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A rolling sphere on a tilted rotating turntable

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There are many experiments and demonstrations in physics in which the response of a system to a driving force is 'intuitively obvious'. By contrast there are very few simple experiments which may easily be set up in a school or first year undergraduate laboratory, in which the response of the system is contrary to almost everyone's expectations. Gyroscope experiments are one of the few which come into this category but in general such experiments are qualitative rather than quantitative. A far better example of this class of non-intuitively obvious experiments is that of the motion of a rolling ball on a rotating table. Not only is this experiment easy and cheap to assemble but it also lends itself to quantitative analysis. Furthermore the mathematical solution of the problem is itself an elegant demonstration of rigid body mechanics and thus the problem provides an ideal combination of applied mathematics, demonstration physics and experimentally observable physics for advanced level students.

Recently several articles have appeared in the *American Journal of Physics* (Weltner 1979, Burns 1981, Romer 1981) which called our attention to this problem. The vectorial mechanics solution is to be found in Milne (1948), a most elegant textbook on vectorial mechanics. However because we feel that this treatment is not readily understood by advanced level students we present a new treatment which leads to solutions which appear more readily interpretable. (For completeness we include the vectorial mechanics solution in an appendix.) Also we present some results from a simple test experiment.

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Uniform sphere rolling on a tilted rotating table

The configuration is shown in figure 1. The rotating plane is inclined so that the normal $\hat{\boldsymbol{n}}$ is at an angle α to the vertical. It is convenient to introduce x and y axes parallel to the plane of the table such that the y axis points up the line of greatest slope with the x axis perpendicular to it. The origin of the axes O is taken as the point in the plane which coincides with the axis of rotation.

The rolling sphere will experience a normal reaction through its centre as well as a frictional force in the plane of the table. Since the weight and normal reaction both act through the centre of mass of the body they do not cause it to rotate and the frictional force is the only source of torque about the centre of mass. Consequently the component of the angular velocity of the ball perpendicular to the plane is conserved and in the following calculation we need only consider the variations in the components of the angular velocity parallel to the plane. We will take $\boldsymbol{\Omega}$, the total angular velocity, to have components Ω_x and Ω_y along the x and y axes (and Ω_n perpendicular to the plane); similarly the frictional force $\boldsymbol{F}$ at any instant will have components F_x and F_y (figure 2).

The equations of motion of the centre of mass are

$$M\ddot{x} = F_x$$
$$M\ddot{y} = F_y - Mg \sin \alpha \qquad (1)$$

where M is the mass of the sphere, g is the acceleration due to gravity and α is the angle of tilt of the table to the horizontal. Likewise there are a pair of torque equations which determine the rotation about the centre of mass, namely

$$I\dot{\Omega}_x = aF_y$$
$$I\dot{\Omega}_y = -aF_x \qquad (2)$$

where a is the radius of the sphere and $I = \frac{2}{5}Ma^2$ is its moment of inertia about an axis through its centre.

To these equations we must add the requirement that the sphere rolls *without slipping*, which implies that the velocity of the point of contact of the sphere with the table is equal to the velocity of the table at that point. The latter has magnitude ωr perpendicular to OP, where r is the radial distance

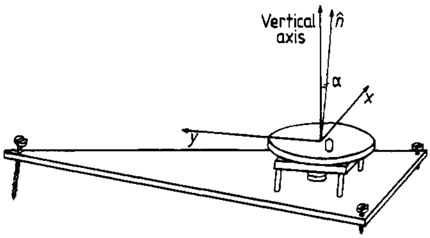

Figure 1 Experimental arrangement

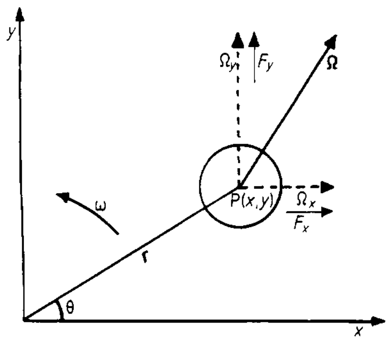

Figure 2 Coordinates for ball on a tilted table

from the axis of rotation to the centre of mass of the sphere (length OP) and ω is the constant angular frequency of rotation of the table.

