(9.1)

Exercise: 1-d wave, Gaussian initial displacement

$$y(x,0) \equiv f(x) = A e^{-\frac{x^2}{2\sigma_x^2}}$$

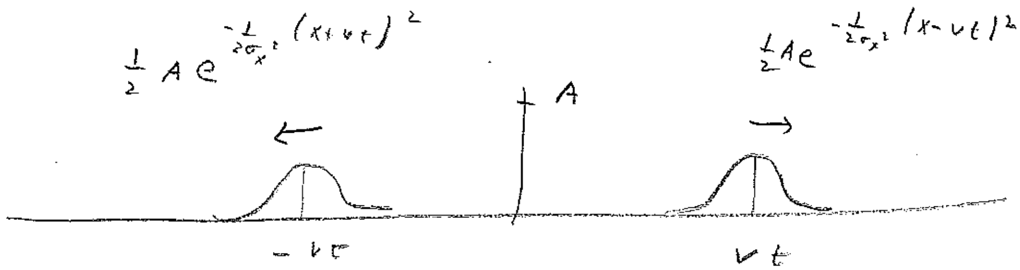
$$\dot{y}(x,0) \equiv g(x) = 0$$

Now:
$$\Psi(x) = \frac{1}{2} f(x) - \frac{1}{2v} \cancel{\int g(x)\,dx}^{\,0}$$
$$= \frac{1}{2} A e^{-\frac{x^2}{2\sigma_x^2}}$$

$$\Phi(x) = \frac{1}{2} f(x) + \frac{1}{2x} \cancel{\int g(x)\,dx}^{\,0}$$
$$= \frac{1}{2} A e^{-\frac{x^2}{2\sigma_x^2}}$$

Thus,
$$y(x,t) = \Psi(x-vt) + \Phi(x+vt)$$
$$= \frac{1}{2} A e^{\frac{-(x-vt)^2}{2\sigma_x^2}} + \frac{1}{2} A e^{-\frac{(x+vt)^2}{2\sigma_x^2}}$$
$$= \frac{1}{2} A \left[e^{-\frac{1}{2\sigma_x^2}(x-vt)^2} + e^{-\frac{1}{2\sigma_x^2}(x+vt)^2} \right]$$

$t = 0$



t

9.2

Exercise: 1-d wave, Gaussian initial velocity

$$y(x,0) \equiv f(x) = 0$$

$$\dot{y}(x,0) \equiv g(x) = A\, e^{-\frac{x^2}{2\sigma_v^2}}$$

Now:

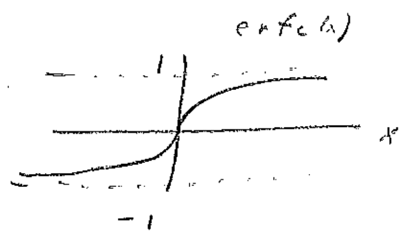
$$\Psi(x) = \cancel{\tfrac{1}{2} f(x)}^{\,0} - \frac{1}{2v}\int g(x)\,dx$$

$$= -\frac{1}{2v} A \int e^{-\frac{x^2}{2\sigma_v^2}}\,dx$$

$$= -\frac{A}{2v}\int_0^x d\bar{x}\; e^{-\frac{\bar{x}^2}{2\sigma_v^2}}$$

NOTE:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$$



Thus,

$$\Psi(x) = -\frac{A}{2v}\int_0^x d\bar{x}\; e^{-\frac{\bar{x}^2}{2\sigma_v^2}} \qquad \text{let} \quad t = \frac{\bar{x}}{\sqrt{2}\,\sigma_v}$$

$$dt = \frac{d\bar{x}}{\sqrt{2}\,\sigma_v}$$

$$= -\frac{A}{2v}\int_0^{\frac{x}{\sqrt{2}\sigma_v}} dt\,\sqrt{2}\,\sigma_v\; e^{-t^2}$$

$$= \frac{-A}{\sqrt{2}\,v}\,\sigma_v \int_0^{\frac{x}{\sqrt{2}\sigma_v}} dt\; e^{-t^2}$$

$$= \frac{-A}{\sqrt{2}\,v}\,\sigma_v\,\frac{\sqrt{\pi}}{2}\,\operatorname{erf}\!\left(\frac{x}{\sqrt{2}\,\sigma_v}\right)$$

$$= \frac{-A}{2}\sqrt{\frac{\pi}{2}}\,\frac{\sigma_v}{v}\,\operatorname{erf}\!\left(\frac{x}{\sqrt{2}\,\sigma_v}\right)$$

$$\Phi(x) = \cancel{\frac{1}{2} f(x)}^{\,0} + \frac{1}{2v}\int g(x)\,dy$$

$$= +\frac{A}{2}\sqrt{\frac{\pi}{2}}\,\frac{\sigma_v}{v}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}\,\sigma_v}\right)$$

