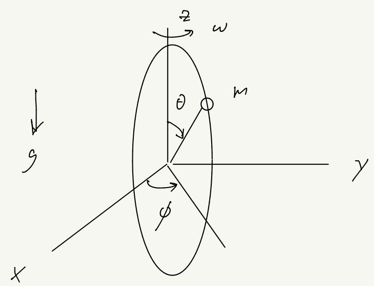
Rotating hoop:



$$\phi = \omega t \quad (\text{specified}) \;\rightarrow\; \dot\phi = \omega$$

$$r = R \;\rightarrow\; \dot r = 0$$

$$T = \frac{1}{2} m(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2)$$

$$= \frac{1}{2} m(R^2\dot\theta^2 + R^2\omega^2\sin^2\theta)$$

$$U = mgz$$

$$= mgR\cos\theta$$

$$L = \frac{1}{2} m(R^2\dot\theta^2 + R^2\omega^2\sin^2\theta) - mgR\cos\theta$$

no explicit t dependence $\rightarrow$ $h = \frac{\partial L}{\partial\dot\theta}\dot\theta - L$

$= \text{const}$

$$\frac{\partial L}{\partial\dot\theta} = mR^2\dot\theta$$

$$\rightarrow\; h = mR^2\dot\theta^2 - \frac{1}{2} mR^2\dot\theta^2 - \frac{1}{2} mR^2\omega^2\sin^2\theta + mgR\cos\theta$$

$$= \frac{1}{2} mR^2\dot\theta^2 - \frac{1}{2} mR^2\omega^2\sin^2\theta + mgR\cos\theta$$

Note: $h = \text{const}$ but $h \neq T + U \equiv E$

Determining constraint force:

$$\varphi_1 \equiv r - R = 0$$

$$\varphi_2 \equiv \phi - \omega t = 0$$

$$\vec F_c = \lambda_1 \vec\nabla\varphi_1 + \lambda_2 \vec\nabla\varphi_2$$

$$= \lambda_1 \hat r + \lambda_2 \frac{1}{r\sin\theta}\hat\phi$$

$$T = \frac{1}{2} m(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2)$$

$$U = mgr\cos\theta$$

$$L = T - U$$

1) $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot r}\right) = \frac{\partial L}{\partial r} + \lambda_1\frac{\partial\varphi_1}{\partial r} + \lambda_2\frac{\partial\varphi_2}{\partial r}$

2) $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = \frac{\partial L}{\partial\theta} + \lambda_1\frac{\partial\varphi_1}{\partial\theta} + \lambda_2\frac{\partial\varphi_2}{\partial\theta}$

3) $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right) = \frac{\partial L}{\partial\phi} + \lambda_1\frac{\partial\varphi_1}{\partial\phi} + \lambda_2\frac{\partial\varphi_2}{\partial\phi}$

4) $r - R = 0 \rightarrow r = R \rightarrow \dot r = 0,\ \ddot r = 0$

5) $\phi - \omega t = 0 \rightarrow \phi = \omega t \rightarrow \dot\phi = \omega,\ \ddot\phi = 0$

$$\frac{d}{dt}(m\dot r) = mr\dot\theta^2 + mr\sin^2\theta\,\dot\phi^2 - mg\cos\theta + \lambda_1$$

$$\rightarrow \boxed{\ddot r = r\dot\theta^2 + r\sin^2\theta\,\dot\phi^2 - g\cos\theta + \frac{\lambda_1}{m}}$$

$$\frac{d}{dt}(mr^2\dot\theta) = mr^2\sin\theta\cos\theta\,\dot\phi^2 + mg\sin\theta$$

$$2mr\dot r\dot\theta + mr^2\ddot\theta = mr^2\sin\theta\cos\theta\,\dot\phi^2 + mg\sin\theta$$

$$\boxed{2r\dot r\dot\theta + r^2\ddot\theta = r^2\sin\theta\cos\theta\,\dot\phi^2 + g\sin\theta}$$

$$\frac{d}{dt}(mr^2\sin^2\theta\,\dot\phi) = +\lambda_2$$

$$\boxed{2mr\dot r\sin^2\theta\,\dot\phi + 2mr^2\sin\theta\cos\theta\,\dot\theta\dot\phi + mr^2\sin^2\theta\,\ddot\phi = \lambda_2}$$

