions* $f(x)$, which are infinitely differentiable and which vanish as $x \to \pm\infty$.

- The defining property of a 1-dimensional Dirac delta function is then

$$\int_a^b dx\,f(x)\delta(x - x') = \begin{cases} f(x') & a < x' < b \\ 0 & \text{otherwise} \end{cases} \tag{2.99}$$

  for any test function $f(x)$.

- Properties (which follow from the above definition):

$$
\begin{aligned}
&(a) \qquad \delta(x-a) = \frac{d}{dx}\left(u(x-a)\right) && (2.100)\\
&(b) \qquad \delta'(-x) = -\delta'(x) && (2.101)\\
&(c) \qquad \delta(-x) = \delta(x) && (2.102)\\
&(d) \qquad \delta(ax) = \frac{1}{|a|}\delta(x) && (2.103)\\
&(e) \qquad \delta\left[f(x)\right] = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} && (2.104)
\end{aligned}
$$

where $u(x)$ is the unit step function and $f(x)$ is such that $f(x_i) = 0$ and $f'(x_i) \neq 0$.

- In 3-dimensions, the defining property of the Dirac delta function is

$$\int_V dV\, f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0) = \begin{cases} f(\mathbf{r}_0) & \text{if } \mathbf{r}_0 \in V \\ 0 & \text{otherwise} \end{cases} \tag{2.105}$$

for any test function $f(\mathbf{r})$.

- This implies

$$\delta(\mathbf{r}-\mathbf{r}_0) = \begin{cases} \delta(x-x_0)\delta(y-y_0)\delta(z-z_0) & \text{in Cartesian coordinates} \\ \frac{1}{r^2\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0) & \text{in spherical polar coordinates} \end{cases} \tag{2.106}$$

Note that we need to include the factor of $1/r^2\sin\theta$ for spherical polar coordinates since $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

- One can show that

$$\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta(\mathbf{r})\,, \qquad \nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{r}) \tag{2.107}$$

which is familiar from electrostatics for the electric field and electric potential for a point charge.

- Fourier transform:

$$\mathcal{F}[\delta(x-a)] = \frac{1}{2\pi}\, e^{-ika}\,, \qquad \delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-a)} \tag{2.108}$$

NOTE: The last expression is only a *formal* expression for the Dirac delta function, since $e^{ik(x-a)}$ is not square integrable and hence does not have a Fourier transform.

- More rigorously

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} \equiv \lim_{L\to\infty}\frac{1}{2\pi}\int_{-L}^{L} dk\, e^{ikx} = \lim_{L\to\infty}\frac{L}{\pi}\,\mathrm{sinc}(Lx) \tag{2.109}$$

- Laplace transform:

$$\mathcal{L}[\delta(t-a)] = \begin{cases} e^{-at} & a > 0 \\ 0 & a < 0 \end{cases} \tag{2.110}$$

- One can use the method of Laplace transforms to solve second-order ODEs with constant coefficients and RHS equal to a Dirac delta function.

- Example:

$$y'' + \omega^2 y = \delta(t-t_0)\,, \quad y_0 = 0\,, \quad y_0' = 0 \tag{2.111}$$

- Solution:

$$y(t) = \begin{cases} \frac{1}{\omega}\sin(\omega(t-t_0)) & 0 < t_0 < t \\ 0 & 0 < t < t_0 \end{cases} \tag{2.112}$$

2.6 Green's functions

- Suppose one wants to solve a *linear* ODE for $y(t)$ with RHS equal to $f(t)$ and specified boundary conditions.
- Then since

$$f(t) = \int_{-\infty}^{\infty} dt'\, \delta(t - t') f(t') \tag{2.113}$$

it follows by linearity that

$$y(t) = \int_{-\infty}^{\infty} dt'\, G(t, t') f(t') \tag{2.114}$$

where $G(t, t')$ is the solution of the ODE with RHS equal to $\delta(t-t')$ and the same boundary conditions.

- Exercise: Prove this last statement.
- The solution $G(t, t')$ of the ODE with RHS equal to $\delta(t-t')$ and specified boundary conditions is called the *Green's function* for the differential equation.
- Example: Find the Green's function for

$$y'' + \omega^2 y = f(t)\,, \quad y_0 = 0\,, \quad y_0' = 0 \tag{2.115}$$

- Solution: This was done in the preceding section using the method of Laplace transforms:

$$G(t, t') = \begin{cases} \frac{1}{\omega} \sin(\omega(t - t')) & 0 < t' < t \\ 0 & 0 < t < t' \end{cases} \tag{2.116}$$

- Alternatively, one can solve for the Green's function by solving the differential equation directly. The method is as follows:

  (i) Solve the equation in the regions $0 < t < t'$ and $0 < t' < t$, where the RHS is zero:

$$G(t, t') = \begin{cases} A(t') \sin \omega t + B(t') \cos \omega t & 0 < t < t' \\ C(t') \sin \omega t + D(t') \cos \omega t & 0 < t' < t \end{cases} \tag{2.117}$$

  (ii) Apply the two boundary conditions:

$$G(t, t') = \begin{cases} 0 & 0 < t < t' \\ C(t') \sin \omega t + D(t') \cos \omega t & 0 < t' < t \end{cases} \tag{2.118}$$

  (iii) Apply the condition that the function is continuous at $t = t'$:

$$0 = C(t') \sin \omega t' + D(t') \cos \omega t' \tag{2.119}$$

  (iv) Apply the condition that the function has a discontinuous first derivative at $t = t'$:

$$1 = \lim_{\epsilon \to 0} \left[\frac{d}{dt} G(t' + \epsilon, t') - \frac{d}{dt} G(t' - \epsilon, t') \right] \tag{2.120}$$

$$= \omega C(t') \cos \omega t' - \omega D(t') \sin \omega t' \tag{2.121}$$

  (iv) Solve (iii) and (iv) for the remaining coefficients:

$$C(t') = \frac{1}{\omega} \cos \omega t'\,, \qquad D(t') = -\frac{1}{\omega} \sin \omega t' \tag{2.122}$$

  (v) Substitute back into the expression for $G(t, t')$:

$$G(t, t') = \begin{cases} 0 & 0 < t < t' \\ \frac{1}{\omega} \sin(\omega(t - t')) & 0 < t' < t \end{cases} \tag{2.123}$$

  This agrees (as it should) with what we found before.

- Example: Find the Green's function for

$$y'' + y = f(x)\,, \quad y(0) = 0\,, \quad y(\pi/2) = 0 \tag{2.124}$$

where $x \in [0, \pi/2]$.

- Solution:

$$G(x, x') = \begin{cases} -\cos x' \sin x & 0 < x < x' < \pi/2 \\ -\sin x' \cos x & 0 < x' < x < \pi/2 \end{cases} \tag{2.125}$$

NOTE: The Laplace transform method is not valid here since the boundary conditions are not of the form where $y(0)$ and $y'(0)$ are specified.

- Example: Solve

$$y'' + 3y' - 4y = e^{-5t}\,, \quad y_0 = 0\,, \quad y_0' = 0 \tag{2.126}$$

using four different methods:

(i) the straight-forward method for linear second-order ODEs with constant coefficients and non-zero RHS;

(ii) the method of Laplace transforms;

(iii) the method of Green's functions, but using Laplace transforms to find $G(t, t')$;

(iv) the method of Green's functions using direct integration to find $G(t, t')$.

- Solution:

$$y(t) = -\frac{1}{5}e^{-4t} + \frac{1}{30}e^{t} + \frac{1}{6}e^{-5t} \tag{2.127}$$

2.7 Useful Kronecker delta and Dirac delta formulae

- The Dirac delta function $\delta(x - y)$ is a generalization of the Kronecker delta δ_{ij} (defined for discrete indices i and j) to continuous variables x and y.

- Fourier representation of Dirac delta function:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{\pm ik(x-x')} = \delta(x - x') \tag{2.128}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{\pm i(k-k')x} = \delta(k - k') \tag{2.129}$$

- Orthogonality:

$$\frac{1}{2L}\int_{-L}^{L} dx\, e^{i(n-m)\pi x/L} = \delta_{nm} \tag{2.130}$$

$$\frac{1}{N}\sum_{l=0}^{N-1} e^{-i2\pi(j-k)l/N} = \delta_{jk} \tag{2.131}$$

- Dirac comb:

$$\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{i2\pi nt/T} = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{2.132}$$

- Completeness (for $\phi, \phi' \in [0, 2\pi]$):

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = \delta(\phi - \phi') \tag{2.133}$$

3 Calculus of variations

- The calculus of variations is an extension of the standard procedure for finding the maximum or minimum of a function $f(x)$—i.e., that value of x for which $f'(x) = 0$.
- Instead of $f(x)$, which is a function of a single real variable x, we consider a *functional* $I[y]$, which is a *function of a function* $y(x)$. The goal is to find that function $y(x)$ for which $\delta I[y] = 0$.
- $\delta I[y]$ denotes the *variation* of the functional $I[y]$ with respect to $y(x)$. It is a generalization of the total differential *df* of an ordinary function $f(x)$ with respect to x.

3.1 Euler-Lagrange equations

- Given the functional

$$I[y] = \int_{x_1}^{x_2} F(x, y, y')\, dx \tag{3.1}$$

we want to find that function $y(x)$ for which $\delta I[y] = 0$.

- The variation of $I[y]$ with respect to y is

$$\delta I[y] = \int_{x_1}^{x_2} \delta F(x, y, y')\, dx = \int_{x_1}^{x_2} \left\{ \left(\frac{\partial F}{\partial y} \right) \delta y + \left(\frac{\partial F}{\partial y'} \right) \delta y' \right\} dx \tag{3.2}$$

where

$$\delta y' = \delta \left(\frac{dy}{dx} \right) = \frac{d}{dx} \delta y \tag{3.3}$$

- Integrating the second term by parts yields

$$\delta I[y] = \left(\frac{\partial F}{\partial y'} \right) \delta y \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left\{ \left(\frac{\partial F}{\partial y} \right) - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right\} \delta y\, dx \tag{3.4}$$

- Thus, for arbitrary variations with fixed endpoints,

$$\delta I[y] = 0 \quad \Leftrightarrow \quad \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0 \tag{3.5}$$

These are the *Euler-Lagrange* equations.

3.2 Some simplifications

- If $F(x, y, y')$ is independent of y, then $\partial F/\partial y = 0$ and the Euler-Lagrange equation can be integrated to yield

$$\frac{\partial F}{\partial y'} = \text{const} \tag{3.6}$$

- If $F(x, y, y')$ is independent of x, then one can make a change of independent variables from x to y such that $x = x(y)$, with

$$y' = 1/x', \quad dx = x'\, dy \tag{3.7}$$

for which

$$I[y] = \int_{x_1}^{x_2} F(x, y, y')\, dx = \int_{y_1}^{y_2} F(y, 1/x')\, x'\, dy \equiv \int_{y_1}^{y_2} G(y, x')\, dy \equiv \bar{I}[x] \tag{3.8}$$

The Euler-Lagrange equation for $\bar{I}[x]$ is then

$$\frac{\partial G}{\partial x'} = \text{const} \tag{3.9}$$

- In addition, if $F(x,y,y')$ is independent of x, then one can show that
$$h \equiv y' \frac{\partial F}{\partial y'} - F \tag{3.10}$$
satisfies $dh/dx = 0$ (i.e., h is constant), as a consequence of the Euler-Lagrange equations.
Exercise: Prove this.
Example: The Lagrangian
$$L \equiv T - V = \frac{1}{2} m\dot{x}^2 - V(x) \tag{3.11}$$
is independent of t. It follows that
$$h \equiv \dot{x} \frac{\partial L}{\partial \dot{x}} - L = \dot{x}\, m\dot{x} - \left(\frac{1}{2} m\dot{x}^2 - V(x)\right) = \frac{1}{2} m\dot{x}^2 + V(x) \tag{3.12}$$
is the total mechanical energy of the system, $E = T + V$.

3.3 Extremization problems subject to constraints

- Suppose we want to extremize some functional
$$I[y] = \int_{x_1}^{x_2} F(x,y,y')\, dx \tag{3.13}$$
subject to a constraint
$$J[y] = \int_{x_1}^{x_2} G(x,y,y')\, dx = \text{const} \tag{3.14}$$
- Then one should extremize the functional
$$\bar{I}[y,\lambda] \equiv I[y] + \lambda\, J[y] = \int_{x_1}^{x_2} \left(F(x,y,y') + \lambda G(x,y,y')\right)\, dx \tag{3.15}$$
with respect to variations in both y and λ, the *Lagrange multiplier*.
- This follows from the Lagrange multiplier method, which is a way of allowing *unconstrained* variations of $y(x)$.

3.4 Examples

- Geodesics: Given two fixed points, find the path that minimizes the distance between them. Examples of geodesics are straight lines for flat space, great circles for the surface of a sphere, etc.
For a flat plane
$$I[y] = \int_A^B ds = \int_A^B \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} dx\, \sqrt{1 + y'^2} \tag{3.16}$$
which leads to $y' = \text{const}$, which is a straight line.
- Fermat's principle of least time: In geometrical optics, light travels between two points in such a way as to minimize the travel time between the two points. From Fermat's principle, one can derive, e.g., Snell's law of refraction
$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \tag{3.17}$$
where v_1, v_2 are the speeds of light in the two different materials and θ_1, θ_2 are the angles that the light rays make wrt the normal to the interface between the two materials.
NOTE: Ordinary calculus will work for this case. One extremizes
$$T(x) = \frac{\sqrt{(x - x_1)^2 + y_1^2}}{v_1} + \frac{\sqrt{(x_2 - x)^2 + y_2^2}}{v_2} \tag{3.18}$$
which is time for light to travel between the two points in materials 1 and 2, respectively.

- Hamilton's principle: Of all possible paths which a system may follow in going from one configuration at time t_1 to another configuration at time t_2, the path followed is the one that extremizes the *action*. The action is the integral of the *Lagrangian*,

$$L(t, x, \dot{x}) = T - V = \frac{1}{2} m \dot{x}^2 - V(x)\,, \tag{3.19}$$

  which is the difference between the kinetic energy and potential energy of the system. The resulting equations of motion are equivalent to Newton's laws for a conservative force:

$$m\ddot{x} = -\frac{\partial V}{\partial x} \tag{3.20}$$

- Soap film: Find the shape of a soap film suspended between two circular loops of wire, ignoring the effects of gravity. The shape minimizes the surface area of revolution

$$I[y] = \int_A^B 2\pi y\, ds = 2\pi \int_{y_1}^{y_2} dy\, y\sqrt{1 + x'^2} \tag{3.21}$$

  It has the mathematical form of a *catenary*, $y \sim \cosh x$.

- Brachistochrone: Find the shape of a wire joining two points such that a bead will slide along the wire under the influence of gravity (without friction) in the shortest amount of time. One extremizes

$$I[y] = \int_A^B \frac{ds}{v} = \int_{y_1}^{y_2} dy\, \frac{\sqrt{1 + x'^2}}{\sqrt{2gy}} \tag{3.22}$$

  where we used conservation of energy

$$\frac{1}{2} m v^2 - mgy = 0 \quad \Rightarrow \quad v = \sqrt{2gy} \tag{3.23}$$

  for the bead released from rest at A (assuming $y = 0$, increasing downward).

  The resulting shape is a *cycloid*:

$$x = \frac{1}{C}(\theta - \sin\theta)\,, \qquad y = \frac{1}{C}(1 - \cos\theta) \tag{3.24}$$

- Hanging cable: Find the shape of a hanging cable. This shape minimizes the gravitational potential energy of the cable

$$I[y] = \int_A^B dm\, gy = \int_A^B \mu ds\, gy = \mu g \int_{y_1}^{y_2} dy\, y\sqrt{1 + x'^2} \tag{3.25}$$

  where μ is the mass per unit length of the cable (assumed constant), subject to the constraint that the cable has a fixed length ℓ:

$$J[y] = \int_A^B ds = \int_{y_1}^{y_2} dy\, \sqrt{1 + x'^2} = \ell \tag{3.26}$$

  The solution has the mathematical form of a catenary, $y \sim \cosh x$.

4 Series solutions of differential equations

- The general form of a homogeneous, linear, second-order ordinary differential equation is

$$y''(x) + p(x)\, y'(x) + q(x)\, y(x) = 0 \tag{4.1}$$

where $p(x)$ and $q(x)$ are arbitrary functions of x.