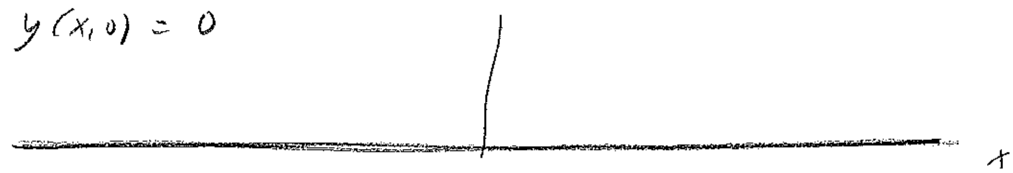
Thus,

$$y(x,t) = \Psi(x-vt) + \Phi(x+vt)$$

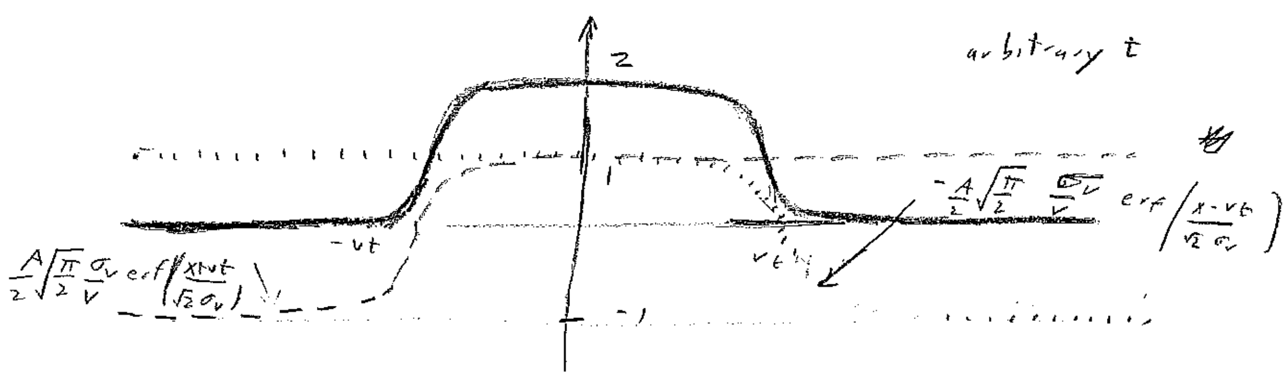
$$= -\frac{A}{2}\sqrt{\frac{\pi}{2}}\,\frac{\sigma_v}{v}\left[\mathrm{erf}\left(\frac{x-vt}{\sqrt{2}\,\sigma_v}\right) - \mathrm{erf}\left(\frac{x+vt}{\sqrt{2}\,\sigma_v}\right)\right]$$

Initial displacement:

$y(x,0) = 0$

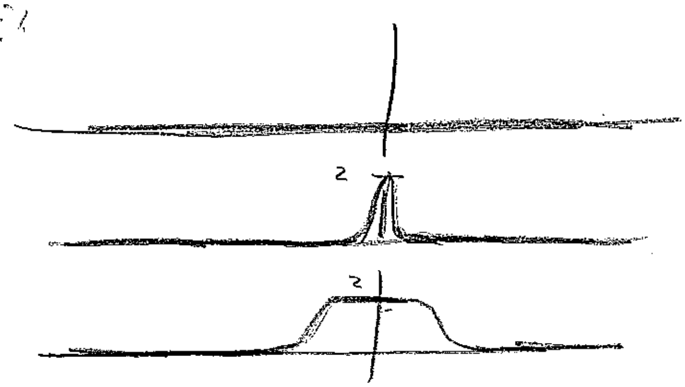


(t=0)



arbitrary t

NOTE:



t = 0

increasing t

(9.3)