Use $\dot r = 0$, $\ddot r = 0$, $r = R$, $\phi = \omega t$, $\dot\phi = \omega$, $\ddot\phi = 0$

$$\rightarrow \quad 0 = R\dot\theta^2 + R\sin^2\theta\,\omega^2 - g\cos\theta + \frac{\lambda_1}{R}$$

$$0 + R^2\ddot\theta = R^2\omega^2\sin\theta\cos\theta + g\sin\theta$$

$$0 + 2mR^2\sin\theta\cos\theta\,\dot\theta\,\omega + 0 = \lambda_2$$

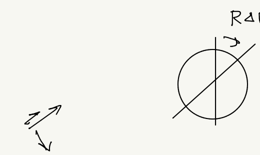
$$\rightarrow \boxed{\lambda_2 = 2mR^2\omega\sin\theta\cos\theta\,\dot\theta}$$

$$\boxed{\lambda_1 = -mR^2\dot\theta^2 - mR^2\sin^2\theta\,\omega^2 + mg\cos\theta}$$

$$\vec F_c = \lambda_1\hat r + \lambda_2\frac{1}{R\sin\theta}\hat\phi$$

$$= (-mR^2\dot\theta^2 - mR^2\sin^2\theta\,\omega^2 + mg\cos\theta)\,\hat r + 2mR\omega\dot\theta\cos\theta\,\hat\phi$$

Virtual displacement: (constant time



$$\delta\vec r = R\,\delta\theta\,\hat\theta$$

$$\rightarrow \vec F_c\cdot\delta\hat r = 0$$

Actual displacement:

$$\delta\vec r = \underset{0}{\cancel{\delta r}}\,\hat r + R\,\delta\theta\,\hat\theta + R\sin\theta\,\delta\phi\,\hat\phi$$

$$= R\dot\theta\,\delta t\,\hat\theta + R\sin\theta\,\omega\,\delta t\,\hat\phi$$

$$= \delta t\left(\dot\theta\hat\theta + R\omega\sin\theta\,\hat\phi\right)$$

$$\vec F_c\cdot\delta\vec r = \delta t\,2mR\omega\dot\theta\cos\theta\,\omega\sin\theta$$

$$= \delta t\,2mR^2\omega^2\dot\theta\sin\theta\cos\theta$$

$$= \delta\left[mR^2\omega^2\sin^2\theta\right]$$

Thus,

$$\vec F_c = -\frac{\partial U_c}{\partial\vec r}, \quad U_c = -mR^2\omega^2\sin^2\theta$$

$$W_c = \Delta T + \Delta U = \Delta E$$

$$E = \frac{1}{2} m \left(R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2\theta \right) + mgR\cos\theta$$

$$\Delta E = \frac{1}{2} m \left(R^2 (\dot{\theta}_2^2 - \dot{\theta}_1^2) + R^2 \omega^2 (\sin^2\theta_2 - \sin^2\theta_1) \right)$$

$$+ mgR(\cos\theta_2 - \cos\theta_1)$$

$$W_c = mR^2\omega^2(\sin^2\theta_2 - \sin^2\theta_1)$$

$$= -\Delta U_c$$

Thus, $\boxed{0 = \Delta T + \Delta U + \Delta U_c = \Delta h}$

where $h = T + U + U_c$

$$= \frac{1}{2} m \left(R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2\theta \right)$$

$$+ mgR\cos\theta - mR^2\omega^2\sin^2\theta$$

$$= \frac{1}{2} m \left(R^2 \dot{\theta}^2 - R^2 \omega^2 \sin^2\theta \right) + mgR\cos\theta$$