- We are interested in *power series solutions* of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{or} \quad y(x) = x^\sigma \sum_{n=0}^{\infty} a_n x^n \tag{4.2}$$

for some value of σ.

- We need to consider the more general power series expansion, called the *method of Frobenius*, if $x = 0$ is a *regular singular point* of the differential equation—i.e., if $p(x)$ or $q(x)$ is singular (i.e., infinite) at $x = 0$, but $xp(x)$ and $x^2 q(x)$ are *finite* at $x = 0$. If $x = 0$ is a regular point of the differential equation, then one can simply set $\sigma = 0$.

- The basic procedure is to differentiate the power series expansion for $y(x)$ term by term, and then substitutes the expansions into the differential equation for $y(x)$. Since the resulting sum must vanish for all values of x, the coefficients of x^n must all equal zero, leading to a *recurrence relation* relating a_n to any of the previous a_r $(r < n)$ and a quadratic equation for σ, called the *indicial equation.*

- Fuch's theorem: The general solution of a differential equation with a regular singular point at $x = 0$ consists of of either: (i) a sum of two Frobenious series $S_1(x)$ and $S_2(x)$, or (ii) the sum of one Frobenious series $S_1(x)$, and a second solution of the form $S_1(x) \ln x + S_2(x)$, where $S_2(x)$ is another Frobenius series. Case (ii) occurs only when the roots of the indicial equation for σ are equal or differ by an integer.

4.1 Trigonometric functions

- The differential equation

$$y'' + k^2 y = 0 \tag{4.3}$$

admits a power series solution with

$$a_{n+2} = \frac{-k^2}{(n+1)(n+2)}\, a_n \tag{4.4}$$

- Thus,

$$y(x) = a_0 \left(1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} - \cdots\right) + \frac{a_1}{k}\left(kx - \frac{(kx)^3}{3!} + \frac{(kx)^5}{5!} - \cdots\right) \tag{4.5}$$

or, equivalently,

$$y(x) = a_0 \cos(kx) + \frac{a_1}{k} \sin(kx) \tag{4.6}$$

- This solution can also be written as

$$y(x) = A\, e^{ikx} + B\, e^{-ikx} \tag{4.7}$$

where

$$A = \frac{1}{2}\left(a_0 + \frac{a_1}{ik}\right) \quad \text{and} \quad B = \frac{1}{2}\left(a_0 - \frac{a_1}{ik}\right) \tag{4.8}$$

4.2 Hyperbolic functions

- The differential equation

$$y'' - k^2 y = 0 \tag{4.9}$$

admits a power series solution with

$$a_{n+2} = \frac{k^2}{(n+1)(n+2)}\, a_n \tag{4.10}$$

- Thus,

$$y(x) = a_0 \left(1 + \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} + \cdots \right) + \frac{a_1}{k} \left(kx + \frac{(kx)^3}{3!} + \frac{(kx)^5}{5!} + \cdots \right) \tag{4.11}$$

or, equivalently,

$$y(x) = a_0 \, \cosh(kx) + \frac{a_1}{k} \, \sinh(kx) \tag{4.12}$$

- This solution can also be written as

$$y(x) = A\, e^{kx} + B\, e^{-kx} \tag{4.13}$$

where

$$A = \frac{1}{2}\left(a_0 + \frac{a_1}{k}\right) \quad \text{and} \quad B = \frac{1}{2}\left(a_0 - \frac{a_1}{k}\right) \tag{4.14}$$

4.3 Legendre polynomials

- Legendre's equation for $y(x)$ is

$$(1-x^2)\, y'' - 2x\, y' + l(l+1)\, y = 0 \tag{4.15}$$

where l is a constant.

- Note that

$$p(x) = -\frac{2x}{1-x^2}\,, \qquad q(x) = \frac{l(l+1)}{1-x^2} \tag{4.16}$$

are finite at $x=0$, so $x=0$ is a regular point of the differential equation.

- This differential equation arises when one does separation of variables for Laplace's equation in spherical polar coordinates (here $x = \cos\theta$).

- One can show that $y(x)$ admits a regular power series solution with recurrence relation

$$a_{n+2} = \frac{n(n+1) - l(l+1)}{(n+1)(n+2)}\, a_n \tag{4.17}$$

- Since the recurrence relation relates a_{n+2} to a_n, the two independent solutions to Legendre's equation are given by setting $a_0 = 1$, $a_1 = 0$ and $a_1 = 1$, $a_0 = 0$. These solutions will be even and odd functions of x, respectively.

- By the ratio test, one can show that the power series solutions converge for $|x| < 1$.

- One can also show that the power series solutions diverge at $x = \pm 1$ (corresponding to the North and South poles of the sphere) unless the series terminates after some finite value of n.

- Example: For $l = 0$, the power series solution proportional to a_1 is

$$y(x) = a_1 \left[x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots \right] \tag{4.18}$$

which diverges at $x = 1$ as can be shown by the integral test.

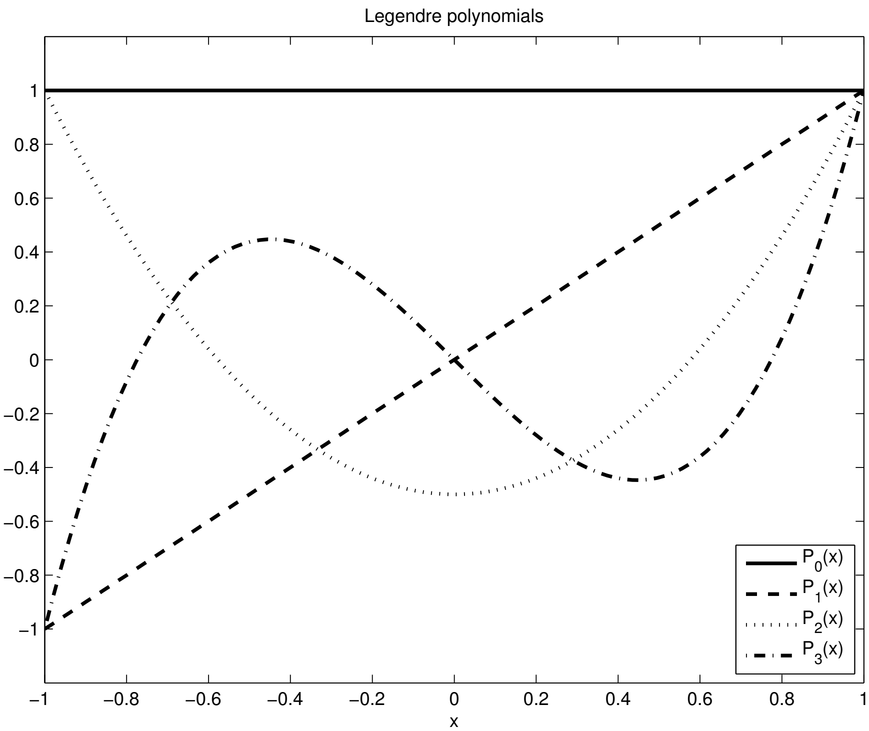

Figure 1: First few Legendre polynomials $P_l(x)$ plotted as functions of $x \in [-1, 1]$.

- From the recurrence relation, we see that if l is a non-negative integer, $l = 0, 1, \cdots$, one of the power series solutions terminates (the even solution if l is even, and the odd solution if l is odd). The other solution can be set to zero (by hand) by setting $a_1 = 0$ or $a_0 = 0$.

- The finite solutions are polynomials of order l. When appropriately normalised, they are called *Legendre polynomials*, denoted $P_l(x)$.

- NOTE: If l is a negative integer, $l = -1, -2, \cdots$, one also obtains a polynomial solution. But these solutions are the *same* as those for l non-negative (e.g., $l = -1$ yields the same solution as $l = 0$, and $l = -2$ yields the same solution as $l = 1$, etc.) so there is no loss of generality in restricting attention to $l = 0, 1, \cdots$.

- Legendre polynomials are normalized by the condition that $P_l(1) = 1$.

- Exercise: Show that the first four Legendre polynomials are given by

$$\begin{aligned} P_0(x) &= 1 && (4.19) \\ P_1(x) &= x && (4.20) \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) && (4.21) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) && (4.22) \end{aligned}$$

  See Figure 1 for a graph of these functions.

- Rodrigues' formula:

$$P_l(x) = \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2 - 1)^l \tag{4.23}$$

- Note that

$$P_l(-x) = (-1)^l P_l(x) \tag{4.24}$$

- Orthonormality:

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'} \tag{4.25}$$

  Thus, Legendre polynomials form a set of *orthogonal* polynomials.

- Exercise: Prove the above. (Hint: The proof of orthogonality is simple if you integrate Legendre's equation times $P_{l'}(x)$. The derivation of the normalization constant is harder, but can proved using mathematical induction and Rodrigues's formula for $P_l(x)$.)

- Completeness: Any square-integrable function $f(x)$ defined on the interval $x \in [-1, 1]$ can be expanded in terms of Legendre polynomials:

$$f(x) = \sum_{l=0}^{\infty} A_l\, P_l(x) \tag{4.26}$$

  where

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} dx\, f(x)\, P_l(x) \tag{4.27}$$

- Exercise: Show that the function

$$f(x) = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases} \tag{4.28}$$

  can be expanded as

$$f(x) = \frac{3}{2}\, P_1(x) - \frac{7}{8}\, P_3(x) + \frac{11}{16}\, P_5(x) + \cdots \tag{4.29}$$

  See Figure 2.

- Generating function:

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n \tag{4.30}$$

- Using the generating function, one can derive the following *recurrence relations*:

$$(n+1)\, P_{n+1} = (2n+1)x\, P_n - n\, P_{n-1} \tag{4.31}$$
$$P_n = P'_{n+1} - 2x\, P'_n + P'_{n-1} \tag{4.32}$$
$$n\, P_n = x\, P'_n - P'_{n-1} \tag{4.33}$$
$$(n+1)\, P_n = P'_{n+1} - x\, P'_n \tag{4.34}$$
$$(2n+1)\, P_n = P'_{n+1} - P'_{n-1} \tag{4.35}$$
$$(1 - x^2)\, P'_n = n(P_{n-1} - x\, P_n) \tag{4.36}$$

- Note that Legendre's equation

$$(1 - x^2)\, P''_n - 2x\, P'_n + n(n+1)\, P_n = 0 \tag{4.37}$$

  can be obtained by differentiating (4.36) wrt x and then using (4.33). In addition, the normalization $P_n(1) = 1$ also follows simply from the generating function.

- Exercise: Prove the above relations by differentiating the generating function wrt t and x separately, and then combining the various expressions.

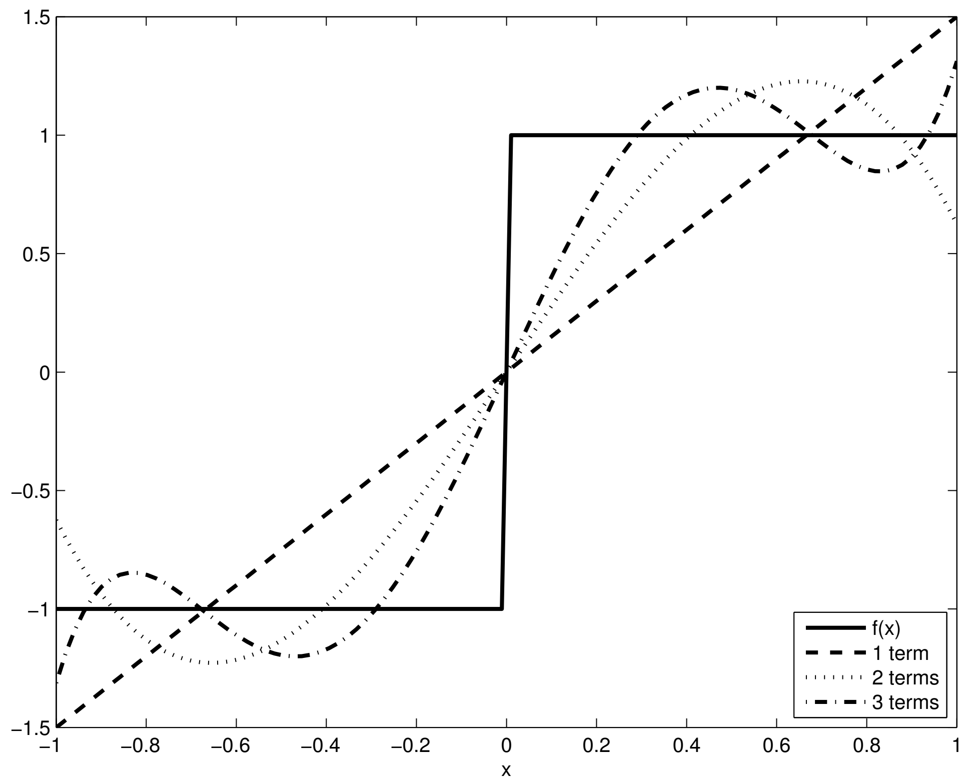

Figure 2: Expansion of the function $f(x) = \pm 1$ for $x \gtrless 0$, in terms Legendre polynomials. This plot shows how the approximation to $f(x)$ improves as more terms in the expansion are used.

- Another important result that follows trivially from the generating function expression is an expansion of the potential of a point charge in terms of Legendre polynomials:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \tag{4.38}$$

where $r_<$ ($r_>$) is the smaller (larger) of r and r', and γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$:

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' \equiv \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\,\cos(\phi-\phi') \tag{4.39}$$

4.4 Associated Legendre functions

- The associated Legendre equation is given by

$$(1-x^2)\,y'' - 2x\,y' + \left[l(l+1) - \frac{m^2}{(1-x^2)}\right] y = 0 \tag{4.40}$$

It differs from the ordinary Legendre equation due to the extra term proportional to m^2.

- It turns out that power series solutions of this differential equation also diverge at the poles ($x = \pm 1$) unless $l = 0, 1, \cdots$ (as before) and $m = -l, -l+1, \cdots, l$.

- The finite solutions are called *associated Legendre functions* are are given by derivatives of the Legendre polynomials:

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \tag{4.41}$$

and

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \tag{4.42}$$

for $m > 0$. (For $m = 0$ we recover the ordinary Legendre polynomials, $P_l(x) = P_l^0(x)$.)

- Exercise: Prove the above by direct substitution.

- NOTE: Boas does not include the factor of $(-1)^m$.

- The associated Legendre functions are *not* polynomials in x on account of the square root factor $(1-x^2)^{m/2}$ for odd m. But since we are ultimately interested in the replacement $x = \cos\theta$, these non-polynomial factors are just proportional to $\sin^m\theta$. Thus, the associated Legendre functions can be written as polynomials in $\cos\theta$ if m is even, and polynomials in $\cos\theta$ multiplied by $\sin\theta$ if m is odd.