Exercise: 1-d wave with sinusoidal displacement and velocity

$$y(x,0) = A \cos kx$$
$$\dot{y}(x,0) = \omega A \sin kx \qquad \text{where} \qquad \omega = kv$$

$$\Psi(x) = \frac{1}{2} f(x) - \frac{1}{2v} \int g(x)\,dx$$
$$= \frac{1}{2} A \cos kx - \frac{1}{2v} \omega A \int \sin kx$$
$$= \frac{1}{2} A \cos kx - \frac{1}{2} \cancel{k} A \left(\frac{-1}{\cancel{k}}\right) \cos kx \qquad (\omega = kv)$$
$$= \frac{1}{2} A \cos kx + \frac{1}{2} A \cos kx$$
$$= A \cos kx$$
$$\Phi(x) = \frac{1}{2} f(x) + \frac{1}{2v} \int g(x)\,dx$$
$$= 0$$

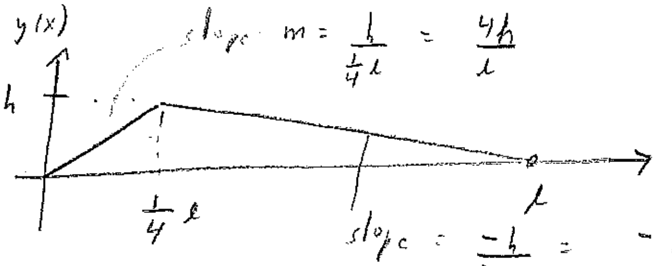
Thus,

$$y(x,t) = \Psi(x-vt) + \Phi(x+vt)$$
$$= A \cos k(x-vt)$$
$$= A \cos(kx - \omega t)$$

9.4

Exercise: Plucked guitar string $\left(\begin{array}{l} m = 2.0\text{ gm} \\ l = 65\text{ cm} \\ \text{Tension} = 100\text{ lb} \end{array} \right)$

slope $m = \dfrac{h}{\frac{1}{4}l} = \dfrac{4h}{l}$



slope $= \dfrac{-h}{\frac{3}{4}l} = -\dfrac{4}{3}\dfrac{h}{l}$

$h = 1\text{ cm}$

$l = 65\text{ cm}$

$$f(x) = y(x,0) = \begin{cases} 4h\left(\frac{x}{l}\right) \\ \frac{4h}{3}\left(1 - \frac{x}{l}\right) \end{cases}, \quad g(x) = \dot{y}(x,0) = 0$$

Check:

$$y - y_1 = m(x - x_1)$$

$$y - h = -\frac{4}{3}\frac{h}{l}\left(x - \frac{l}{4}\right)$$

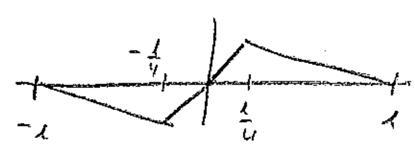
$$y = -\frac{4}{3}\frac{h}{l}x + \left(\frac{h}{3} + h\right) = -\frac{4}{3}\frac{h}{l}x + \frac{4}{3}h = \frac{4}{3}h\left(1 - \frac{x}{l}\right)$$

$$y(x,t) = \sum_{n=1}^{\infty} |c_n| \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi v t}{l} - \phi_n\right)$$ general sol'n

$$c_n = a_n + i b_n, \quad \phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

since $g(x) = 0 \rightarrow b_n = 0 \rightarrow \phi_n = 0$

$$a_n = \frac{1}{l}\int_{-l}^{l} dx\, f(x) \sin\left(\frac{n\pi x}{l}\right)$$



$$= \frac{2}{l}\int_0^l dx\, f(x) \sin\left(\frac{n\pi x}{l}\right)$$

$$\frac{2}{l}\int_0^l$$

$$a_n = \frac{2}{l}\left[\int_0^{l/4} dx\; 4h\frac{x}{l}\sin\left(\frac{n\pi x}{l}\right) + \int_{l/4}^{l} dx\; \frac{4h}{3}\left(1-\frac{x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{2}{l}\left[\frac{4h}{l}\int_0^{l/4} dx\; x\sin\left(\frac{n\pi x}{l}\right) + \frac{4h}{3}\int_{l/4}^{l} dx\; \sin\left(\frac{n\pi x}{l}\right) - \frac{4h}{3l}\int_{l/4}^{l} dx\; x\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{2}{l}\left[\; A + B + C \;\right]$$