Equating the x and y components of the velocities gives

$$\omega r \cos\theta = \omega x = \dot{y} + a\Omega_x$$
$$-\omega r \sin\theta = -\omega y = \dot{x} - a\Omega_y$$

Since equations (2) contain $\dot{\Omega}_x$ and $\dot{\Omega}_y$ it is convenient to differentiate the rolling conditions giving

$$\omega\dot{x} = \ddot{y} + a\dot{\Omega}_x$$
$$-\omega\dot{y} = \ddot{x} - a\dot{\Omega}_y \qquad (3)$$

Eliminating F_x, F_y, $\dot{\Omega}_x$ and $\dot{\Omega}_y$ between the three pairs of equations (1), (2) and (3) gives

$$[(Ma^2/I)+1]\ddot{x} = -\omega\dot{y}$$
$$[(Ma^2/I)+1]\ddot{y} = \omega\dot{x} - (Ma^2/I)\, g \sin\alpha$$

or

$$\ddot{x} = -\omega_r\dot{y}$$
$$\ddot{y} = \omega_r\dot{x} - g'\sin\alpha \qquad (4)$$

where

$$\omega_r = \frac{\omega I}{I+Ma^2} \qquad \text{and} \qquad g' = \frac{gMa^2}{I+Ma^2}$$

Integrating equations (4) and introducing the initial conditions that at time $t=0$ the centre of mass of the sphere is at the point (x_0, y_0) and is moving with velocity (v_x, v_y) gives

$$\dot{x} - v_x = -\omega_r(y-y_0)$$
$$\dot{y} - v_y = \omega_r(x-x_0) - g't\sin\alpha \qquad (5)$$

The substitution of equation (5) into equation (4) to eliminate $\dot{x}$ and $\dot{y}$ gives

$$\ddot{x} = -\omega_r v_y - \omega_r^2(x-x_0) + g'\omega_r t\sin\alpha$$
$$\ddot{y} = \omega_r v_x - \omega_r^2(y-y_0) - g'\sin\alpha \qquad (6)$$

These equations are now uncoupled in that they only contain either x or y or their respective derivatives. They may readily be solved by defining

$$X = x - x_0 + \frac{v_y}{\omega_r} - \frac{g't}{\omega_r}\sin\alpha$$
$$Y = y - y_0 - \frac{v_x}{\omega_r} + \frac{g'}{\omega_r^2}\sin\alpha$$

and noting that $\ddot{X} = \ddot{x}$, $\ddot{Y} = \ddot{y}$. Then equations (6) become

$$\ddot{X} = -\omega_r^2 X$$
$$\ddot{Y} = -\omega_r^2 Y$$

which are both equations of simple harmonic motion with solutions of the form

$$X = A\sin(\omega_r t + \phi)$$
$$Y = A'\sin(\omega_r t + \phi')$$

where A, ϕ and A', ϕ' are constants determined by the initial conditions.

Consequently

$$x = A\sin(\omega_r t+\phi) + x_0 - \frac{v_y}{\omega_r} + \frac{g't}{\omega_r}\sin\alpha$$
$$y = A'\sin(\omega_r t+\phi') + y_0 + \frac{v_x}{\omega_r} - \frac{g'}{\omega_r^2}\sin\alpha$$

and substitution of these expressions into the first equation of (4) gives

$$-A\omega_r^2\sin(\omega_r t+\phi) = -A'\omega_r^2\cos(\omega_r t+\phi')$$

which must be true for all values of t. This is only possible if $A = A'$ and $\phi' = \phi - \pi/2$, so that

$$x = A\sin(\omega_r t+\phi) + x_0 - \frac{v_y}{\omega_r} + \frac{g't}{\omega_r}\sin\alpha$$
$$y = -A\cos(\omega_r t+\phi) + y_0 + \frac{v_x}{\omega_r} - \frac{g'}{\omega_r^2}\sin\alpha$$

At $t=0$, $x = x_0$ so that

$$A\sin\phi = \frac{v_y}{\omega_r}$$

and $y = y_0$, giving

$$A\cos\phi = -\frac{g'}{\omega_r^2}\sin\alpha + v_x/\omega_r$$

Hence

$$x - x_0 = \frac{v_y}{\omega_r}(\cos\omega_r t - 1) - \left(\frac{g'}{\omega_r^2}\sin\alpha - \frac{v_x}{\omega_r}\right)$$

$$\times\sin\omega_r t + \frac{g' t}{\omega_r}\sin\alpha$$

and

$$y - y_0 = \left(\frac{g'}{\omega_r^2}\sin\alpha - \frac{v_x}{\omega_r}\right)(\cos\omega_r t - 1)$$

$$+\frac{v_y}{\omega_r}\sin\omega_r t \qquad (7)$$

These equations represent, in parametric form, the trajectory of the sphere for any initial conditions (x_0, y_0) and (v_x, v_y) and we now examine the motion of the sphere for various conditions.