- Exercise: Show that the first few associated Legendre functions are given by:

$l = 0$:

$$P_0^0(\cos\theta) = 1 \tag{4.43}$$

$l = 1$:

$$P_1^0(\cos\theta) = \cos\theta \tag{4.44}$$
$$P_1^1(\cos\theta) = -\sin\theta \tag{4.45}$$

$l = 2$:

$$P_2^0(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right) \tag{4.46}$$
$$P_2^1(\cos\theta) = -3\sin\theta\,\cos\theta \tag{4.47}$$
$$P_2^2(\cos\theta) = 3(1-\cos^2\theta) \tag{4.48}$$

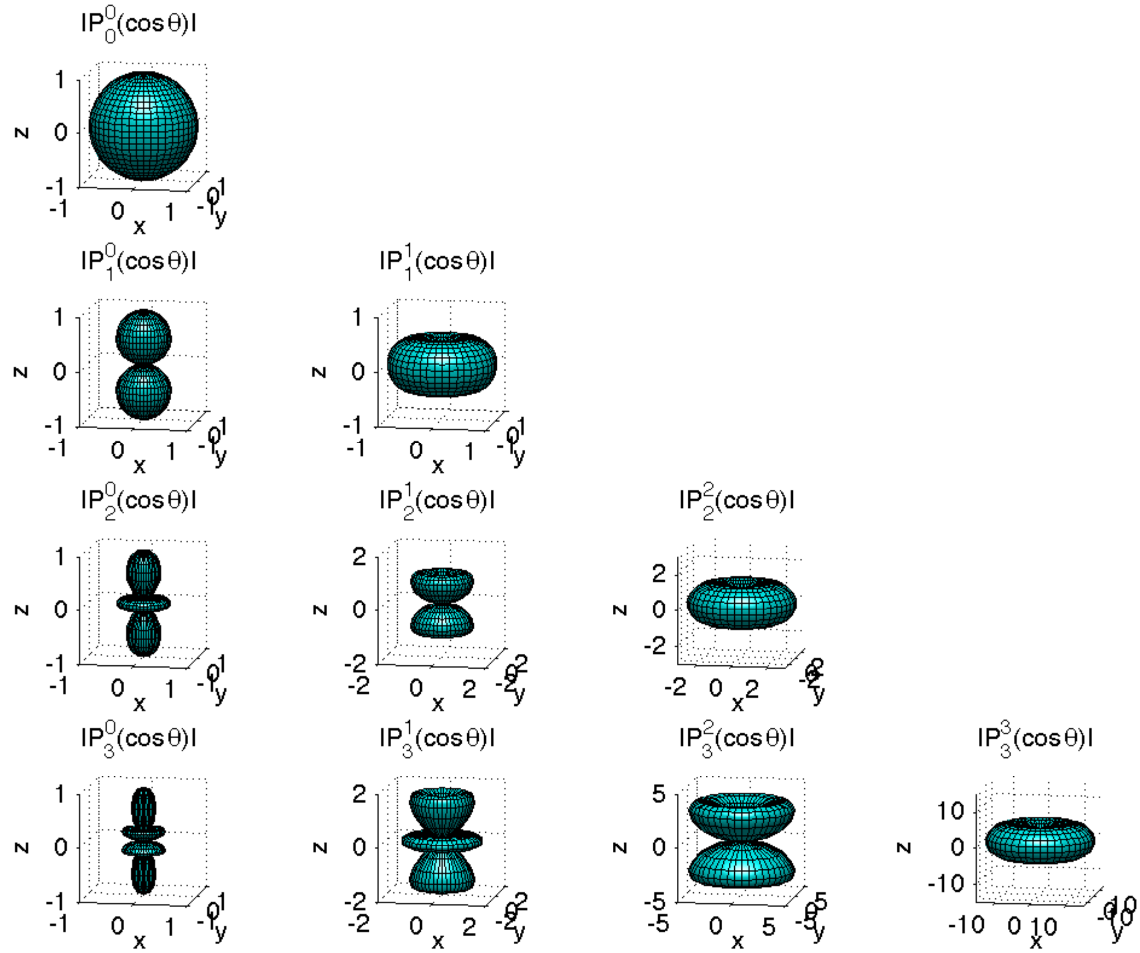

Figure 3: The *magnitude* $|P_l^m(\cos\theta)|$ of the first few associated Legendre functions plotted as surfaces of revolution. Note that the sign (i.e., $\pm$) of the associated Legendre functions $P_l^m(\cos\theta)$ is lost in this graphical representation.

<u>$l=3$</u>:

$$P_3^0(\cos\theta) = \frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right) \tag{4.49}$$

$$P_3^1(\cos\theta) = -\frac{3}{2}\sin\theta\,\left(5\cos^2\theta - 1\right) \tag{4.50}$$

$$P_3^2(\cos\theta) = 15\left(\cos\theta - \cos^3\theta\right) \tag{4.51}$$

$$P_3^3(\cos\theta) = -15\sin\theta\,\left(1 - \cos^2\theta\right) \tag{4.52}$$

See Figure 3 for plots of the magnitude of the first few of these functions.

- Using Rodrigues' formula, we can write down a formula valid for both positive and negative values of m:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\,\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \tag{4.53}$$

- <u>Orthonormality</u>: For each m

$$\int_{-1}^{1} dx\, P_l^m(x) P_{l'}^m(x) = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\,\delta_{ll'} \tag{4.54}$$

- Completeness: For each m, the associated Legendre functions form a complete set (in the index l) for square-integrable functions on $x \in [-1, 1]$:

$$f(x) = \sum_{l=0}^{\infty} A_l \, P_l^m(x) \tag{4.55}$$

where

$$A_l = \frac{2l+1}{2} \frac{(l-m)!}{(l+m)!} \int_{-1}^{1} dx \, f(x) \, P_l^m(x) \tag{4.56}$$

4.5 Bessel functions

- Separation of variables of Laplaces's equation in cylindrical coordinates leads to either of the following two differential equations for the radial function $R(\rho)$:

$$R''(\rho) + \frac{1}{\rho} R'(\rho) + \left(k^2 - \frac{\nu^2}{\rho^2} \right) R(\rho) = 0 \tag{4.57}$$

and

$$R''(\rho) + \frac{1}{\rho} R'(\rho) - \left(k^2 + \frac{\nu^2}{\rho^2} \right) R(\rho) = 0 \tag{4.58}$$

- These equations can be put into more standard form by making a change of variables

$$\rho \to x = k\rho \, , \qquad y(x)\Big|_{x=k\rho} \equiv R(\rho) \tag{4.59}$$

The two different equations corresponding to different choices of the sign for the constant $\pm k^2$ become:

$$y''(x) + \frac{1}{x} \, y'(x) + \left(1 - \frac{\nu^2}{x^2} \right) y(x) = 0 \tag{4.60}$$

and

$$y''(x) + \frac{1}{x} \, y'(x) - \left(1 + \frac{\nu^2}{x^2} \right) y(x) = 0 \tag{4.61}$$

- The first equation is called *Bessel's equation of order* ν; the second is called the *modified Bessel's equation of order* ν.

- Note that if $y(x)$ is a solution of Bessel's equation, then $\bar{y}(x) \equiv y(ix)$ is a solution of the modified Bessel's equation.

- Exercise: Prove the above.

- To solve Bessel's equation, note that $x = 0$ is a regular singular point of the differential equation. (The functions $p(x) \equiv 1/x$ and $q(x) \equiv (1 - \nu^2/x^2)$, which multiply $y'(x)$ and $y(x)$, respectively, are singular at $x = 0$, but $x \, p(x)$ and $x^2 \, q(x)$ are both finite at $x = 0$.)

- The method of Frobenius says that such a differential equation will admit a power series solution of the form

$$y(x) = x^\sigma \sum_{n=0}^{\infty} a_n x^n \tag{4.62}$$

- Substituting this expansion into Bessel's equation and equating the coefficients multiplying like powers of x leads to

$$a_0(\sigma^2 - \nu^2) = 0 \tag{4.63}$$

$$a_1(1 + 2\sigma + \sigma^2 - \nu^2) = 0 \tag{4.64}$$

$$a_{n+2} = -\frac{a_n}{(n+2+\sigma)^2 - \nu^2} \tag{4.65}$$

- The first of the above equations is called the *indicial equation*. For $a_0 \neq 0$ it has the solutions:

$$\sigma = \pm\nu \tag{4.66}$$

- Substituting these solutions for σ into the second equation leads to

$$a_1(1 \pm 2\nu) = 0 \tag{4.67}$$

- For $\nu \neq \pm 1/2$, this equation implies $a_1 = 0$. But even for $\nu = \pm 1/2$, we can set $a_1 = 0$.
- Thus, $a_1 = 0$ together with the recurrence relation implies $a_n = 0$ for all *odd* values of n.
- For the even expansion coefficients, we can rewrite the recurrence relation for $\sigma = \nu$ as

$$a_{2n} = a_0 \, \frac{(-1)^n \Gamma(1+\nu)}{2^{2n} n! \Gamma(n+1+\nu)} \tag{4.68}$$

for $n = 0, 1, 2, \cdots$, where the gamma function is defined by

$$\Gamma(z) = \int_0^\infty dx \, x^{z-1} e^{-x} \tag{4.69}$$

for $\mathrm{Re}(z) > 0$.

- The gamma function generalizes the factorial function to non-integer arguments in the sense that

$$\Gamma(n+1) = n! \quad \text{for } n = 0, 1, \cdots \tag{4.70}$$

$$\Gamma(z+1) = z\,\Gamma(z) \quad \text{for } \mathrm{Re}(z) > 0 \tag{4.71}$$

- If the normalization constant a_0 is chosen to be

$$a_0 = \frac{1}{2^\nu \Gamma(1+\nu)} \tag{4.72}$$

then

$$a_{2n} = \frac{(-1)^n}{2^{2n+\nu} n! \Gamma(n+1+\nu)} \tag{4.73}$$

- The power series solution is thus

$$J_\nu(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n! \Gamma(n+1+\nu)} \left(\frac{x}{2}\right)^{2n+\nu} \tag{4.74}$$

$J_\nu(x)$ is called *Bessel's function of the 1st kind.*

- Asymptotic form:

$$x \ll 1: \qquad J_\nu(x) \to \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \tag{4.75}$$

$$x \gg 1, \nu: \qquad J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{4.76}$$

- Thus, $J_0(0) = 1$, $J_\nu(0) = 0$ for all $\nu \neq 0$; while for large x, $J_\nu(x)$ behaves like a damped sinusoid, and has infinitely many zeros $x_{\nu n}$:

$$J_\nu(x_{\nu n}) = 0\,, \qquad \text{for } n = 1, 2, \cdots \tag{4.77}$$

See Figure 4.

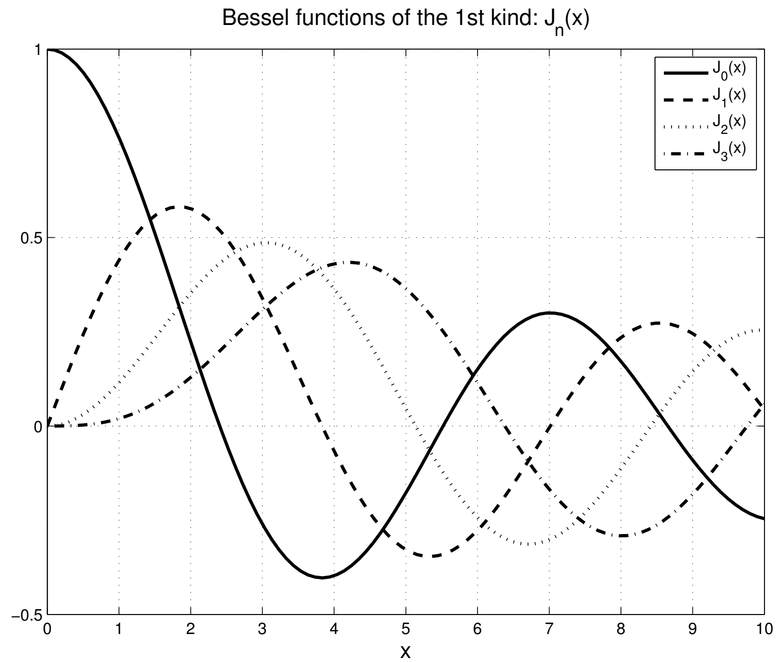

Figure 4: First few Bessel functions of the 1st kind for integer ν.

- Exercise: Show that the zeros of $J_\nu(x)$ are given by

$$x_{\nu n} \simeq n\pi + \left(\nu - \frac{1}{2}\right)\frac{\pi}{2} \tag{4.78}$$

- If ν is not an integer, then $J_{-\nu}(x)$ is the second independent solution to Bessel's equation.
- If $\nu = m$ is an integer, then one can show that $J_{-m}(x)$ is proportional to $J_m(x)$:

$$J_{-m}(x) = (-1)^m J_m(x) \tag{4.79}$$

  so $J_{-m}(x)$ is not an independent solution for this case.
- Exercise: Prove the above.
- A second solution, which is independent of $J_\nu(x)$ for all ν (integer or not), is

$$N_\nu(x) := \frac{J_\nu(x)\,\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \tag{4.80}$$

  $N_\nu(x)$ is called a *Neumann function* or *Bessel's function of the 2nd kind*, and is sometimes denoted by $Y_\nu(x)$.
- For $\nu = m$ an integer, one needs to use L'Hopital's rule to show that the RHS of the expression defining $N_m(x)$ is well-defined.
- Asymptotic form:

$$x \ll 1: \qquad N_\nu(x) \to \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + 0.5772\cdots\right], & \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu, & \nu \neq 0 \end{cases} \tag{4.81}$$

$$x \gg 1, \nu: \qquad N_\nu(x) \to \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{4.82}$$

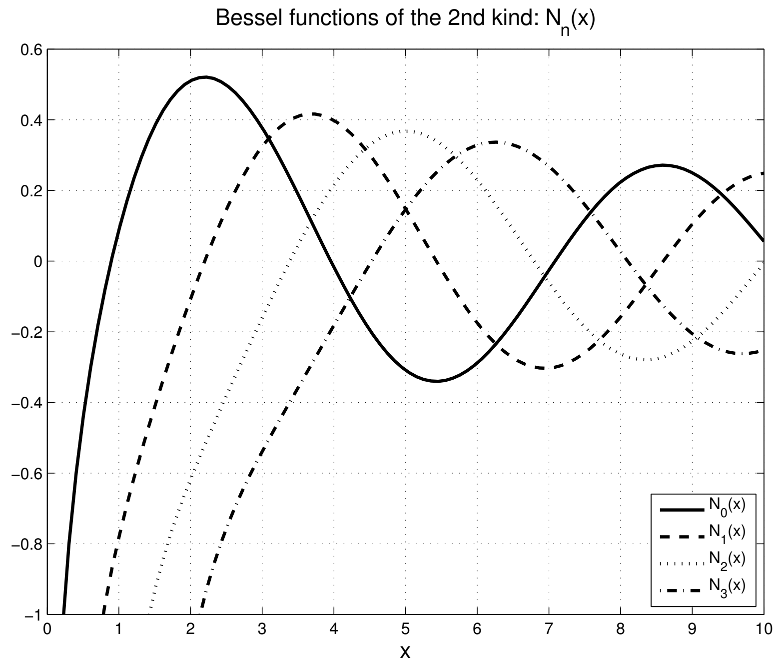

Figure 5: First few Bessel functions of the 2nd kind for integer ν.

- Note that for all ν, $N_\nu(x) \to -\infty$ as $x \to 0$.
- As we saw for $J_\nu(x)$, for large x, $N_\nu(x)$ behaves like a damped sinusoid, 90° out of phase with $J_\nu(x)$. See Figure 5.
- Thus, the most general solution to the radial part of Laplace's equation is

$$R(\rho) = A\, J_\nu(k\rho) + B\, N_\nu(k\rho) \tag{4.83}$$

- Since $N_\nu(x)$ blows up at $x = 0$, if $\rho = 0$ is in the region of interest, then all of the B coefficients must vanish to yield a finite value of the solution to Laplace's equation on the axis.
- *Hankel functions* (or *Bessel functions of the 3rd kind*) are defined by

$$H_\nu^{(1)}(x) := J_\nu(x) + iN_\nu(x) \tag{4.84}$$
$$H_\nu^{(2)}(x) := J_\nu(x) - iN_\nu(x) \tag{4.85}$$

- Recurrence relations:

$$\frac{d}{dx}\left(x^\nu J_\nu(x)\right) = x^\nu J_{\nu-1}(x) \tag{4.86}$$
$$\frac{d}{dx}\left(x^{-\nu} J_\nu(x)\right) = -x^{-\nu} J_{\nu+1}(x) \tag{4.87}$$
$$J'_\nu(x) = -\frac{\nu}{x} J_\nu(x) + J_{\nu-1}(x) \tag{4.88}$$
$$J'_\nu(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x) \tag{4.89}$$
$$2J'_\nu(x) = J_{\nu-1}(x) - J_{\nu+1}(x) \tag{4.90}$$
$$\frac{2\nu}{x} J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x) \tag{4.91}$$

- Exercise: Prove the above.

- Note that the recurrence relations also hold for $N_\nu(x)$, $H_\nu^{(1)}(x)$, $H_\nu^{(2)}(x)$, since they are relatively simple linear combinations of $J_\nu(x)$ and $J_{-\nu}(x)$.

- Orthogonality:

$$\int_a^b d\rho\, \rho J_\nu(k\rho) J_\nu(k'\rho) = 0 \quad \text{for } k \neq k' \tag{4.92}$$

where

$$\rho \left[J_\nu(k\rho) \frac{dJ_\nu}{d\rho}(k'\rho) - J_\nu(k'\rho) \frac{dJ_\nu}{d\rho}(k\rho) \right] \Bigg|_{\rho=a}^{b} = 0 \tag{4.93}$$

- Exercise: Prove this. (Hint: Let $f(\rho) = J_\nu(k\rho)$, $g(\rho) = J_\nu(k'\rho)$, write down Bessel's equation for f and g, multiply these equations by g and f, subtract, and then integrate.)