Need to evaluate:

$$\boxed{\int_a^b dx\, \sin\left(\frac{n\pi x}{l}\right)} = -\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\Big|_a^b$$

$$= -\frac{l}{n\pi}\left[\cos\left(\frac{n\pi b}{l}\right) - \cos\left(\frac{n\pi a}{l}\right)\right]$$

$$\boxed{\int_a^b dx\; x\sin\left(\frac{n\pi x}{l}\right)} = -\frac{l}{n\pi}\, x\cos\left(\frac{n\pi x}{l}\right)\Big|_a^b + \frac{l}{n\pi}\int_a^b dx\, \cos\left(\frac{n\pi x}{l}\right)$$

$u = x \qquad dv = \sin\left(\frac{n\pi x}{l}\right)dx$

$du = dx \qquad v = -\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)$

$$= -\frac{l}{n\pi}\left(b\cos\left(\frac{n\pi b}{l}\right) - a\cos\left(\frac{n\pi a}{l}\right)\right) + \left(\frac{l}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{l}\right)\Big|_a^b$$

$$= -\frac{l}{n\pi}\left(b\cos\left(\frac{n\pi b}{l}\right) - a\cos\left(\frac{n\pi a}{l}\right)\right) + \left(\frac{l}{n\pi}\right)^2\left(\sin\left(\frac{n\pi b}{l}\right) - \sin\left(\frac{n\pi a}{l}\right)\right)$$

Thus,

$$Ⓐ = \frac{4h}{l} \int_0^{l/4} dx \; x \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{4h}{l} \left\{ \frac{-l}{n\pi} \left(\frac{l}{4} \cos\left(\frac{n\pi}{4}\right)\right) + \left(\frac{l}{n\pi}\right)^2 \sin\left(\frac{n\pi}{4}\right) \right\}$$

$$= -\frac{hl}{n\pi} \cos\left(\frac{n\pi}{4}\right) + \frac{4hl}{n^2\pi^2} \sin\left(\frac{n\pi}{4}\right)$$

$$Ⓑ = \frac{4h}{3} \int_{l/4}^{l} dx \; \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{4h}{3} \left(\frac{-l}{n\pi}\right) \left[\cos\left(n\pi\right) - \cos\left(\frac{n\pi}{4}\right) \right]$$

$$= \frac{-4hl}{3n\pi} \left[(-1)^n - \cos\left(\frac{n\pi}{4}\right) \right]$$

$$Ⓒ = -\frac{4h}{3l} \int_{l/4}^{l} dx \; x \sin\left(\frac{n\pi x}{l}\right)$$

$$= -\frac{4h}{3l} \left\{ -\frac{l}{n\pi} \left(l \underbrace{\cos(n\pi)}_{(-1)^n} - \frac{l}{4} \cos\left(\frac{n\pi}{4}\right)\right) + \left(\frac{l}{n\pi}\right)^2 \left(\cancel{\sin(n\pi)}_{0} - \sin\left(\frac{n\pi}{4}\right)\right)\right\}$$

$$= \frac{4hl}{3n\pi} (-1)^n - \frac{hl}{3n\pi} \cos\left(\frac{n\pi}{4}\right) + \frac{4hl}{3n^2\pi^2} \sin\left(\frac{n\pi}{4}\right)$$

Thus,

$$a_n = \frac{2}{l}\left[A + B + C \right]$$

$$= \frac{2}{l}\left\{ -\frac{hl}{n\pi}\cos\left(\frac{n\pi}{4}\right) + \frac{4hl}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right) \right.$$