Examples of types of motion

(a) *Horizontal table,* $\alpha = 0$. From equation (7)

$$x - x_0 + \frac{v_y}{\omega_r} = \frac{v_y}{\omega_r}\cos\omega_r t + \frac{v_x}{\omega_r}\sin\omega_r t$$

$$y - y_0 - \frac{v_x}{\omega_r} = -\frac{v_x}{\omega_r}\cos\omega_r t + \frac{v_y}{\omega_r}\sin\omega_r t \qquad (8)$$

and eliminating t by squaring and adding gives

$$\left(x - x_0 + \frac{v_y}{\omega_r}\right)^2 + \left(y - y_0 - \frac{v_x}{\omega_r}\right)^2 = \frac{v_x^2 + v_y^2}{\omega_r^2} = \frac{v_0^2}{\omega_r^2}$$

where v_0 is the starting speed of the ball. Thus the ball executes a circular orbit on the table of radius v_0/ω_r and centre $(x_0 - v_y/\omega_r,\ y_0 + v_x/\omega_r)$. Thus we have the remarkable *and non-obvious result that this rolling sphere describes perfect circles about a centre not coincident with the centre of rotation of the table and, regardless of starting conditions, its frequency around the orbit is* $\frac{2}{7}$ *of the frequency of rotation of the table.* As is discussed below, this result is not only extremely easy to demonstrate but also is quantitatively verifiable with simple equipment. Before discussing the experimental side it is worthwhile taking the problem a stage further to look at the behaviour of this same sphere when the table is tilted—in this circumstance the result is even less obvious than the previous case.

(b) *Table tilted, ball starts at rest from the origin.* Now the initial conditions are $x_0 = y_0 = v_x = v_y = 0$ so that

$$x = \frac{g'\sin\alpha}{\omega_r^2}(\omega_r t - \sin\omega_r t)$$

$$y = \frac{g'\sin\alpha}{\omega_r^2}(\cos\omega_r t - 1) \qquad (9)$$

Putting $g'\sin\alpha/\omega_r^2 = L$ and eliminating the sines and cosines by squaring and adding gives

$$(x - \omega_r t L)^2 + (y + L)^2 = L^2$$

This corresponds to motion in a circle of radius L with the centre drifting uniformly *in the x direction, across the table,* with velocity $\omega_r L = g'\sin\alpha/\omega_r = \frac{5}{2}(g/\omega)\sin\alpha$. Note that the radius of the circle is small and for a frequency of 1 Hz is only $\sim\sin\alpha$ cm. Thus for low angles of tilt as used below, this remarkable cycloidal motion across the table approximates closely to the even more extraordinary straight line drift across and not down the table. To achieve a perfect straight line drift one needs a slightly modified initial condition as discussed next.

(c) *Straight line motion on a tilted table.* Inspection of equation (7) shows that if we start the ball with $v_y = 0$ and $v_x = (g'/\omega_r)\sin\alpha$ then the resultant solution is

$$x = x_0 + (g' t/\omega_r)\sin\alpha$$

$$y = y_0$$

giving the surprising result that the ball moves horizontally across the tilted table with uniform velocity $(g'/\omega_r)\sin\alpha$. In practice it is difficult to establish this motion, while it is relatively simple to establish the condition (b) which leads to a very close approximation to (c) for small tilt of the table. In general, however, for any starting condition the motion of the ball is not so simple, as we discuss below.