- An explicit example satifying the above boundary condition is to choose $a = 0$, rename $b = a$, and then choose k and k' so that $J_\nu(ka) = 0 = J_\nu(k'a)$. For this case k and k' take on discrete values

$$k \equiv k_{\nu n} = \frac{x_{\nu n}}{a}\,, \qquad k' \equiv k_{\nu n'} = \frac{x_{\nu n'}}{a}\,, \qquad n, n' = 1, 2, \cdots \tag{4.94}$$

where $x_{\nu n}$ and $x_{\nu n'}$ are the nth and n'th zeroes of $J_\nu(x)$.

- Note that the orthogonality of Bessel functions is wrt to different arguments $x = k\rho$ and $x' = k'\rho$ of a *single* function $J_\nu(x)$, and not wrt *different* functions $J_\nu(x)$ and $J_{\nu'}(x)$ of the same argument $x = k\rho$. (This latter case held for the Legendre polynomials $P_l(x)$ and $P_{l'}(x)$.)

- The orthogonality of Bessel functions is similar to the orthogonality of the sine functions $\sin(n\pi x/L)$ for different values of n.

- Normalization:

$$\int_a^b d\rho\, \rho J_\nu(k\rho) J_\nu(k\rho) = \frac{1}{2} \left[\left(\rho^2 - \frac{\nu^2}{k^2} \right) J_\nu^2(k\rho) + \rho^2 [J_\nu'(k\rho)]^2 \right] \Bigg|_{\rho=a}^{b} \tag{4.95}$$

where $J_\nu'(k\rho)$ denotes derivative wrt to its argument $x = k\rho$.

- Exercise: Prove the normalization condition. (Note: This is a rather tricky proof, requiring some clever integration by parts and the use of Bessel's equation to substitute for $x^2 J_\nu(x)$ in one of the integrals.)

- Again the RHS can be simplified for the case described above where we set $a = 0$, rename $b = a$, and take $k = x_{\nu n}/a$ for some integer n. Then

$$\int_0^a d\rho\, \rho J_\nu(x_{\nu n}\rho/a) J_\nu(x_{\nu n}\rho/a) = \frac{1}{2} a^2 [J_\nu'(x_{\nu n})]^2 = \frac{1}{2} a^2 J_{\nu+1}^2(x_{\nu n}) \tag{4.96}$$

where a recurrence relation was used to get the last equality.

- Exercise: Prove this.

- We can put the orthogonality and normalisation equations together as a single equation:

$$\int_0^a d\rho\, \rho J_\nu(x_{\nu n}\rho/a) J_\nu(x_{\nu n'}\rho/a) = \frac{1}{2} a^2 J_{\nu+1}^2(x_{\nu n})\, \delta_{nn'} \tag{4.97}$$

where we have explicitly indicated the zeroes $x_{\nu n}$ and $x_{\nu n'}$ of $J_\nu(x)$.

- If the interval $[0, a]$ becomes infinite $[0, \infty)$, then the orthogonality and normalisation conditions actually become simpler

$$\int_0^\infty d\rho\, \rho J_\nu(k\rho) J_\nu(k'\rho) = \frac{1}{k}\delta(k - k') \tag{4.98}$$

where k now takes on a continuous range of values.

- This is similar to the transition from Fourier series (basis functions $e^{ik_n x}$ with $k_n = n2\pi/a$) to Fourier transforms (basis functions e^{ikx} with k a real variable):

$$\int_{-a/2}^{a/2} dx\, e^{i2\pi(n-n')x/a} = a\,\delta_{nn'} \longrightarrow \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = 2\pi\,\delta(k - k') \tag{4.99}$$

4.6 Modified Bessel functions

- As mentioned previously, the *modified* Bessel's equation of order ν is given by:

$$y''(x) + \frac{1}{x} y'(x) - \left(1 + \frac{\nu^2}{x^2}\right) y(x) = 0 \tag{4.100}$$

It differs from the ordinary Bessel's equation only in the sign of one of the terms multiplying $y(x)$.

- *Modified* (or *hyperbolic*) *Bessel functions* are solutions to the above equation. They are defined by

$$I_\nu(x) := i^{-\nu} J_\nu(ix) \tag{4.101}$$

$$K_\nu(x) := \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \tag{4.102}$$

Note the pure imaginary arguments on the RHS, consistent with our earlier statement that if $y(x)$ is a solution of Bessel's equation then $y(ix)$ is a solution of the modified Bessel's equation.

- See Figures 6 and 7 for graphs of the first few modified Bessel functions of the first and second kind, $I_\nu(x)$ and $K_\nu(x)$, for integer values of ν.

- Asymptotic form:

$$x \ll 1: \qquad I_\nu(x) \to \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \tag{4.103}$$

$$K_\nu(x) \to \begin{cases} -\left[\ln\left(\frac{x}{2}\right) + 0.5772\cdots\right], & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu, & \nu \neq 0 \end{cases} \tag{4.104}$$

$$x \gg 1, \nu: \qquad I_\nu(x) \to \frac{1}{\sqrt{2\pi x}}\, e^x \left[1 + O\left(\frac{1}{x}\right)\right] \tag{4.105}$$

$$K_\nu(x) \to \sqrt{\frac{\pi}{2x}}\, e^{-x} \left[1 + O\left(\frac{1}{x}\right)\right] \tag{4.106}$$

- Thus, $I_0(0) = 1$, $I_\nu(0) = 0$ for all $\nu \neq 0$, while $K_\nu(x) \to \infty$ as $x \to 0$ for all ν.

- For large x, $I_\nu(x) \to \infty$ while $K_\nu(x) \to 0$ for all ν.

- Thus, the most general solution to the radial part of Laplace's equation for the choice of negative separation constant $-k^2$ is

$$R(\rho) = A\, I_\nu(k\rho) + B\, K_\nu(k\rho) \tag{4.107}$$

- Since $K_\nu(x)$ blows up at $x = 0$, if $\rho = 0$ is in the region of interest, then all of the B coefficients must vanish to yield a finite value of the solution to Laplace's equation on the axis.

- Similarly, since $I_\nu(x)$ blows up as $x \to \infty$, if the solution to Laplace's equation is to vanish as $\rho \to \infty$, then all of the A coefficients must vanish.

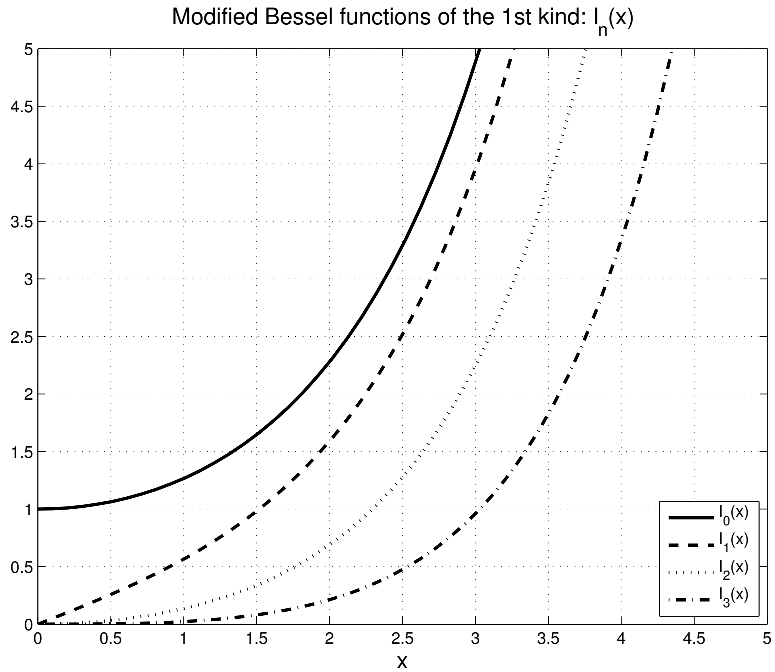

Figure 6: First few modified Bessel functions of the 1st kind for integer ν.

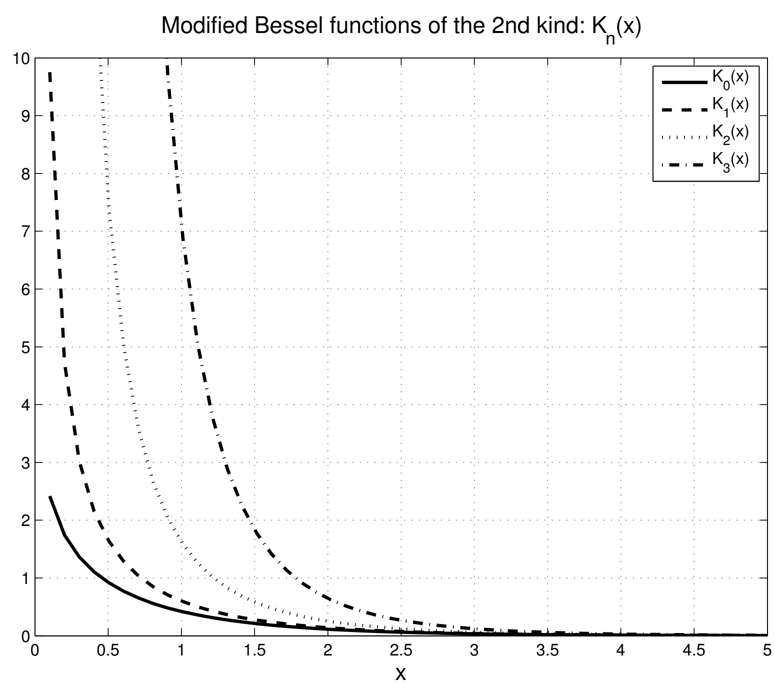

Figure 7: First few modified Bessel functions of the 2nd kind for integer ν.

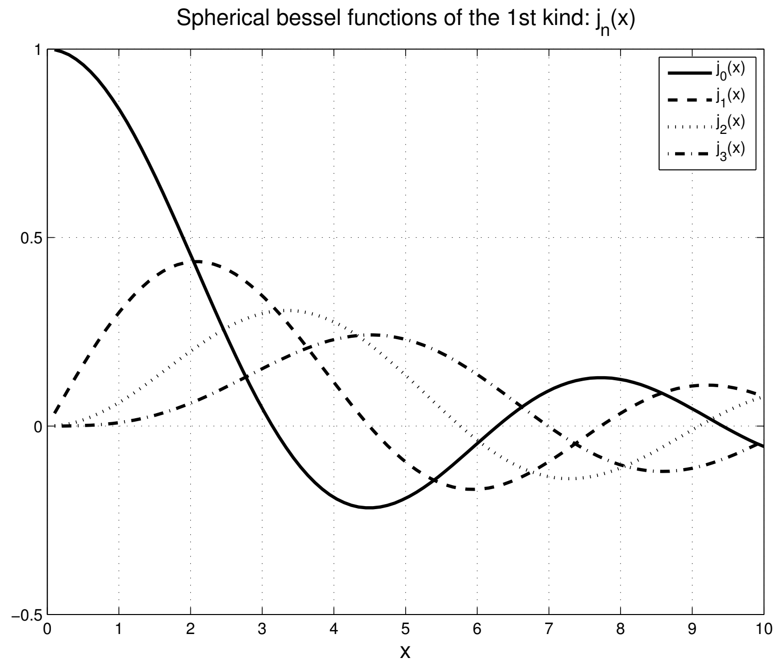

Figure 8: First few spherical Bessel functions of the 1st kind.

4.7 Spherical Bessel functions

- *Spherical Bessel functions* are defined by

$$j_n(x) := \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x) \tag{4.108}$$

$$n_n(x) := \sqrt{\frac{\pi}{2x}}\, N_{n+\frac{1}{2}}(x) \tag{4.109}$$

where $n = 0, 1, 2, \cdots$.

- One can also define

$$h_n^{(1)}(x) := j_n(x) + i n_n(x) \tag{4.110}$$

$$h_n^{(2)}(x) := j_n(x) - i n_n(x) \tag{4.111}$$

- Given the explicit form of $J_{n+\frac{1}{2}}(x)$ one can show that

$$j_n(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right) \tag{4.112}$$

$$n_n(x) = -x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right) \tag{4.113}$$

- In particular, it follows that

$$j_0(x) = \frac{\sin x}{x}\,, \qquad n_0(x) = -\frac{\cos x}{x} \tag{4.114}$$

See Figures 8 and 9 for plots of the first few spherical Bessel functions.

- Exercise: Prove the above expression for $j_0(x)$ directly from its definition in terms of the ordinary Bessel function $J_{\frac{1}{2}}(x)$.

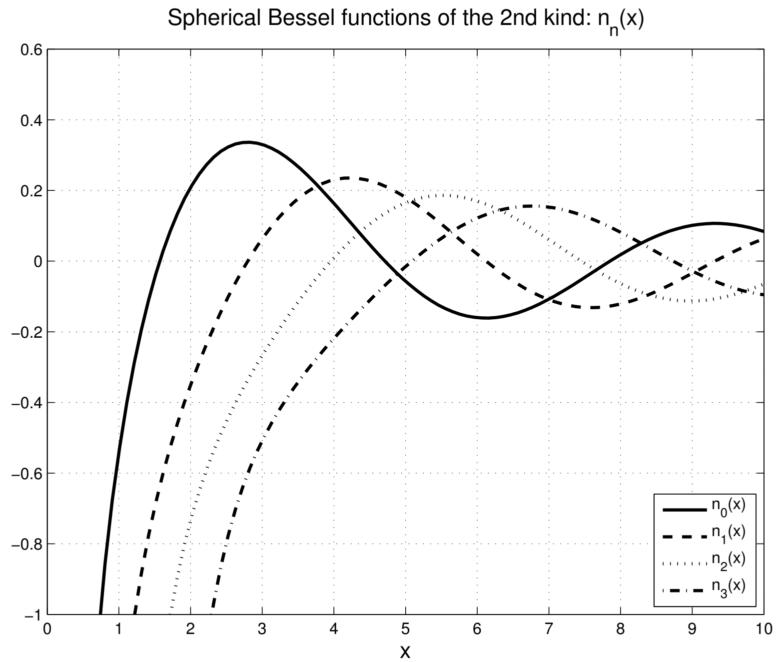

Figure 9: First few spherical Bessel functions of the 2nd kind.

- Given the relationship between $j_n(x)$ and $J_{n+\frac{1}{2}}(x)$, one can show that the spherical Bessel functions satisfy the differential equation

$$j_n''(x) + \frac{2}{x} j_n'(x) + \left[1 - \frac{n(n+1)}{x^2}\right] j_n(x) = 0 \tag{4.115}$$

- Exercise: Prove this.

- Alternatively, one arrives at the same differential equation by using separation of variables in *spherical polar coordinates* to solve the *Helmholtz equation*:

$$\nabla^2 \Phi(r, \theta, \phi) + k^2 \Phi(r, \theta, \phi) = 0 \tag{4.116}$$

- The ϕ equation is the standard harmonic oscillator equation with separation constant $-m^2$; the θ equation is the associated Legendre's equation with separation constants l and m; and the radial equation is

$$R''(r) + \frac{2}{r} R'(r) + \left[k^2 - \frac{l(l+1)}{r^2}\right] R(r) = 0 \tag{4.117}$$

- Making the change of variables $x = kr$ with $y(x)|_{x=kr} = R(r)$, leads to

$$y''(x) + \frac{2}{x} y'(x) + \left[1 - \frac{l(l+1)}{x^2}\right] y(x) = 0 \tag{4.118}$$

  which is the differential equation (4.115) we found earlier with solution $y(x) = j_l(x)$.