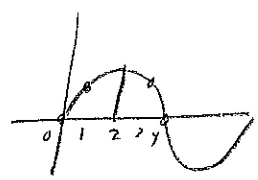
$$\cancel{-\frac{4hl}{3n\pi}(-1)^n} + \frac{4hl}{3n\pi}\cos\left(\frac{n\pi}{4}\right)$$

$$\left. \cancel{+\frac{4hl}{3n\pi}(-1)^n} - \frac{hl}{3n\pi}\cos\left(\frac{n\pi}{4}\right) + \frac{4hl}{3n^2\pi^2}\sin\left(\frac{n\pi}{4}\right) \right\}$$

$$= \frac{2}{l}\left\{ \frac{hl}{n\pi}\cos\left(\frac{n\pi}{4}\right)\left[\underbrace{\cancel{-1+\frac{4}{3}-\frac{1}{3}}}_{0}\right] + \frac{4hl}{n^2\pi^2}\underbrace{\left(1+\frac{1}{3}\right)}_{\frac{4}{3}}\sin\left(\frac{n\pi}{4}\right) \right\}$$

$$= \frac{2}{\cancel{l}}\,\frac{16}{3}\,\frac{h\cancel{l}}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right)$$

$$= \frac{32}{3}\left(\frac{h}{n^2\pi^2}\right)\sin\left(\frac{n\pi}{4}\right)$$



$$= \frac{32}{3}\left(\frac{h}{n^2\pi^2}\right)\begin{cases} \frac{\sqrt{2}}{2} & n=1 \\ 1 & n=2 \\ \frac{\sqrt{2}}{2} & n=3 \\ 0 & n=4 \\ -\sqrt{2}/2 & n=5 \\ -1 & n=6 \\ -\sqrt{2}/2 & n=7 \\ 0 & n=8 \end{cases}$$

NOTE:

No contributions from the $4^{th}, 8^{th}, 12^{th}, \ldots$ harmonics since $a_n = 0$.

$$|C_n| = |a_n| = \frac{32}{3}\left(\frac{h}{n^2\pi^2}\right)\left|\sin\left(\frac{n\pi}{4}\right)\right|$$

$$= \frac{32}{3}\left(\frac{h}{n^2\pi^2}\right)\begin{cases} 0 & n = 0, 4, 8, 12, \ldots \\ 1 & n = 2, 6, 10, \ldots \\ \frac{\sqrt{2}}{2} & n = 1, 3, 5 \end{cases}$$

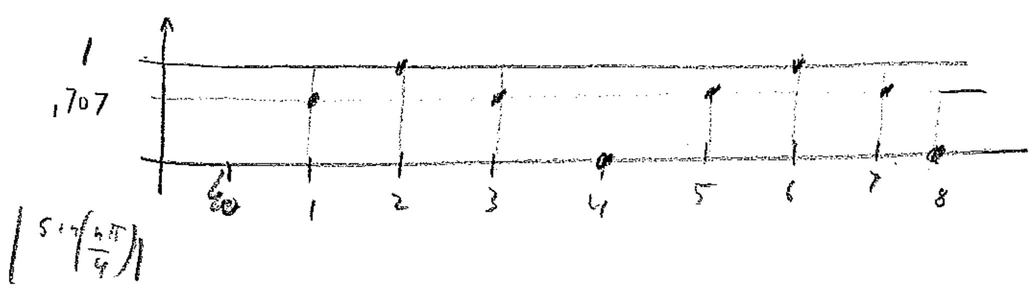
Fundamental Freq:

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\cdot\frac{\pi v}{l} = \boxed{\frac{v}{2l}}$$

(Need to know wave velocity) (see next page)

$$|a_n| = \frac{32}{3}\left(\frac{h}{n^2\pi^2}\right)\left|\sin\left(\frac{n\pi}{4}\right)\right|$$

$$\propto \frac{1}{n^2} \quad \left(\text{higher harmonics get smaller by } \frac{1}{n^2}\right)$$



wave velocity

$$v = \sqrt{\frac{T}{\mu}} \qquad T = 100\,lb \left(\frac{4.45\,N}{lb} \right)$$

$$\mu = 2\,gm/65\,cm = \frac{0.002\,kg}{.65\,m}$$

$$\rightarrow \quad V = \sqrt{\frac{100 \cdot 4.45}{\frac{0.002}{.65}}} = 380\,m/s$$