(d) *General motion.* We may rearrange equation (7) to give

$$\left(x - x_0 + \frac{v_y}{\omega_r} - \omega_r L t\right) = \left[\frac{v_y}{\omega_r}\cos\omega_r t + \left(\frac{v_x}{\omega_r} - L\right)\sin\omega_r t\right]$$

$$\left(y - y_0 - \frac{v_x}{\omega_r} + L\right) = \left[\frac{v_y}{\omega_r}\sin\omega_r t - \left(\frac{v_x}{\omega_r} - L\right)\cos\omega_r t\right]$$

where squaring and adding yields

$$\left(x - x_0 + \frac{v_y}{\omega_r} - \omega_r L t\right)^2 + \left(y - y_0 - \frac{v_x}{\omega_r} + L\right)^2$$

$$= \left(\frac{v_y}{\omega_r}\right)^2 + \left(\frac{v_x}{\omega_r} - L\right)^2$$

This is readily interpreted as motion of the ball in a circle, again with frequency ω_r, of radius $\{(v_y/\omega_r)^2 + [(v_x/\omega_r) - L]^2\}^{1/2}$ with the centre moving horizontally across the table with velocity $\omega_r L$, i.e. the ball performs an epicycloidal motion across the table.

All of the above motions may be readily demonstrated and quantitative verification of the frequency of rotation of the ball and its drift speed as related

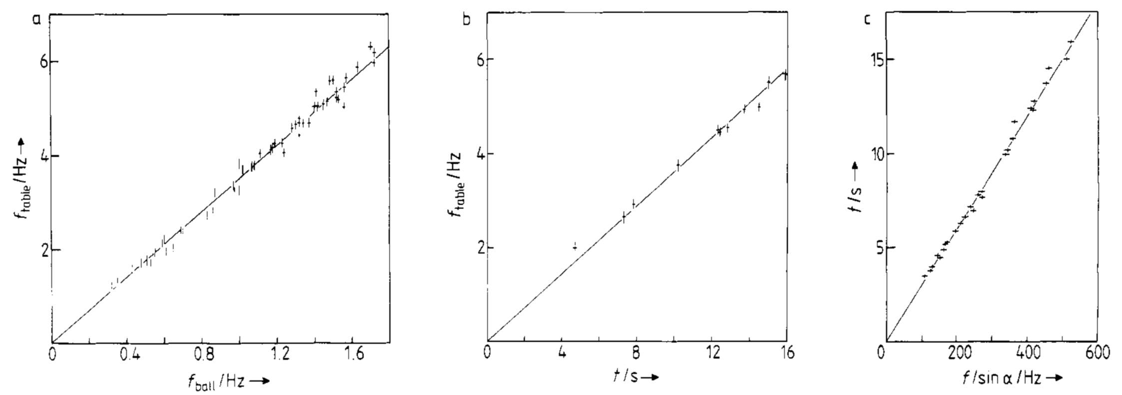

Figure 3 a, Graph showing frequency of rotation of horizontal table against frequency of rotation of ball about its orbit. The full line is $f_{table} = \frac{7}{2} f_{ball}$. **b,** Graph showing frequency of rotation of table against time for ball to travel one radius of the table ($r = 0.1175$ m) for fixed angle of tilt of the table, 0.622°. The full line is the theoretical prediction $f = t[(5g/4\pi r)\sin\alpha]$. **c,** Graph showing time for ball to travel one radius of the table ($r = 0.1175$ m) against the frequency of rotation of the table divided by the sine of the angle of tilt. In this case the frequency of the table is kept nearly constant at about 4 Hz while the angle of tilt is varied. The full line is the theoretical prediction $t = (f/\sin\alpha)(4\pi r/5g)$

to the frequency and angle of tilt of the table may easily be achieved as shown below.

Basic equipment

The essential requirement for this experiment is a flat *smooth* circular table. In order to establish the rolling condition it is not necessary to have a surface with a high coefficient of friction—in fact a rough surface leads to 'bouncing' and violation of the rolling condition. Initially we used a 23 cm diameter polished aluminium plate which was superseded by a 38 cm diameter plastic-laminated glass sheet. This was mounted on a central vertical bearing with some means of driving it at various frequencies of rotation. In our experiment we used a simple voltage controlled variable speed DC motor. The bearing on which the plate is mounted is rigidly attached to one end of a long base plate which has two adjusting screws at the rear end and one at the far end (figure 2). These screws are for adjusting horizontality of the table, the single screw being used to adjust the angle of tilt.

The ball we used, as in Romer's experiment (Romer 1981), is a 1 in (2.5 cm) diameter steel ball bearing; other steel ball bearings were used but this proved to be the most successful.