4.8 Similarities between Bessel and trig / hyperbolic functions

- Trigonometric functions:

$$y''(x) + k^2 y(x) = 0 \quad \Rightarrow \quad y(x) = A\sin(kx) + B\cos(kx) \tag{4.119}$$

- Hyperbolic functions:

$$y''(x) - k^2 y(x) = 0 \quad \Rightarrow \quad y(x) = A\sinh(kx) + B\cosh(kx) \tag{4.120}$$

- Bessel functions:

$$R''(\rho) + \frac{1}{\rho}R'(\rho) + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R(\rho) = 0 \quad \Rightarrow \quad R(\rho) = AJ_\nu(k\rho) + BN_\nu(k\rho) \tag{4.121}$$

- Modified Bessel functions:

$$R''(\rho) + \frac{1}{\rho}R'(\rho) - \left(k^2 + \frac{\nu^2}{\rho^2}\right)R(\rho) = 0 \quad \Rightarrow \quad R(\rho) = AI_\nu(k\rho) + BK_\nu(k\rho) \tag{4.122}$$

- Hyperbolic functions are related to trig functions via:

$$\sinh(kx) = -i\sin(ikx)\,, \qquad \cosh(kx) = \cos(ikx) \tag{4.123}$$

- Modified Bessel functions are related to Bessel functions via:

$$I_\nu(k\rho) = i^{-\nu}J_\nu(ik\rho)\,, \qquad K_\nu(k\rho) = \frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(ik\rho) \tag{4.124}$$

- Hankel functions:

$$H_\nu^{(1)}(k\rho) = J_\nu(k\rho) + iN_\nu(k\rho) \tag{4.125}$$
$$H_\nu^{(2)}(k\rho) = J_\nu(k\rho) - iN_\nu(k\rho) \tag{4.126}$$

  are analogous to complex exponentials:

$$e^{ikx} = \cos(kx) + i\sin(kx) \tag{4.127}$$
$$e^{-ikx} = \cos(kx) - i\sin(kx) \tag{4.128}$$

- At $x = L$, the trig function $\sin(kx)$ has zeroes at

$$kL = n\pi \quad \Rightarrow \quad k = \frac{n\pi}{L}\,, \qquad \text{for } n = 1, 2, 3, \cdots \tag{4.129}$$

- At $\rho = a$, the Bessel function $J_\nu(k\rho)$ has zeroes at

$$ka = x_{mn} \quad \Rightarrow \quad k = \frac{x_{\nu n}}{a}\,, \qquad \text{for } n = 1, 2, 3, \cdots \tag{4.130}$$

- Orthogonality of trig functions:

$$\int_0^L dx\ \sin(n\pi x/L)\sin(n'\pi x/L) = \frac{L}{2}\delta_{nn'} \tag{4.131}$$

- Orthogonality of Bessel functions:

$$\int_0^a d\rho\,\rho J_\nu(x_{\nu n}\rho/a)J_\nu(x_{\nu n'}\rho/a) = \frac{1}{2}a^2 J_{\nu+1}^2(x_{\nu n})\,\delta_{nn'} \tag{4.132}$$

4.9 Hermite polynomials

- Differential equation:

$$y'' - 2xy' + 2my = 0 \tag{4.133}$$

- Recurrence relation for power series solution:

$$a_{n+2} = \frac{2(n-m)}{(n+1)(n+2)} a_n \tag{4.134}$$

- Polynomial solutions for integer m.

  For m even:

$$y(x) = a_0 \left[1 + (-2)\frac{m}{2!}\, x^2 + (-2)^2 \frac{m(m-2)}{4!}\, x^4 + \cdots + (-2)^{m/2} \frac{m(m-2)\cdots 2}{m!}\, x^m \right] \tag{4.135}$$

  For m odd:

$$y(x) = a_1 \Bigg[x + (-2)\frac{(m-1)}{3!}\, x^3 + (-2)^2 \frac{(m-1)(m-3)}{5!}\, x^5 + \cdots \tag{4.136}$$

$$+ (-2)^{(m-1)/2} \frac{(m-1)(m-3)\cdots 2}{m!}\, x^m \Bigg] \tag{4.137}$$

- First few Hermite polynomials normalized so that $a_m = 2^m$:

$$H_0(x) = 1\,, \quad H_1(x) = 2x\,, \quad H_2(x) = 4x^2 - 2\,, \quad \cdots \tag{4.138}$$

- Rodrigues formula for Hermite polynomials:

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} e^{-x^2} \tag{4.139}$$

- <u>Orthogonality</u>:

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} H_m(x) H_n(x) = \sqrt{\pi}\, 2^m m!\, \delta_{mn} \tag{4.140}$$

- <u>Generating function</u>:

$$e^{2xt - t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \tag{4.141}$$

- <u>Recurrence relations</u>:

$$H_n'(x) = 2nH_{n-1}(x) \tag{4.142}$$

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{4.143}$$

4.10 Hermite functions

- Hermite functions are obtained from Hermite polynomials via:

$$y_n(x) = (-1)^n e^{-x^2/2} H_n(x) \tag{4.144}$$

- They satisfy the differential equation:

$$y_n'' - x^2 y_n = -(2n+1) y_n \tag{4.145}$$

  which is an *eigenvalue* equation with discrete eigenvalues $\lambda_n = -(2n+1)$, where n is an integer.

- One can also show that

$$y_{n+1} = (D - x)y_n \,, \qquad y_{n-1} = (D + x)y_n \tag{4.146}$$

where $D = d/dx$ is derivative with respect to x.

- Exercise: Prove the above.

- The operators $(D \mp x)$ are called *raising* and *lowering* operators, since they take one Hermite function to another by simply *increasing* or *decreasing* the value of the index n. These operators are also sometimes called *ladder* operators.

- In quantum mechanics, they are called *creation* and *annihilation* operators for particle states.

- The so-called *ground state* is the Hermite function having $n = 0$:

$$y_0(x) = e^{-x^2/2} \tag{4.147}$$

- It satisfies

$$(D + x)y_0 = 0 \quad \Rightarrow \quad y_0'' - x^2 y_0 = -y_0 \tag{4.148}$$

which means that you can't get to a lower state by having the lowering operator act on the ground state.

4.11 Laguerre polynomials

- Differential equation:

$$xy'' + (1 - x)y' + my = 0 \tag{4.149}$$

- Polynomial solution for integer m:

$$L_m(x) = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{x^n}{n!} \,, \quad \text{where} \quad \binom{m}{n} = \frac{m!}{(m-n)!n!} \tag{4.150}$$

- First few Laguerre polynomials:

$$L_0(x) = 1 \,, \quad L_1(x) = 1 - x \,, \quad L_2(x) = 1 - 2x + \frac{1}{2}x^2 \,, \quad \cdots \tag{4.151}$$

- Rodrigues formula for Hermite polynomials:

$$L_m(x) = \frac{1}{m!} e^x \frac{d^m}{dx^m} \left(x^m e^{-x} \right) \tag{4.152}$$

- Orthogonality:

$$\int_0^\infty dx\, e^{-x} L_m(x) L_n(x) = \delta_{mn} \tag{4.153}$$

- Generating function:

$$\frac{e^{xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} L_n(x) t^n \tag{4.154}$$

- Recurrence relations:

$$L_{n+1}'(x) - L_n'(x) + L_n(x) = 0 \tag{4.155}$$

$$(n+1)L_{n+1}(x) - (2n + 1 - x)L_n(x) + nL_{n-1}(x) = 0 \tag{4.156}$$

$$xL_n'(x) - nL_n(x) + nL_{n-1}(x) = 0 \tag{4.157}$$

4.12 Associated Laguerre polynomials

- Differential equation:

$$xy'' + (k + 1 - x)y' + my = 0 \tag{4.158}$$

- Polynomial solution for integer m and k:

$$L_m^k(x) = (-1)^k \frac{d^k}{dx^k} L_{m+k}(x) \tag{4.159}$$

- Note that the associated Laguerre polynomials are obtained by simply differentiating the ordinary Laguerre polynomials.

- Rodrigues formula for the associated Laguerre polynomials:

$$L_m^k(x) = \frac{x^{-k} e^x}{m!} \frac{d^m}{dx^m} \left(x^{m+k} e^{-x} \right) \tag{4.160}$$

- NOTE: This last form is actually valid for *non-integer* k, and can be used to define $L_m^k(x)$ for any real $k > -1$.

- Orthogonality:

$$\int_0^\infty dx\, x^k e^{-x} L_m^k(x) L_n^k(x) = \frac{(m+k)!}{m!} \delta_{mn} \tag{4.161}$$

- Recurrence relations:

$$(n+1)L_{n+1}^k(x) - (2n + k + 1 - x)L_n^k(x) + (n+k)L_{n-1}^k(x) = 0 \tag{4.162}$$

$$x \frac{d}{dx} L_n^k(x) - nL_n^k(x) + (n+k)L_{n-1}^k(x) = 0 \tag{4.163}$$

5 Partial differential equations

5.1 Examples of partial differential equations

- Laplace's equation:
$$\nabla^2 u = 0 \tag{5.1}$$

- Helmholtz equation:
$$\nabla^2 u + k^2\, u = 0 \tag{5.2}$$

- Diffusion equation:
$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \quad \text{(heat flow)} \tag{5.3}$$
$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\,\psi = i\hbar\frac{\partial\psi}{\partial t} \quad \text{(quantum mechanics)} \tag{5.4}$$

- Source-free wave equation:
$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{5.5}$$

- Vector wave equations (E&M):
$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0\,, \quad \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = 0 \tag{5.6}$$

- Poisson's equation:
$$\nabla^2 u = f(\mathbf{r}) \tag{5.7}$$
where f is some source (e.g., charge density in electrostatics, mass density in Newtonian gravity).

- Navier-Stokes equation: (non-linear, fluid mechanics)
$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla}p}{\rho} + \nu\nabla^2\mathbf{v} + \mathbf{g} \tag{5.8}$$

5.2 Separation of variables

- The most common method of solution of a PDE is called *separation of variables*. It amounts to assuming that the function u can be written as a product of functions, which depend on only a single variable—e.g., $u(x,y,z) = X(x)Y(y)Z(z)$.

- Separation of variables converts a single PDE into separate ODEs (the number of ODEs equal to the number of independent variables).

- Not all PDEs can be solved using separation of variables. Only certain types of PDEs can be separated, and only in certain coordinate systems—e.g., Laplace's equation in Cartesian, cylindrical polar coordinates, spherical polar coordinates, ... The BCs must also have the appropriate symmetry and homogeneity in order for the separation of variables method to work.

5.3 Laplace's equation

- Laplace's equation is
$$\nabla^2 u = 0 \tag{5.9}$$
where u is a function of the spatial coordinates x, y, z or ρ, ϕ, z, etc.

- We are interested in solving the above equation for u subject to specified boundary conditions, using separation of variables.

- The choice of coordinates is usually dictated by the geometry of the problem.

- In the following subsections, we solve Laplace's equation in the context of steady-state temperature distributions in: (i) a rectangular plate, (ii) a cylinder, and (iii) a sphere, subject to different BCs.

5.3.1 Steady-state temperature in a rectangular plate

- Example: Find the steady-state temperature distribution $u(x,y)$ in a two-dimensional rectangular plate $0 \le x \le a$ and $0 \le y \le b$, subject to BCs that the temperature is zero along three of the edges ($x=0$, $x=a$, $y=b$) and is equal to a constant u_0 along the other edge ($y=0$).

- Answer:

$$u(x,y) = \frac{4u_0}{\pi} \sum_{n=1,3,\cdots}^{\infty} \frac{1}{n\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left[\frac{n\pi(b-y)}{a}\right] \tag{5.10}$$

- Solution method:

  Differential equation:

$$0 = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{5.11}$$

  Separation of variables:

$$u(x,y) \equiv X(x)Y(y) \tag{5.12}$$

  Resultant set of ordinary differential equations:

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2 \tag{5.13}$$

  where k^2 is a separation constant, whose sign was chosen to be able to satisfy the BCs.

  Solutions of the x-equation:

$$X(x) = A_0 + B_0\, x\,, \quad \text{for } k=0 \tag{5.14}$$

$$X(x) = A\,\sin(kx) + B\,\cos(kx)\,, \quad \text{for } k \neq 0 \tag{5.15}$$

  Solutions of the y-equation:

$$Y(y) = C_0 + D_0\, y\,, \quad \text{for } k=0 \tag{5.16}$$

$$Y(y) = C\, e^{ky} + D\, e^{-ky}\,, \quad \text{for } k \neq 0 \tag{5.17}$$

  BCs at $x=0$ and $x=a$ imply:

$$X(x) = A\sin kx\,, \quad \text{where } k = \frac{n\pi}{a}\,, \quad n = 1,2,\cdots \tag{5.18}$$

  BC at $y=b$ implies:

$$Y(y) = E\sinh\left[\frac{n\pi(b-y)}{a}\right] \tag{5.19}$$

  Notes:

  1) There is no non-zero $k=0$ solution which satisfies the BCs.

  2) We need only consider positive integers n, since negative n introduces only an overall sign change from positive n, which can be absorbed in the multiplicative constant.

  General solution satisfying these three BCs:

$$u(x,y) = \sum_{n=1}^{\infty} E_n\,\sin\left(\frac{n\pi x}{a}\right)\sinh\left[\frac{n\pi(b-y)}{a}\right] \tag{5.20}$$

  Apply final BC at $y=0$:

$$u_0 = u(x,0) = \sum_{n=1}^{\infty} E_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \tag{5.21}$$

Orthogonality of sinusoids:

$$\int_0^a dx\, \sin\left(\frac{n\pi x}{a}\right)\, \sin\left(\frac{m\pi x}{a}\right) = \frac{a}{2}\,\delta_{nm} \tag{5.22}$$

implies

$$C_n \equiv E_n\, \sinh\left(\frac{n\pi b}{a}\right) = \begin{cases} 4u_0/n\pi & n = 1, 3, \cdots \\ 0 & n = 2, 4, \cdots \end{cases} \tag{5.23}$$

Thus,

$$u(x, y) = \frac{4u_0}{\pi} \sum_{n=1,3,\cdots}^{\infty} \frac{1}{n \sinh\left(\frac{n\pi b}{a}\right)}\, \sin\left(\frac{n\pi x}{a}\right) \sinh\left[\frac{n\pi(b-y)}{a}\right] \tag{5.24}$$

NOTE: To solve Laplace's equation inside the same rectangular region for more complicated BCs (e.g., where more than one edge has non-zero values), we can simply superimpose the 'single-edge' solutions, which all have a form similar to the above solution.

5.3.2 Steady-state temperature in a cylinder

- Example: Find the steady-state temperature distribution $u(\rho, \phi, z)$ in a cylinder of radius a and height b, subject to the BCs that the temperature is zero on the top of the cylinder ($z = b$) and the curved surface ($\rho = a$), and is equal to a constant u_0 on the bottom ($z = 0$).

- Answer:

$$u(\rho, \phi, z) = 2u_0 \sum_{n=1}^{\infty} \frac{1}{x_{0n} J_1(x_{0n}) \sinh\left(\frac{x_{0n} b}{a}\right)}\, J_0\left(\frac{x_{0n}\rho}{a}\right) \sinh\left[\frac{x_{0n}(b-z)}{a}\right] \tag{5.25}$$

where x_{0n} is the nth zero of the Bessel function $J_0(x)$.

- Solution method:

Differential equation:

$$0 = \nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2} \tag{5.26}$$

Separation of variables:

$$u(\rho, \phi, z) \equiv R(\rho)Q(\phi)Z(z) \tag{5.27}$$

leads to

$$0 = \frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2}\frac{1}{Q}\frac{d^2 Q}{d\phi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} \tag{5.28}$$

where we've divided the whole equation by $u = RQZ$.

If the separation constants are chosen so that

$$Z''(z) = k^2 Z(z)\,, \tag{5.29}$$

$$Q''(\phi) = -\nu^2 Q(\phi) \tag{5.30}$$

then

$$R''(\rho) + \frac{1}{\rho}\, R'(\rho) + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R(\rho) = 0 \tag{5.31}$$

which is Bessel's equation of order ν.