Fundamental Freq:

$$f = \frac{v}{2l} = \frac{380\,m/s}{2(0.65\,m)} = \boxed{292\,Hz}$$

9.5

Exercise: odd functions

Suppose

$$f(-x) = -f(x) \qquad (a)$$

$$g(-x) = -g(x) \qquad (b)$$

$$f(l-x) = -f(l+x) \qquad (c)$$

$$g(l-x) = -g(l+x) \qquad (d)$$

Show: $f(x+2l) = f(x)$, $g(x+2l) = g(x)$

Proof: consider only $f(x)$

$$\begin{aligned} f(x+2l) &= f(l+(x+l)) \\ &= -f(l-(x+l)) && \text{using (c)} \\ &= -f(l-x-l) \\ &= -f(-x) \\ &= +f(x) && \text{using (a)} \end{aligned}$$

EXERCISE 9.7 (EXERCISE 9.6 included) ①

Equivalence of eigenfunction and normal form solutions:

for periodic BCs:

$$y(x,t) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[C_n\, e^{i(k_n x - \omega_n t)} + C_{-n}^{*}\, e^{i(k_n x + \omega_n t)}\right]$$

$$= \frac{1}{2}\left(C_0 + C_{-0}^{*}\right)$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty}\left[C_n\, e^{ik_n(x-vt)} + C_{-n}^{*}\, e^{ik_n(x+vt)}\right]$$

$$+ \frac{1}{2}\sum_{n=-1}^{-\infty}\left[C_n\, e^{ik_n(x+vt)} + C_{-n}^{*}\, e^{ik_n(x-vt)}\right]$$

$$= \frac{1}{2l}\int_{-l}^{l} dx\, f(x)$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{2l}\int_{-l}^{l} du\left(f(u) + \frac{i}{\underset{= k_n v}{\omega_n}}\, g(u)\right)\overbrace{e^{-ik_n u}\, e^{ik_n \xi}}^{\xi - u}\right.$$

$$\left. + \frac{1}{2l}\int_{-l}^{l} du\left(f(u) - \frac{i}{\underset{= k_n v}{\omega_n}}\, g(u)\right)\overbrace{e^{-ik_n u}\, e^{ik_n \eta}}^{\eta - u}\right]$$

$$+ \frac{1}{2}\sum_{n=-1}^{-\infty}\left[\frac{1}{2l}\int_{-l}^{l} du\left(f(u) + \frac{i}{\underset{= -k_n v}{\omega_n}}\, g(u)\right)\overbrace{e^{-ik_n u}\, e^{ik_n \eta}}^{\eta - u}\right.$$

$$\left. + \frac{1}{2l}\int_{-l}^{l} du\left(f(u) - \frac{i}{\underset{= -k_n}{\omega_n}}\, g(u)\right)\overbrace{e^{-ik_n u}\, e^{ik_n \xi}}^{\xi - u}\right]$$

$$= \frac{1}{2}\left\{ \int_{-l}^{l} du\, f(u) \frac{1}{2l} \sum_{n=-\infty}^{\infty} \left(e^{i k_n (\xi - u)} + e^{i k_n (\eta - u)} \right) \right.$$

$k_n = \frac{n\pi}{l}$

$$\left. + \int_{-l}^{l} du\, g(u) \frac{1}{2l} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(\frac{i}{k_n v} \right) \left(e^{i k_n (\xi - u)} - e^{i k_n (\eta - u)} \right) \right\}$$

$k_n = \frac{n\pi}{l}$

Dirac delta:

$$\boxed{\delta(x - x') = \frac{1}{2l} \sum_{n=-\infty}^{\infty} e^{\pm i n \pi (x - x')/l}}$$

Proof: $\delta(\phi - \phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{\pm i m (\phi - \phi')}$