Horizontal table

Initially the table is set horizontal with a certain amount of care and the verification of equation (8) is attempted. We found that rotation frequencies above about 6 Hz were not very satisfactory as there was then a much greater tendency for the ball to fly off the table rather quickly (due to imperfect rolling contact with the table). To start the ball in motion we used a paper tube just a little bigger than the ball. Placing the ball anywhere on the rotating table inside this tube (held vertically) and then releasing pressure from the ball allowed it to come into rolling equilibrium with the table. After a short while, typically a few seconds, the tube can be carefully removed and the ball is seen to remain with its centre stationary, rolling in contact with the table. If now a slight motion is imparted to the ball by tapping gently with the paper tube it will be seen to perform circles as described by equation (8). It is then a simple matter to time the rate of rotation of the ball around the circle. After several orbits the circle will be seen to be increasing in size (due to non-perfect rolling) and eventually the ball will fall off the edge of the table.

This occurrence is made more frequent if the orbit of the ball takes it very near to the centre of the table. Varying the frequency of the table and repeating this experiment confirms that $\omega_r = \frac{2}{7}\omega$. Typical results of such an experiment are shown in figure 3a. These data were recorded using a stopwatch to time the ball while a stroboscope was used to monitor the frequency of the table. It is apparent from the results that agreement between theory and experiment is good and such agreement, or better, should easily be achieved. Notice that below rates of rotation of 1 Hz and above about 6 Hz we found that it was very difficult to obtain data.

Tilted table

If the table is tilted at a *small* angle and the ball is set in motion as in the horizontal table experiment it is easy to verify the extraordinary cycloidal motion of the ball *across* and not *down* the table. It is also a simple matter to verify quantitatively expression (7). We found it best to start with the ball at the very centre of the table, condition (b), and to measure s, its speed of horizontal motion, by timing how long it took to fall off the edge of the table. This we did at an approximately fixed frequency, varying α, and also at fixed α, varying frequency. Again, for frequencies above 6 Hz, problems arose because of non-perfect rolling. Typical results for both types of experiment are presented in figures 3b and 3c. Figure 3b presents data for a fixed angle of tilt, 0.622°; it shows the somewhat surprising result that as the frequency of rotation of the table is raised it takes longer for the ball to drift across the radius. In figure 3c the data are for an approximately fixed table frequency of ~4 Hz with various angles of tilt. In both these figures the full line is the theoretical prediction, showing very close agreement between theory and experiment.

Conclusions

The ease with which this experiment may be performed, the elegance of the mechanics describing the motion of the ball, the most remarkable non-obviousness of the behaviour and the straightforward manner in which quantitative results can be obtained make this a very suitable experiment for advanced A-level students and first year undergraduates in physics. Also the experiment, as a demonstration, provides a very simple illustration of how the intuitively obvious answer—in this case the ball 'falling' off the tilted table—is not the correct one.

Acknowledgments

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Appendix: Vectorial theory

Sphere rolling on a horizontal rotating table. Consider a solid sphere, radius a and mass M, rolling in perfect rolling contact on a horizontal table which is rotating at a constant angular frequency ω ($2\pi\times$ frequency) about a vertical axis (figure 4). Let $\hat{\boldsymbol{z}}$ be a unit vector in the upward direction and $\boldsymbol{r}$ the horizontal vector distance of the centre of the rolling sphere from the axis of rotation. The angular velocity of the rotating plane is $\omega\hat{\boldsymbol{z}}$ and the angular velocity of the sphere, about its centre, is undefined; we label this $\boldsymbol{\Omega}$. There will be a reaction force $\boldsymbol{R}$ at the point of contact of the sphere with the table. We can now write down the two equations of motion for the sphere.

Firstly, we have an equation involving linear momentum

$$M(d^2\boldsymbol{r}/dt^2) = \boldsymbol{R} - Mg\hat{\boldsymbol{z}} = M\ddot{\boldsymbol{r}} \qquad \text{(A1)}$$

Next there is an equivalent expression for the rate of change of angular momentum

$$I(d\boldsymbol{\Omega}/dt) = -a\hat{\boldsymbol{z}}\times\boldsymbol{R} = I\dot{\boldsymbol{\Omega}} \qquad \text{(A2)}$$

where I is the moment of inertia of the sphere about its centre. Substituting equation (A1) into (A2) gives

$$I\dot{\boldsymbol{\Omega}} = -a\hat{\boldsymbol{z}}\times(M\ddot{\boldsymbol{r}} + Mg\hat{\boldsymbol{z}})$$

or

$$\dot{\boldsymbol{\Omega}} = (Ma/I)\ddot{\boldsymbol{r}}\times\boldsymbol{z} \qquad \text{(A3)}$$