Solutions of the z-equation:

$$Z(z) = A_0 + B_0\, z\,, \quad \text{for } k = 0 \tag{5.32}$$

$$Z(z) = A\, e^{kz} + B\, e^{-kz}\,, \quad \text{for } k \neq 0 \tag{5.33}$$

Solutions to the ϕ-equation:

$$Q(\phi) = C_0 + D_0\phi\,, \quad \text{for } \nu = 0 \tag{5.34}$$
$$Q(\phi) = C\sin(\nu\phi) + D\cos(\nu\phi)\,, \quad \text{for } \nu \neq 0 \tag{5.35}$$

Periodic boundary condition:

$$Q(\phi + 2\pi n) = Q(\phi) \tag{5.36}$$

implies $D_0 = 0$ and $\nu = m$ an integer, so

$$Q(\phi) = Ce^{im\phi}\,, \quad m = 0, \pm 1, \pm 2, \cdots \tag{5.37}$$

where C is a complex multiplicative constant.
For $\nu = m$, the ρ-equation becomes:

$$R''(\rho) + \frac{1}{\rho}R'(\rho) + \left(k^2 - \frac{m^2}{\rho^2}\right)R(\rho) = 0 \tag{5.38}$$

Solutions to the ρ-equation:

$$R(\rho) = E_0\rho^{|m|} + F_0\rho^{-|m|}\,, \quad \text{for } k = 0 \tag{5.39}$$
$$R(\rho) = EJ_m(k\rho) + FN_m(k\rho)\,, \quad \text{for } k \neq 0 \tag{5.40}$$

Apply BC that the solution should be finite on the axis of the cylinder, $\rho = 0$:

$$F_0 = 0\,, \quad F = 0 \tag{5.41}$$

Apply BC that $u = 0$ when $\rho = a$:

$$R(\rho) = EJ_m(k\rho)\,, \quad \text{where } k = \frac{x_{mn}}{a}\,, \quad n = 1, 2, \cdots \tag{5.42}$$

where x_{mn} is the nth zero of the mth Bessel function $J_m(x)$.
Note that there is no non-zero $k = 0$ solution which satifies the boundary condition.
BC at $z = b$ implies:

$$Z(z) = A\sinh\left[\frac{x_{mn}(b-z)}{a}\right] \tag{5.43}$$

General solution satifying all the BCs except the one at $z = 0$:

$$u(\rho, \phi, z) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} E_{mn}\,J_m\left(\frac{x_{mn}\rho}{a}\right)\sinh\left[\frac{x_{mn}(b-z)}{a}\right]e^{im\phi} \tag{5.44}$$

Apply final BC at z=0:

$$u_0 = u(\rho, \phi, 0) = \sum_{n=1}^{\infty} E_n\,\sinh\left(\frac{x_{0n}b}{a}\right)J_0\left(\frac{x_{0n}\rho}{a}\right) \tag{5.45}$$

Note that we are able to set $m = 0$ since $u_0 = \text{const}$ is independent of ϕ.
Orthogonality of Bessel functions:

$$\int_0^a d\rho\,\rho J_0\left(\frac{x_{0n}\rho}{a}\right)J_0\left(\frac{x_{0n'}\rho}{a}\right) = \frac{a^2}{2}J_1^2(x_{0n})\,\delta_{nn'} \tag{5.46}$$

implies

$$C_n \equiv E_n\sinh\left(\frac{x_{0n}b}{a}\right) = \frac{2u_0}{a^2J_1^2(x_{0n})}\int_0^a d\rho\,\rho J_0\left(\frac{x_{0n}\rho}{a}\right) \tag{5.47}$$

To do the integral, use the recurrence relation:

$$\frac{d}{dx}\left[xJ_1(x)\right] = xJ_0(x) \tag{5.48}$$

to get

$$C_n = \frac{2u_0}{x_{0n}J_1(x_{0n})} \tag{5.49}$$

Thus,

$$u(\rho,\phi,z) = 2u_0 \sum_{n=1}^{\infty} \frac{1}{x_{0n}J_1(x_{0n})\sinh\left(\frac{x_{0n}b}{a}\right)} J_0\left(\frac{x_{0n}\rho}{a}\right)\sinh\left[\frac{x_{0n}(b-z)}{a}\right] \tag{5.50}$$

5.3.3 Steady-state temperature in a sphere

- Example: Find the steady-state temperature distribution $u(r,\theta,\phi)$ in a sphere of radius a, subject to the BC that the temperature on the bottom hemisphere is zero, while the temperature on the upper hemisphere is a constant $u = u_0$.

- Answer:

$$u(r,\theta,\phi) = u_0\left[\frac{1}{2}P_0(\cos\theta) + \frac{3}{4}\left(\frac{r}{a}\right)P_1(\cos\theta) - \frac{7}{16}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \cdots\right] \tag{5.51}$$

- Solution method:

Differential equation:

$$0 = \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \tag{5.52}$$

Separation of variables:

$$u(r,\theta,\phi) \equiv R(r)P(\theta)Q(\phi) \tag{5.53}$$

leads to

$$Q''(\phi) = -m^2\,Q(\phi) \tag{5.54}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P(\theta) = 0 \tag{5.55}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)\,R(r) \tag{5.56}$$

where l and m are (at this stage) arbitrary real separation constants.

Solutions of the ϕ-equation are

$$Q(\phi) = A_0 + B_0\phi\,, \qquad \text{for } m = 0 \tag{5.57}$$

$$Q(\phi) = A\,e^{im\phi} + B\,e^{-im\phi}\,, \qquad \text{for } m \neq 0 \tag{5.58}$$

Periodic BCs for $Q(\phi)$ imply:

$$Q(\phi) = C\,e^{im\phi}\,, \quad m = 0, \pm 1, \pm 2, \cdots \tag{5.59}$$

just as we saw for the cylinder case.

The θ-equation can be put into more standard form by making a change of variables from θ to $x = \cos\theta$ and then expanding the derivative:

$$(1-x^2)P''(x) - 2x\,P'(x) + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P(x) = 0 \tag{5.60}$$

This is just the associated Legendre's equation.

Recall that in order for the solutions $P_l^m(x)$ to be finite at the poles ($x = \pm 1$), the constants l and m must be restricted to:

$$l = 0, 1, 2, \cdots , \quad m = -l, -l+1, \cdots , l \tag{5.61}$$

Solutions to the radial equation:

$$R(r) = Dr^l + Er^{-(l+1)} \tag{5.62}$$

Finiteness of the solution at the center of the sphere ($r = 0$) implies $E = 0$, so

$$R(r) = Dr^l \tag{5.63}$$

General solution satisfying all of the BCs except the one at $r = a$:

$$u(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} E_{lm}\, r^l P_l^m(\cos\theta) e^{im\phi} \tag{5.64}$$

NOTE: Ignorning an overall normalization factor, the combination $P_l^m(\cos\theta)e^{im\phi}$ is a *spherical harmonic*, usually denoted by $Y_l^m(\theta, \phi)$.

Apply final BC at $r = a$:

$$u(a, \theta, \phi) = \left\{ \begin{array}{ll} u_0 & 0 \le \theta < \pi/2 \\ 0 & \pi/2 < \theta \le \pi \end{array} \right\} = \sum_{l=0}^{\infty} E_l\, a^l P_l(\cos\theta) \tag{5.65}$$

Note that $m = 0$ again since the BC is independent of ϕ.

Orthogonality of Legendre polynomials

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'} \tag{5.66}$$

implies

$$C_l \equiv E_l a^l = \frac{2l+1}{2} u_0 \int_0^1 dx\, P_l(x) \tag{5.67}$$

Evaluating the first few integrals using explicit expressions for the Legendre polynomials yields:

$$C_0 = \frac{1}{2}u_0\,, \quad C_1 = \frac{3}{4}u_0\,, \quad C_2 = 0\,, \quad C_3 = -\frac{7}{16}u_0\,, \cdot \tag{5.68}$$

Thus,

$$u(r, \theta, \phi) = u_0 \left[\frac{1}{2}P_0(\cos\theta) + \frac{3}{4}\left(\frac{r}{a}\right) P_1(\cos\theta) - \frac{7}{16}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \cdots \right] \tag{5.69}$$

5.4 Diffusion equation

- The diffusion equation is

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \tag{5.70}$$

  where α^2 is the diffusion constant, which is a property of the material, and u is a function of both spatial coordinates and time.

- In the following subsections, we will solve the diffusion equation in the context of: (i) heat flow through a rectangular slab of infinite extent in the y and z directions, and (ii) a quantum-mechanical "particle in a box".

5.4.1 Heat flow through a rectangular slab

- Example: Find the temperature distribution as a function of time $u(x,t)$ in a rectangular slab of thickness L and infinite in extent in the other directions, subject to the initial condition:

$$u(x,0) = \frac{u_0 x}{L}\,, \quad \text{for } 0 \le x \le L \tag{5.71}$$

and BCs:

$$u(0,t) = 0\,, \quad u(L,t) = 0\,, \quad \text{for } t > 0 \tag{5.72}$$

NOTE: By having a slab of infinite in extent in the y and z directions, we are able to neglect any heat flow in those directions. Equivalently, we could consider a rectangular slab of *finite* extent in the y and z directions, provided we insulate the faces having $y = \text{const}$ and $z = \text{const}$.

- Answer:

$$u(x,t) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2\alpha^2 t/L^2} \tag{5.73}$$

- Solution method:

Differential equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \tag{5.74}$$

Separation of variables:

$$u(x,t) = X(x)T(t) \tag{5.75}$$

leads to

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -k^2 \tag{5.76}$$

Solutions of the x-equation:

$$X(x) = A_0 + B_0\,x\,, \quad \text{for } k = 0 \tag{5.77}$$

$$X(x) = A\sin(kx) + B\cos(kx)\,, \quad \text{for } k \ne 0 \tag{5.78}$$

Solutions of the t-equation:

$$T(t) = Ce^{-k^2\alpha^2 t} \tag{5.79}$$

BCs at $x = 0$ and $x = L$ imply:

$$X(x) = A\sin(kx)\,, \quad \text{where } k \equiv \frac{n\pi}{L}\,, \quad n = 1, 2, \cdots \tag{5.80}$$

Note that there is no non-zero $k = 0$ solution which satisfies the BCs.

Substituting for k:

$$T(t) = Ce^{-n^2\pi^2\alpha^2 t/L^2} \tag{5.81}$$

General solution satisfying all of the BCs except the initial condition:

$$u(x,t) = \sum_n C_n \sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2\alpha^2 t/L^2} \tag{5.82}$$

Apply initial condition:

$$\frac{u_0 x}{L} = u(x,0) = \sum_n C_n \sin\left(\frac{n\pi x}{L}\right) \tag{5.83}$$

Orthogonality of sinusoids leads to:

$$C_n = \frac{2}{L}\int_0^L dx\,\frac{u_0 x}{L}\sin\left(\frac{n\pi x}{L}\right) = \frac{2u_0}{n\pi}(-1)^{n+1} \tag{5.84}$$

Thus,

$$u(x,t) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2\alpha^2 t/L^2} \tag{5.85}$$

5.4.2 Quantum-mechanical particle in a box

- Example: Find the allowed energies for a quantum mechanical particle of mass m in a 1-d box

$$V(x) = \begin{cases} 0 & \text{for } 0 \le x \le L \\ \infty & \text{otherwise} \end{cases} \tag{5.86}$$

- Answer: The allowed energies are quantized:

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}\,, \quad \text{where } n = 1, 2, \cdots \tag{5.87}$$

- Solution method:

Differential equation (Schrödinger's equation):

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\,\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{5.88}$$

For the specified form of the potential V, we need to solve

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial\Psi}{\partial t} \tag{5.89}$$

for $0 \le x \le L$, subject to the BCs that $\Psi(x,t)$ vanish at $x = 0$ and $x = L$ for all t.

Separation of variables:

$$\Psi(x,t) = X(x)T(t) \tag{5.90}$$

leads to

$$X'' = -k^2 X \tag{5.91}$$

$$T' = -\frac{iE}{\hbar}T\,, \quad \text{where } E \equiv \frac{\hbar^2k^2}{2m} \tag{5.92}$$

Solutions of the x-equation:

$$X(x) = A_0 + B_0\,x\,, \quad \text{for } k = 0 \tag{5.93}$$

$$X(x) = A\sin(kx) + B\cos(kx)\,, \quad \text{for } k \neq 0 \tag{5.94}$$

Solutions of the t-equation:

$$T(t) = Ce^{-iEt/\hbar} \tag{5.95}$$

BCs at $x = 0$ and $x = L$ imply:

$$X(x) = A\sin(kx)\,, \quad \text{where } k \equiv \frac{n\pi}{L}\,, \quad n = 1, 2, \cdots \tag{5.96}$$

Note that there is no non-zero $k = 0$ solution which satisfies the BCs.

Thus, the most general solution is

$$\Psi(x,t) = \sum_n C_n \sin\left(\frac{n\pi x}{L}\right) e^{-iE_nt/\hbar} \tag{5.97}$$

where

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \tag{5.98}$$

5.5 Wave equation

- The wave equation is

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{5.99}$$

where v is the velocity of the wave.

- It differs from the diffusion equation in that there is a *second* partial derivative wrt time, instead of a single partial derivative wrt time.

- In the following subsections, we will solve the wave equation in the context of: (i) a vibrating guitar string, and (ii) a vibrating circular drum head.

5.5.1 Vibrating guitar string

- Example: Solve for the motion of a guitar string, initially plucked in the middle and let go.

The guitar string is fixed at both ends:

$$u(0,t) = 0\,, \quad u(L,t) = 0 \tag{5.100}$$

The initial displacement of the string plucked in the middle is

$$u(x,0) = \begin{cases} 2hx/L\,, & 0 \le x \le L/2 \\ 2h\left(1 - x/L\right)\,, & L/2 \le x \le L \end{cases} \tag{5.101}$$

and has zero initial velocity

$$\frac{\partial u}{\partial t}(x,0) = 0 \tag{5.102}$$

- Answer:

$$u(x,t) = \frac{8h}{\pi^2}\left[\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi vt}{L}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi vt}{L}\right) + \frac{1}{25}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{5\pi vt}{L}\right) - \cdots\right] \tag{5.103}$$

- Solution method:

Differential equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{5.104}$$

Separation of variables:

$$u(x,t) \equiv X(x)T(t) \tag{5.105}$$

leads to:

$$\frac{X^{''}}{X} = \frac{1}{v^2}\frac{T^{''}}{T} = -k^2 \tag{5.106}$$

Solutions to the x-equation:

$$X(x) = A_0 + B_0\,x\,, \quad \text{for } k = 0 \tag{5.107}$$
$$X(x) = A\sin(kx) + B\cos(kx)\,, \quad \text{for } k \neq 0 \tag{5.108}$$

Solutions to the t-equation:

$$T(t) = C_0 + D_0\,t\,, \quad \text{for } k = 0 \tag{5.109}$$
$$T(t) = C\sin(kvt) + D\cos(kvt)\,, \quad \text{for } k \neq 0 \tag{5.110}$$

BCs at $x = 0$ and $x = L$ imply:

$$X(x) = A\sin(kx)\,, \quad \text{where } k \equiv \frac{n\pi}{L}\,, \quad n = 1, 2, \cdots \tag{5.111}$$

Note that there is no non-zero $k = 0$ solution which satisfies the BCs.

Zero initial velocity implies:

$$T(t) = D\cos\left(\frac{n\pi vt}{L}\right) \tag{5.112}$$

Thus, the most general solution satisfying all of the BCs except the initial condition on the displacement is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right) \tag{5.113}$$

which is a sum of *standing waves* with discrete wavelengths and frequencies

$$\lambda_n = \frac{2L}{n}\,, \quad f_n = \frac{v}{\lambda_n} = \frac{nv}{2L}\,, \quad \text{where } n = 1, 2, \cdots \tag{5.114}$$

The standing wave solutions for different values of n correspond to different *modes* of vibration of the guitar string. The vibrational frequency of the nth mode is simply $f_n = nf_1$, where $f_1 = v/2L$ is the fundamental frequency.

Note that each standing wave can be written as the sum of a right-moving and left-moving wave

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right) = \frac{1}{2}\left[\sin\left(k_n(x-vt)\right) + \sin\left(k_n(x+vt)\right)\right] \tag{5.115}$$

where $k_n = 2\pi/\lambda_n = n\pi/L$.

Apply the initial condition on the displacement:

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \tag{5.116}$$

Orthogonality of sinusoids leads to:

$$C_n = \frac{2}{L}\int_0^L dx\, u(x,0)\sin\left(\frac{n\pi x}{L}\right) = \begin{cases} \frac{8h}{n^2\pi^2}(-1)^{(n-1)/2} & n = 1, 3, 5, \cdots \\ 0 & n = \text{even} \end{cases} \tag{5.117}$$

Thus,

$$u(x,t) = \frac{8h}{\pi^2}\left[\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi vt}{L}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi vt}{L}\right) + \frac{1}{25}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{5\pi vt}{L}\right) - \cdots\right] \tag{5.118}$$

5.5.2 Vibrating drum head

- Example: Solve for the vibrational modes of a circular drum head of radius a, fixed at the circumference.
- Answer: The vibrational modes are given by

$$u_{mn}(\rho,\phi,t) = J_m\left(\frac{x_{mn}\rho}{a}\right)e^{im\phi}\left[A_{mn}\sin\left(\frac{x_{mn}vt}{a}\right) + B_{mn}\cos\left(\frac{x_{mn}vt}{a}\right)\right] \tag{5.119}$$

with frequency

$$f_{mn} = \frac{x_{mn}v}{2\pi a}\,, \quad \text{where } m = 0, 1, \cdots\,, \quad \text{and } n = 1, 2, \cdots \tag{5.120}$$

where x_{mn} is the nth zero of the mth Bessel function $J_m(x)$.