Now we add in the final constraint that the sphere is in perfect rolling contact with the table. This constraint is expressed mathematically by equating the velocity of the point of contact between the sphere and the table (the point at $\boldsymbol{r} - a\hat{\boldsymbol{z}}$) with the velocity of the sphere at that point, i.e.

$$\omega\hat{\boldsymbol{z}}\times(\boldsymbol{r} - a\hat{\boldsymbol{z}}) = \dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times(-a\boldsymbol{z})$$

$\dot{\boldsymbol{r}}$ is just the velocity of the centre of the sphere and $\boldsymbol{\Omega}\times(-a\hat{\boldsymbol{z}})$ is that velocity which has to be added to $\dot{\boldsymbol{r}}$ to give the velocity of the point in contact with the table. This simplifies to

$$\omega\hat{\boldsymbol{z}}\times\boldsymbol{r} = \dot{\boldsymbol{r}} - a\boldsymbol{\Omega}\times\hat{\boldsymbol{z}}$$

Differentiating this with respect to time gives

$$\omega\hat{\boldsymbol{z}}\times\dot{\boldsymbol{r}} = \ddot{\boldsymbol{r}} - a(\dot{\boldsymbol{\Omega}}\times\hat{\boldsymbol{z}}) \qquad \text{(A4)}$$

Substituting from equation (A3) into (A4) for $\dot{\boldsymbol{\Omega}}$ we have

$$\omega\hat{\boldsymbol{z}}\times\dot{\boldsymbol{r}} = \ddot{\boldsymbol{r}} - (Ma^2/I)(\ddot{\boldsymbol{r}}\times\hat{\boldsymbol{z}})\times\hat{\boldsymbol{z}}$$

or

$$\omega\hat{\boldsymbol{z}}\times\dot{\boldsymbol{r}} = [1 + (Ma^2/I)]\ddot{\boldsymbol{r}}$$

This may now be integrated with respect to time giving

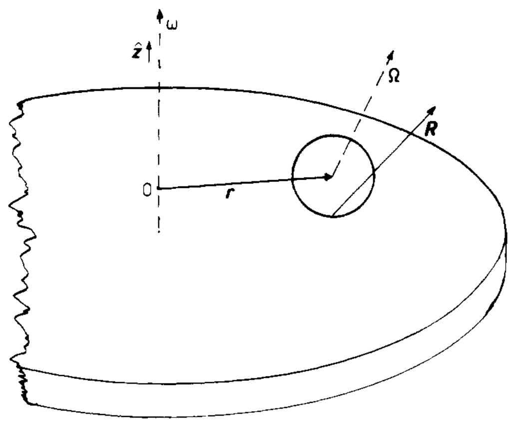
$$\omega\hat{\boldsymbol{z}}\times(\boldsymbol{r} - \boldsymbol{r}_0) = [1 + (Ma^2/I)]\dot{\boldsymbol{r}} \qquad \text{(A5)}$$

where $\boldsymbol{r}_0$ is an arbitrary constant of integration which is defined by the starting condition imposed upon the sphere. This equation is readily interpreted. It means that the centre of the sphere defined by $\boldsymbol{r}$ describes a circle of angular frequency

$$\frac{\omega}{1 + (Ma^2/I)} = \frac{2}{7}\omega$$

because $I = \frac{2}{5}Ma^2$.

Since $\boldsymbol{r}_0$ is defined by the starting conditions, the



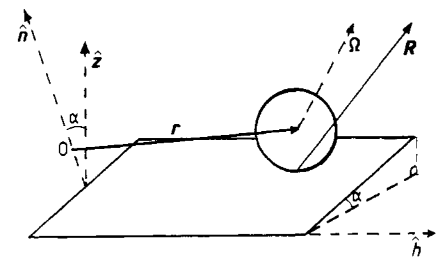


Figure 5 Coordinates for a ball on a tilted table

Figure 4 Coordinates for a ball on a horizontal table

circle has an arbitrary centre, not necessarily the centre of the table (note $\boldsymbol{r}_0$ does not move round with the table). Moreover because $\boldsymbol{r}$ appears on both sides of the equation, the radius of the circle described by the centre of the rolling sphere is also arbitrary being defined by the starting velocity. Furthermore, regardless of ω, the size of the sphere or the distance from the centre of the table, the frequency of rotation of the sphere around its circle is always $\frac{2}{7}$ times the frequency of rotation of the table.