NOTE: The vibrational modes are labeled by two integers (m, n), where m labels the number of nodal diameters of the mode, and n labels the number of nodal circles. (There is always one nodal circle, located at the circumference of the drum head.)

- <u>Solution method</u>:

  Differential equation:

$$0 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{5.121}$$

  Separation of variables:

$$u(\rho,\phi,t) \equiv R(\rho)Q(\phi)T(t) \tag{5.122}$$

  leads to

$$T''(t) = -k^2v^2Z(t)\,, \tag{5.123}$$
$$Q''(\phi) = -\nu^2Q(\phi) \tag{5.124}$$
$$R''(\rho) + \frac{1}{\rho}R'(\rho) + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R(\rho) = 0 \tag{5.125}$$

  Solutions of the t-equation:

$$T(t) = A_0 + B_0\,t\,, \quad \text{for } k = 0 \tag{5.126}$$
$$T(t) = A\,\sin(kvt) + B\,\cos(kvt)\,, \quad \text{for } k \neq 0 \tag{5.127}$$

  In order to have oscillatory solutions for the vibrational modes, we must have $k \neq 0$.

  Solutions to the ϕ-equation:

$$Q(\phi) = C_0 + D_0\phi\,, \quad \text{for } \nu = 0 \tag{5.128}$$
$$Q(\phi) = C\sin(\nu\phi) + D\cos(\nu\phi)\,, \quad \text{for } \nu \neq 0 \tag{5.129}$$

  Periodic BCs for $Q(\phi)$ imply:

$$Q(\phi) = Ce^{im\phi}\,, \quad m = 0, \pm 1, \pm 2, \cdots \tag{5.130}$$

  just as we saw for the cylinder case.

  For $\nu = m$, the ρ-equation becomes:

$$R''(\rho) + \frac{1}{\rho}R'(\rho) + \left(k^2 - \frac{m^2}{\rho^2}\right)R(\rho) = 0 \tag{5.131}$$

  which is Bessel's equation of order m.

  Solutions to the ρ-equation:

$$R(\rho) = EJ_m(k\rho) + FN_m(k\rho)\,, \quad \text{for } k \neq 0 \tag{5.132}$$

  Finiteness of $R(\rho)$ at $\rho = 0$ implies $F = 0$.

  The BC that $u = 0$ at $\rho = a$ implies:

$$R(\rho) = EJ_m(k\rho)\,, \quad \text{where } k = \frac{x_{mn}}{a}\,, \quad n = 1, 2, \cdots \tag{5.133}$$

  where x_{mn} is the nth zero of $J_m(x)$.

  Thus, the most general solution satisfying all of the above BCs is:

$$u(\rho,\phi,t) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} E_{mn}u_{mn}(\rho,\phi,t) \tag{5.134}$$

  where

$$u_{mn}(\rho,\phi,t) \equiv J_m\left(\frac{x_{mn}\rho}{a}\right)e^{im\phi}\left[A_{mn}\sin\left(\frac{x_{mn}vt}{a}\right) + B_{mn}\cos\left(\frac{x_{mn}vt}{a}\right)\right] \tag{5.135}$$

Note that the frequency of the vibrational mode labeled by (m,n) is:

$$f_{mn} = \frac{x_{mn}v}{2\pi a}\,, \quad \text{where } m = 0, 1, \cdots\,, \quad \text{and } n = 1, 2, \cdots \tag{5.136}$$

where x_{mn} is the nth zero of the mth Bessel function $J_m(x)$.

The fundamental (i.e., lowest) frequency is

$$f_{01} = \frac{x_{01}v}{2\pi a} \tag{5.137}$$

The next five lowest frequencies, expressed as multiples of the fundamental frequency f_{01}, are

$$f_{11} = 1.5933 f_{01}\,, \quad f_{21} = 2.1355 f_{01}\,, \quad f_{02} = 2.2954 f_{01}\,, \quad f_{31} = 2.6531 f_{01}\,, \quad f_{12} = 2.9173 f_{01}\,, \tag{5.138}$$

Note that the frequencies of the vibrational modes are not harmonically related as we saw for the guitar string where $f_n = nf_1$, with $f_1 = v/2L$.

5.6 Poisson's equation

- Poisson's equation is

$$\nabla^2 u = f \tag{5.139}$$

  where $f = f(\mathbf{r})$ is a source term.

- In electrostatics, $f(\mathbf{r})$ is proportional to the charge density $\rho(\mathbf{r})$. In Newtonian gravity, $f(\mathbf{r})$ is proportional to the mass density $\mu(\mathbf{r})$.

- The most general solution to Poisson's equation is given by

$$u = u_c + u_p \tag{5.140}$$

  where u_c (the *complementary function*) is the general solution to the homogeneous equation $\nabla^2 u_c = 0$, and u_p (a *particular solution*) is *any* solution to the inhomogeneous equation.

- The constants of integration that appear in the complementary function will be determined by the boundary conditions.

5.6.1 Point source exterior to a grounded sphere

- Example: Determine the electrostatic potential $V(r,\theta,\phi)$ due to a point charge q exterior to a grounded sphere of radius R. Choose coordinates so that the center of the sphere is at the origin, and the point charge is located on the z-axis a distance a from the center of the sphere.

- Answer:

$$V(r,\theta,\phi) = \frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{qR/a}{\sqrt{r^2 + (R^2/a)^2 - 2r(R^2/a)\cos\theta}} \tag{5.141}$$

- Solution method:

  Differential equation:

$$\nabla^2 V = -4\pi\rho(\mathbf{r}) \quad \text{(in Gaussian units)} \tag{5.142}$$

  where $\rho(\mathbf{r}) = q\,\delta(\mathbf{r} - a\hat{\mathbf{z}})$.

  The general solution will be

$$V = V_p + V_c \tag{5.143}$$

  where V_p is a particular solution to the above equation, and V_c is the general solution to the homogeneous equation $\nabla^2 V_c = 0$ exterior to the sphere.

Particular solution:

$$V_p(r,\theta,\phi) = \frac{q}{|\mathbf{r} - a\hat{\mathbf{z}}|} = \frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} \tag{5.144}$$

To prove this claim we will show that

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\,\delta(\mathbf{r}) \tag{5.145}$$

Proof: For $r \neq 0$,

$$\nabla^2 \left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{-1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(-1) = 0 \tag{5.146}$$

To determine the behavior at $r = 0$, we integrate $\nabla^2(1/r)$ over a spherical volume of radius ϵ, and then take the limit $\epsilon \to 0$:

$$\int_{\text{sphere}} dV\, \nabla^2\left(\frac{1}{r}\right) = \oint_{\text{boundary}} da\,\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}\left(\frac{1}{r}\right) = \epsilon^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \left.\frac{-1}{r^2}\right|_{r=\epsilon} = -4\pi \tag{5.147}$$

where we used the divergence theorem to get the first equality, and $\hat{\mathbf{n}}\,da = \hat{\mathbf{r}}\,\epsilon^2 \sin\theta\, d\theta d\phi$ to get the second and third equalities. Note that this result is independent of ϵ, and thus holds for any spherical volume that includes the origin, no matter how small.

Thus, $\nabla^2(1/r) = 0$ for all $\mathbf{r} \neq 0$, but integrates to -4π for any volume that contains the origin $\mathbf{r} = 0$. This means that $\nabla^2(1/r) = -4\pi\,\delta(\mathbf{r})$ as claimed.

Complementary function:

$$V_c(r,\theta,\phi) = \sum_{l=0}^{\infty} C_l\, r^{-(l+1)} P_l(\cos\theta) \tag{5.148}$$

Notes:

1) This expansion can be taken almost directly from an earlier subsection where we considered the steady-state temperature distribution in a sphere.

2) Like that problem, there is no m-dependence in the above expansion since the problem is azimuthally symmetric (i.e., there is no dependence on ϕ).

3) Unlike that problem, the finite solution to the radial equation exterior to the sphere is

$$R(r) = Ar^{-(l+1)} \tag{5.149}$$

as opposed to $R(r) = Ar^l$, which was appropriate when solving Laplace's equation interior to a sphere.

Thus,

$$V(r,\theta,\phi) = \frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} + \sum_{l=0}^{\infty} C_l\, r^{-(l+1)} P_l(\cos\theta) \tag{5.150}$$

Apply BC that the sphere is grounded—i.e., $V(R,\theta,\phi) = 0$.

$$\begin{aligned}
0 &= \frac{q}{\sqrt{R^2 + a^2 - 2Ra\cos\theta}} + \sum_{l=0}^{\infty} C_l\, R^{-(l+1)} P_l(\cos\theta) && (5.151)\\
&= \frac{q}{a}\sum_{l=0}^{\infty}\left(\frac{R}{a}\right)^l P_l(\cos\theta) + \sum_{l=0}^{\infty} C_l\, R^{-(l+1)} P_l(\cos\theta) && (5.152)\\
&= \sum_{l=0}^{\infty}\left[\frac{q}{a}\left(\frac{R}{a}\right)^l + C_l\, R^{-(l+1)}\right] P_l(\cos\theta) && (5.153)
\end{aligned}$$

where we used the generating function for Legendre polynomials

$$\frac{1}{\sqrt{1-2xh+h^2}} = \sum_{l=0}^{\infty} h^l P_l(x) \tag{5.154}$$

to get the second equality.

Thus, setting the coefficients multiplying $P_l(\cos\theta)$ equal to zero implies

$$C_l = -q\frac{R^{2l+1}}{a^{l+1}} \tag{5.155}$$

so that

$$V(r,\theta,\phi) = \frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \left(\frac{qR}{a}\right)\sum_{l=0}^{\infty}\left(\frac{R^2}{a}\right)^l \frac{1}{r^{l+1}}P_l(\cos\theta) \tag{5.156}$$

$$= \frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{qR/a}{\sqrt{r^2+(R^2/a)^2-2r(R^2/a)\cos\theta}} \tag{5.157}$$

where we used the generating function for Legendre polynomials again to get the last line.

The interpretation of the second term is the potential due to an *image* point charge $q_I \equiv -qR/a$ located at $\mathbf{r}_I \equiv (R^2/a)\hat{\mathbf{z}}$. Note that the image charge is opposite in sign to the actual charge and is located *inside* the sphere, along the same line that connects the origin to the actual point charge.

Thus, the electrostatic potential exterior to a grounded sphere sphere due to a single point charge outside the sphere is equivalent to the electrostatic potential due to the original point charge *plus* an image point charge, without any grounded sphere present!

5.6.2 Green's function methods

- Suppose you want to solve an inhomogeneous equation, like Poisson's equation,

$$\nabla^2 u = f(\mathbf{r}) \tag{5.158}$$

but already know the solution $G(\mathbf{r},\mathbf{r}')$ to the same equation with a Dirac delta function source:

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') \tag{5.159}$$

- $G(\mathbf{r},\mathbf{r}')$ is called a *Green's function* of the differential equation. One thinks of the argument $\mathbf{r}$ as the *field point* and the argument $\mathbf{r}'$ as the *source point*.

- If the BCs are such that both $u(\mathbf{r})$ and $G(\mathbf{r},\mathbf{r}')$ vanish on the boundary, then the solution to the original equation is given by

$$u(\mathbf{r}) = \int dV'\, G(\mathbf{r},\mathbf{r}')f(\mathbf{r}') \tag{5.160}$$

- Proof:

$$\nabla^2 u(\mathbf{r}) = \int dV'\, (\nabla^2 G(\mathbf{r},\mathbf{r}'))f(\mathbf{r}') = \int dV'\, \delta(\mathbf{r}-\mathbf{r}')f(\mathbf{r}') = f(\mathbf{r}) \tag{5.161}$$

- If $G(\mathbf{r},\mathbf{r}')$ vanishes on the boundary but $u(\mathbf{r})$ does not, then the solution to the original equation is given by

$$u(\mathbf{r}) = \int_V dV'\, G(\mathbf{r},\mathbf{r}')f(\mathbf{r}') + \oint_S da'\, u(\mathbf{r}')\,\hat{\mathbf{n}}'\cdot\boldsymbol{\nabla}'G(\mathbf{r},\mathbf{r}') \tag{5.162}$$

where S is the boundary of V and $u(\mathbf{r}')$ are the values of u specified on the boundary. The vector $\hat{\mathbf{n}}'$ is the unit normal to the surface S that points *outward* from the volume V.

- Such a Green's function is called a *Dirichlet* Green's function.

- From the previous subsection, we see that:

  (i) the Green's function for Poisson's equation, which vanishes at infinity is

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \quad \text{(vanishes at infinity)} \tag{5.163}$$

  (ii) the Dirichlet Green's function for Poisson's equation exterior to a sphere of radius R is

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} + \frac{R/r'}{4\pi|\mathbf{r} - (R^2/r'^2)\mathbf{r}'|} \quad \text{(vanishes on surface of sphere)} \tag{5.164}$$

- NOTE: Recall that to obtain (ii), we let

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} + W(\mathbf{r}, \mathbf{r}') \tag{5.165}$$

  where $W(\mathbf{r}, \mathbf{r}')$ was the general solution to the homogeneous equation $\nabla^2 W = 0$, and then chose the arbitrary constants in the expression for W such that

$$G(\mathbf{r}, \mathbf{r}')\Big|_S = \left[-\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} + W(\mathbf{r}, \mathbf{r}')\right]\Bigg|_S = 0 \tag{5.166}$$

6 Complex analysis

6.1 Analytic functions

- A function $f(z)$ of a complex variable $z = x + iy$ can be written as

$$f(z) = u(x, y) + iv(x, y) \tag{6.1}$$

where $u(x, y)$ and $v(x, y)$ are real-valued functions of (x, y). f is a mapping from complex numbers $z = x + iy$ to complex numbers $w = u + iv$.

- Definition: The *derivative* of $f(z)$ is defined as

$$f'(z) \equiv \frac{df}{dz} \equiv \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{6.2}$$

The above definition is identical in form to that for a function of a real variable $f(x)$.

- Definition: A function $f(z)$ of a complex variable is *analytic* in a region of the complex plane if it has a unique derivative at every point of the region. (By unique we mean that the limit should be independent of how Δz approaches 0.)

- Theorem: $f(z) = u(x, y) + iv(x, y)$ is analytic in a region if and only if u, v and their partial derivatives with respect to x and y are continuous and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{6.3}$$

- Proof: Using the chain rule for $f(z) = f(x + iy)$:

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}\,, \qquad \frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y} = i\frac{df}{dz} \tag{6.4}$$

it follows that

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \tag{6.5}$$

In addition, since $f = u + iv$:

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\,, \qquad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \tag{6.6}$$

Thus,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \tag{6.7}$$

Equating the real and imaginary parts gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{6.8}$$

- Theorem: If $f = u + iv$ is analytic, then both u and v satisfy the 2-dimensional Laplace equation:

$$\nabla^2 u = 0\,, \qquad \nabla^2 v = 0 \tag{6.9}$$

Conversely, any function u (or v) that satisfies the 2-dimensional Laplace equation in a simply-connected region is the real (or imaginary) part of an analytic function $f(z) = u + iv$.

- Terminology: Any function that satisfies the 2-dimensional Laplace equation is said to be a *harmonic* function.

- Exercise: Show that $u(x,y) = x^2 - y^2$ satisfies the 2-dimensional Laplace equation, and then find an analytic function f for which u is the real part. (The corresponding u and v are called *conjugate* harmonic functions.)

- Answer: $v(x,y) = 2xy + C$, where C is a constant. The resulting function is $f(z) = z^2 + C$.

- Definitions:

  (i) z_0 is a *regular point* of $f(z)$ if f is analytic at z_0.

  (ii) z_0 is a *singular point* of $f(z)$ if f is not analytic at z_0.

  (iii) z_0 is an *isolated singular point* of $f(z)$ if f is analytic at every other point inside some small circle centered at z_0.

- Theorem: If $f(z)$ is analytic in a region, then it has derivatives of all orders at points inside the region, and it can be expanded in a Taylor series about any point z_0 inside the region. The Taylor series converges inside the circle about z_0 that extends to the nearest singular point of $f(z)$.

- The above theorem is a very powerful result. For functions of a real variable, being differentiable does not imply infinitely differentiable. An example is

$$f(x) = \begin{cases} 0 & x \le 0 \\ x^2 & x \ge 0 \end{cases} \tag{6.10}$$

  This function is continuous and once differentiable with

$$f'(x) = \begin{cases} 0 & x \le 0 \\ 2x & x \ge 0 \end{cases} \tag{6.11}$$

  But because $f'(x)$ has a kink at $x = 0$ it is not differentiable there (the derivative from the left is zero; the derivative from the right is two). Thus, the second derivative of $f(x)$ at $x = 0$ does not exist even though the first derivative did exist.

6.2 Contour integration

- Definition: A *simple closed curve* is a curve that does not intersect itself and has at most a finite number of kinks.

- Cauchy's theorem: If $f(z)$ is analytic inside and on a simple closed curve C, then

$$\oint_C f(z)\, dz = 0 \tag{6.12}$$

- Proof: (We will assume that $f'(z)$ is continuous, which is not necessary, but makes the proof easier.) We will also use Green's theorem in the plane

$$\oint_C P\, dx + Q\, dy = \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \tag{6.13}$$

  which is valid for continuous P, Q, and their partial derivatives, and which follows from Stokes' theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \int_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, da \tag{6.14}$$

  for

$$\mathbf{F}(x,y) = P(x,y)\, \hat{\mathbf{x}} + Q(x,y)\, \hat{\mathbf{y}}\,, \qquad d\mathbf{s} = dx\, \hat{\mathbf{x}} + dy\, \hat{\mathbf{y}} \tag{6.15}$$

Thus,

$$\oint_C f(z)\,dx = \oint_C (u+iv)(dx+idy) \tag{6.16}$$

$$= \oint_C [u\,dx - v\,dy] + i\oint_c [v\,dx + u\,dy] \tag{6.17}$$

$$= \int_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\int_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \tag{6.18}$$

$$= 0 \tag{6.19}$$

where the third equality follows from Green's theorem and the fourth equality follows from the Cauchy-Reimann equations, since $f(z)$ is analytic.

- Cauchy's integral formula: If $f(z)$ is analytic inside and on a simple closed curve C, then the value of $f(z)$ at a point $z=a$ inside C is given by

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \tag{6.20}$$

- Proof: Consider a simple closed curve Γ which consists of C (traversed counter-clockwise) and a small circle C' centered at a (traversed clockwise) connected by a straight line, which is traversed in both directions. Then inside and on Γ, the function $f(z)/(z-a)$ is analytic so

$$0 = \oint_\Gamma \frac{f(z)}{z-a}\,dz = \oint_{C,\mathrm{ccw}} \frac{f(z)}{z-a}\,dz + \oint_{C',\mathrm{cw}} \frac{f(z)}{z-a}\,dz = \oint_{C,\mathrm{ccw}} \frac{f(z)}{z-a}\,dz - \oint_{C',\mathrm{ccw}} \frac{f(z)}{z-a}\,dz \tag{6.21}$$

Thus

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C'} \frac{f(z)}{z-a}\,dz \tag{6.22}$$

where both C and C' are traversed counter-clockwise. But since C' is a small circle centered at a, we can write

$$z-a = \rho e^{i\phi}\,, \qquad dz = i\rho e^{i\phi}\,d\phi \tag{6.23}$$

for which

$$\oint_{C'} \frac{f(z)}{z-a}\,dz \approx f(a)\oint_{C'} \frac{dz}{z-a} = f(a)\int_0^{2\pi} \frac{i\rho e^{i\phi}}{\rho e^{i\phi}} d\phi = 2\pi i f(a) \tag{6.24}$$

Note that we used the fact that ρ can be arbitrarily small to approximate $f(z)$ by $f(a)$ around C'. Thus,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \tag{6.25}$$

which is the desired result.

- One often rewrites Cauchy's integral formula as

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw \tag{6.26}$$

which has the following interpretation: A function $f(z)$ which is analytic inside a region bounded by a simple closed curve C is completely determined by its values on the boundary C.

- Exercise: Show that

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n=1 \\ 0 & \text{for any other integral value of } n \end{cases} \tag{6.27}$$

for C a circle of radius ρ centered at z_0.

6.3 Laurent series

- Theorem: Let C_1, C_2 be two circles centered at z_0. Suppose $f(z)$ is analytic in the region between C_1 and C_2. Then we can expand $f(z)$ as a Laurent series about z_0:

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + b_1(z - z_0)^{-1} + b_2(z - z_0)^{-2} + \cdots \tag{6.28}$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}\,, \qquad b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}} \tag{6.29}$$

and C is any simple closed curve surrounding z_0 and lying in the region between C_1 and C_2.

- Proof: To derive the above expressions for a_n and b_n, divide $f(z)$ by the appropriate power of $(z - z_0)$ and integrate around C. Use the result of the exercise at the end of the previous section to evaluate the integrals.

- Notes:

(i) The part of the expansion involving the b terms is called the *principal part* of the Laurent series.

(ii) The inner circle C_1 might be a point and the outer circle C_2 might have infinite radius.

- Important series expansions:

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots\,, \quad |z| < 1 \tag{6.30}$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots\,, \quad |z| < 1 \tag{6.31}$$

- Exercise: Show that the function

$$f(z) = \frac{12}{z(2-z)(1+z)} \tag{6.32}$$

can be expanded in three different ways:

(i) For $0 < |z| < 1$:

$$f(z) = \frac{6}{z} - 3 + \frac{9}{2}z - \frac{15}{4}z^2 + \frac{33}{8}z^3 + \cdots \tag{6.33}$$

(ii) For $1 < |z| < 2$:

$$f(z) = 1 + \frac{z}{2} + \frac{z^2}{4} + \cdots + \frac{2}{z} + 4\left(\frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) \tag{6.34}$$

(iii) For $2 < |z|$:

$$f(z) = -\frac{12}{z^3}\left(1 + \frac{1}{z} + \frac{3}{z^2} + \frac{5}{z^3} + \cdots\right) \tag{6.35}$$

Hint: Use partial fractions to write

$$f(z) = \frac{12}{z(2-z)(1+z)} = \frac{4}{z}\left(\frac{1}{1+z} + \frac{1}{2-z}\right) \tag{6.36}$$

and then expand $1/(1+z)$ for $|z| < 1$ and $|z| > 1$, and $1/(2-z)$ for $|z| < 2$ and $|z| > 2$, etc.

- Definitions:

Let $f(z)$ be expanded in a Laurent series that converges around z_0.

(i) If all the b's are zero, then $f(z)$ is analytic at z_0 and z_0 is a *regular point* of $f(z)$.

(ii) If $b_n \neq 0$, but all the b's after b_n equal zero, then $f(z)$ is said to have a *pole of order* n at z_0. If $n = 1$, z_0 is called a *simple pole*.

(iii) If an infinite number of b's are non-zero, then $f(z)$ is said to have an *essential singularity* at $z = z_0$.

(iv) The coefficient b_1 of $1/(z - z_0)$ is called the *residue* of $f(z)$ at z_0, denoted $\mathrm{Res}(f, z_0)$.

- Example: The function
$$f(z) = e^z = 1 + z + \frac{1}{2!}z^2 + \cdots \tag{6.37}$$
is analytic at $z = 0$. The residue of e^z at $z = 0$ is 0.

- Example: The function
$$f(z) = \frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} + \cdots \tag{6.38}$$
has a pole of order 3 at $z = 0$. The residue of e^z/z^3 at $z = 0$ is $1/2!$.

- Example: The function
$$f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots \tag{6.39}$$
has an essential singularity at $z = 0$. The residue of e^z at $z = 0$ is 1.

6.4 Residue theorem

- Theorem: Let $f(z)$ be analytic inside and on some simple closed curve C except for possibly a finite number of isolated singularities. Then
$$\oint_C f(z)\,dz = 2\pi i \sum_i \text{ residues of } f(z) \text{ inside } C \tag{6.40}$$
where the summation is over all the singular points of f inside C.

- Proof: Consider a simple closed curve Γ consisting of C (traversed counter-clockwise) and small circles C_1, C_2, C_3, ... centered at the singularities z_1, z_2, z_3, ... (traversed clockwise) connected by cuts, and proceed as in the proof of Cauchy's integral formula. One finds
$$\oint_C f(z)\,dz = \sum_i \oint_{C_i} f(z)\,dz \tag{6.41}$$
where each of the curves are traversed in the counter-clockwise direction. For each of the integrals on the RHS, make a Laurent series expansion of $f(z)$ about the singularity z_i inside C_i, which leads to
$$\oint_{C_i} f(z)\,dz = 2\pi i b_1 = 2\pi i \cdot (\text{residue of } f(z) \text{ at } z_i) \tag{6.42}$$
Thus,
$$\oint_C f(z)\,dz = 2\pi i \sum_i \text{ residues of } f(z) \text{ inside } C \tag{6.43}$$
as claimed.

- Methods for finding residues of $f(z)$ at an isolated singular point z_0:

  (a) Laurent series: If one has a Laurent series expansion of $f(z)$ about z_0, then the residue of $f(z)$ at z_0 is simply the coefficient b_1 of the $1/(z - z_0)$ term in the expansion.

  (b) Simple pole: If z_0 is a simple pole of $f(z)$, then
$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z) \tag{6.44}$$
  The proof of this follows from the Laurent series expansion of $f(z)$ about z_0.

  (c) Pole of order n: If z_0 is a pole of order n of $f(z)$, then
$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right\} \tag{6.45}$$
  where m is any integer $m \geq n$. The proof of this follows again from the Laurent series expansion of $f(z)$ about z_0.

6.5 Evaluation of definite integrals

6.5.1 Integrals of trig functions

- Let $G(\sin\theta, \cos\theta)$ be any rational function of $\sin\theta$ and $\cos\theta$, whose denominator is never zero for any value of θ between 0 and 2π. Then the integral

$$I = \int_0^{2\pi} d\theta\, G(\sin\theta, \cos\theta) \tag{6.46}$$

can be cast as a contour integral around the unit circle by making the substitutions

$$z = e^{i\theta}\,, \quad dz = zi\, d\theta\,, \quad \cos\theta = \frac{1}{2}(z + z^{-1})\,, \quad \sin\theta = \frac{1}{2i}(z - z^{-1}) \tag{6.47}$$

and can then be evaluated using the residue theorem.

- Example:

$$I = \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta} = \frac{2\pi}{3} \tag{6.48}$$

6.5.2 Indefinite integrals

- Consider the rational function $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials with the degree of Q at least two greater than the degree of P, and assume that $Q(z)$ has no zeros on the real axis. Then the indefinite integral

$$I = \int_{-\infty}^{\infty} dx\, \frac{P(x)}{Q(x)} \tag{6.49}$$

can be evaluated as a contour integral around the closed boundary of a semi-circular region extending to infinity. The contribution to the integral from the semi-circular part of the contour is zero in the limit that the radius $\rho \to \infty$.

- Example:

$$I = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi \tag{6.50}$$

- Theorem: Suppose $P(x)$ and $Q(x)$ are polynomials with the degree of Q at least one greater than the degree of P. (Note that $Q(z)$ may have zeros on the real axis.) Then the above method can be used to calculate the indefinite integrals

$$I = \int_{-\infty}^{\infty} dx\, \frac{P(x)}{Q(x)} \cos(mx)\,, \qquad I = \int_{-\infty}^{\infty} dx\, \frac{P(x)}{Q(x)} \sin(mx) \tag{6.51}$$

provided one adds *half* of the residues of the simple poles on the real axis to the residues due to singularities inside the contour.

- Example:

$$\int_{-\infty}^{\infty} dx\, \frac{\cos x}{x} = 0\,, \qquad \int_{-\infty}^{\infty} dx\, \frac{\sin x}{x} = \pi \tag{6.52}$$

6.5.3 Principal value of an integral

- Suppose we want to evaluate the definite integral $\int_a^b dx\, F(x)$, but the integrand $F(x)$ is singular at $x = x_0 \in [a, b]$. We define the *principal value* of the integral to be

$$PV \int_a^b dx\, F(x) = \lim_{\epsilon \to 0} \left[\int_a^{x_0 - \epsilon} dx\, F(x) + \int_{x_0 - \epsilon}^b dx\, F(x) \right] \tag{6.53}$$

- NOTE: The calculation of the integral from the previous example

$$\int_{-\infty}^{\infty} dx\, \frac{\cos x}{x} = 0 \tag{6.54}$$

where we used a contour that avoided the singularity at $x = 0$ is an example of calculating the principal value of an integral.

- Example:

$$PV \int_0^5 \frac{dx}{x-3} = \ln(2/3)\,, \tag{6.55}$$

- Example:

$$PV \int_{-\infty}^{\infty} \frac{dx}{x^3} = 0 \tag{6.56}$$

6.5.4 Integrals of multivalued functions

- Multivalued functions like z^p can be integrated provided we chose a contour that stays within a single branch of the function. For example, integrals of certain real-valued functions from 0 to ∞ can be evaluated by chosing a circular contour in the complex z-plane with a branch cut along the positive x-axis that encircles the origin. [On the portion of the cut just above the positive x-axis, $\theta = 0$, so $z = re^{i0} = r$. Just below the positive x-axis, we have $\theta = 2\pi$, so $z = re^{i2\pi}$.]

- Example:

$$\int_0^{\infty} dx\, \frac{\sqrt{x}}{1+x^2} = \frac{\pi}{\sqrt{2}} \tag{6.57}$$

- Example:

$$\int_0^{\infty} dr\, \frac{r^{p-1}}{1+r} = \frac{\pi}{\sin \pi p}\,, \qquad 0 < p < 1 \tag{6.58}$$

6.5.5 Argument principle

- Consider a function $f(z)$ which has a finite number of poles inside some simple closed curve C and does not vanish anywhere on C. Then

$$\oint_c \frac{f'(z)\, dz}{f(z)} = 2\pi i(N-P) = i\Delta\Theta_C \tag{6.59}$$

where N (and P) are the total number of zeros (and poles) of $f(z)$, and $\Delta\Theta_C$ is the change in the angle of $f(z)$ around the contour. (Zeros (poles) of order n are counted as n zeros (poles).)

- Example: The function $f(z) = z^3 + 4z + 1$ has exactly one zero in the first quadrant. [HINT: To prove this, evaluate the change in the angle of $f(z)$ around the boundary of a quarter circle in the first quadrant, extending to infinity.]

6.5.6 Inverse Laplace transform

- Recall that the Laplace transform $F(p)$ of a function $f(t)$ is defined as:

$$F(p) = \int_0^{\infty} dt\, f(t)e^{-pt}\,, \qquad \mathrm{Re}(p) > k \tag{6.60}$$

- Theorem: The inverse Laplace transform of $F(p)$ is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dp\, F(p)e^{pt}\,, \quad t>0 \tag{6.61}$$

$$= \sum_i \text{residues of } F(p)e^{pt} \text{ at all poles} \tag{6.62}$$

where $c = \text{const} > k$. To obtain the last equality, we consider a contour that is the boundary of a semi-circular region to the left of the line $x = c$, in the limit that the radius of the semi-circle goes to infinity.

- Proof: To derive the above expression for the inverse Laplace transform, substitute $p = x + iy$ in the definition of the Laplace transform and compare the resulting expression with the Fourier transform of

$$\phi(t) \equiv \begin{cases} f(t)e^{-xt}\,, & t>0 \\ 0 & t<0 \end{cases} \tag{6.63}$$

Then take the inverse Fourier transform to solve for $\phi(t)$ and then $f(t)$.

- The above integral for the inverse Laplace transform is called the *Bromwich integral*.

- Example:

$$F(p) = \frac{5}{(p+2)(p^2+1)} \tag{6.64}$$

has inverse Laplace transform

$$f(t) = e^{-2t} + 2\sin t - \cos t\,, \qquad t>0 \tag{6.65}$$