Sphere rolling on a tilted rotating table. Let α be the angle of inclination of the plane to the horizontal (figure 5). Take an origin O which is on the axis of rotation of the table and at a distance a up the axis. This axis is now in a direction $\hat{\boldsymbol{n}}$ which, being the normal to the table, is at an angle α to the $\hat{\boldsymbol{z}}$ axis. $\boldsymbol{r}$ is now a vector from the origin O to the centre of the sphere. We can once more write down the equation involving rate of change of linear momentum to give

$$M\ddot{\boldsymbol{r}} = \boldsymbol{R} - Mg\hat{\boldsymbol{z}}$$

while the equivalent equation for angular momentum is

$$I\dot{\boldsymbol{\Omega}} = -a\hat{\boldsymbol{n}} \times \boldsymbol{R} = a\boldsymbol{R} \times \hat{\boldsymbol{n}}$$

We may again eliminate $\boldsymbol{R}$ to obtain

$$\dot{\boldsymbol{\Omega}} = (aM/I)(\ddot{\boldsymbol{r}} + g\hat{\boldsymbol{z}}) \times \hat{\boldsymbol{n}} \qquad \text{(A6)}$$

Once more we introduce the equation of rolling contact, i.e.

$$\omega\hat{\boldsymbol{n}} \times (\boldsymbol{r} - a\hat{\boldsymbol{n}}) = \dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times (-a\hat{\boldsymbol{n}})$$

or

$$\omega\hat{\boldsymbol{n}} \times \boldsymbol{r} = \dot{\boldsymbol{r}} - a\boldsymbol{\Omega} \times \hat{\boldsymbol{n}}$$

Differentiating with respect to time gives

$$a\dot{\boldsymbol{\Omega}} \times \hat{\boldsymbol{n}} = -\omega\hat{\boldsymbol{n}} \times \dot{\boldsymbol{r}} + \ddot{\boldsymbol{r}}$$

Then substituting from equation (A6) for $\dot{\boldsymbol{\Omega}}$ we have

$$(a^2M/I)[(\ddot{\boldsymbol{r}} + g\hat{\boldsymbol{z}}) \times \hat{\boldsymbol{n}}] \times \hat{\boldsymbol{n}} = -\omega\hat{\boldsymbol{n}} \times \dot{\boldsymbol{r}} + \ddot{\boldsymbol{r}}$$

or because $\boldsymbol{r}$ is perpendicular to $\hat{\boldsymbol{n}}$

$$[1 + (Ma^2/I)]\ddot{\boldsymbol{r}} - (Ma^2/I)g(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}) \times \hat{\boldsymbol{n}} = \omega\hat{\boldsymbol{n}} \times \dot{\boldsymbol{r}}$$

but $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}} = -\hat{\boldsymbol{h}} \sin\alpha$ where $\hat{\boldsymbol{h}}$ is a horizontal unit vector perpendicular to $\hat{\boldsymbol{z}}$ and at right angles to the maximum slope of the plane. Integrating with respect to time gives

$$[1 + (Ma^2/I)]\dot{\boldsymbol{r}} + (Ma^2/I)gt \sin\alpha\, \hat{\boldsymbol{h}} \times \hat{\boldsymbol{n}} = \omega\hat{\boldsymbol{n}} \times (\boldsymbol{r} - \boldsymbol{r}_0)$$

where $\boldsymbol{r}_0$ is a constant of integration. Therefore

$$\dot{\boldsymbol{r}} = \frac{\omega}{1 + (Ma^2/I)}\, \hat{\boldsymbol{n}} \times [(\boldsymbol{r} - \boldsymbol{r}_0) + (Ma^2/\omega I)gt \sin\alpha\hat{\boldsymbol{h}}] \qquad \text{(A7)}$$

This is a very similar result to equation (A5) except that now superimposed on the circular motion of the ball is a uniform velocity in the direction $\hat{\boldsymbol{h}}$ with a drift speed s, given by

$$s = (Ma^2/\omega I)g \sin\alpha = \tfrac{5}{2}(g/\omega)\sin\alpha \qquad \text{(A8)}$$

Thus the motion of the sphere is now an epicycloid which moves *across* the tilt of the table and not *down*, a certainly non-intuitive solution.