$\phi \in [0, 2\pi]$

$x \in [-l, l]$

$x = \phi \frac{l}{\pi}$

$\phi = \frac{x\pi}{l}$

$$\delta(\phi - \phi') = \delta\left(\frac{x\pi}{l} - \frac{x'\pi}{l} \right)$$

$$= \delta\left(\frac{\pi}{l} (x - x') \right)$$

$$= \frac{l}{\pi} \delta(x - x')$$

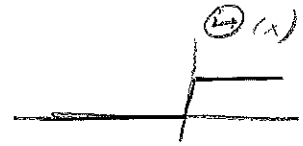
$$\rightarrow \delta(x - x') = \frac{\pi}{l} \delta(\phi - \phi')$$

$$= \frac{\cancel{\pi}}{l} \frac{1}{2\cancel{\pi}} \sum_{m=-\infty}^{\infty} e^{\pm i m \left(\frac{x\pi}{l} - \frac{x'\pi}{l} \right)}$$

$$= \frac{1}{2l} \sum_{m=-\infty}^{\infty} e^{\pm i \frac{m\pi}{l} (x - x')}$$

Heaviside step function (EXERCISE 9.6)

$$\Theta(x) = \int_{-\infty}^{x} du\, \delta(u) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$



$$\Theta(x) = \int_{-\infty}^{x} du\, \delta(u)$$

$$= \int_{-\infty}^{x} du\, \frac{1}{2l} \sum_{n=-\infty}^{\infty} e^{\pm i n\pi u/l}$$

$$= \frac{1}{2l} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{x} du\, e^{\pm i n\pi u/l}$$

$$= \frac{1}{2l} \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{l}{\pm i n\pi} e^{\pm i n \frac{\pi u}{l}} \Big|_{-\infty}^{x}$$

$$= \frac{1}{2l} \sum_{\substack{n=-\infty \\ \neq 0}}^{\infty} \frac{l}{\pm i n\pi} e^{\pm i n\pi \frac{x}{l}}$$

$$= \sum_{\substack{n=-\infty \\ \neq 0}}^{\infty} \frac{\mp i l}{2n\pi} e^{\pm i \frac{n\pi x}{l}}$$

$$= \frac{\mp i}{2l} \sum_{\substack{n=-\infty \\ \neq 0}}^{\infty} \left(\frac{l}{n\pi}\right) e^{\pm i \frac{n\pi x}{l}}$$

so $$\boxed{\Theta(x) = \frac{\mp i}{2l} \sum_{\substack{n=-\infty \\ \neq 0}}^{\infty} \frac{l}{n\pi} e^{\pm \frac{i n\pi x}{l}}}$$

Thus,

$$y(x,t) = \frac{1}{2}\left\{ \int_{-l}^{l} du\ f(u)\,\big(\delta(\xi-u) + \delta(\eta-u)\big) + \int_{-l}^{l} du\ g(u)\left[-\frac{1}{v}\big(\theta(\xi-u) - \theta(\eta-u)\big)\right]\right\}$$

$$= \frac{1}{2}\big(f(\xi) + f(\eta)\big) - \frac{1}{2v}\int_{-l}^{l} du\ g(u)\,\theta(\xi-u) + \frac{1}{2v}\int_{-l}^{l} du\ g(u)\,\theta(\eta-u)$$

$$= \frac{1}{2}\big(f(\xi) + f(\eta)\big) - \frac{1}{2v}\int_{-l}^{\xi} du\ g(u) + \frac{1}{2v}\int_{-l}^{\eta} du\ g(u)$$

$$= \frac{1}{2}\left[f(\xi) - \frac{1}{v}\int_{-l}^{\xi} du\ g(u)\right] + \frac{1}{2}\left[f(\eta) + \frac{1}{v}\int_{-l}^{\eta} du\ g(u)\right]$$

$$= \Psi(\xi) + \Phi(\eta)$$

9.8

Exercise: Show $F(-k) = F^*(k)$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du\, f(u)\, e^{-iku}$$

$$F(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du\, f(u)\, e^{iku}$$

$$= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du\, f(u)\, e^{-iku} \right)^*$$

since $f(u)$ is real

Heaviside Functions

$$\Theta(x) = \int_{-\infty}^{x} dy\, \delta(y) \qquad \delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{\pm ik(x-x')}$$

$$= \int_{-\infty}^{x} dy\, \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{\pm iky}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{x} dy\, e^{\pm iky}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{1}{\pm ik}\, e^{\pm iky}\Big|_{-\infty}^{x}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{\mp i}{k}\, e^{\pm ikx}$$

$$= \frac{\mp i}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{1}{k}\, e^{\pm ikx}$$

Thus, $\boxed{\Theta(x-x') = \frac{\mp i}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{1}{k}\, e^{\pm ik(x-x')}}$

9.9

Exercise: Equivalence of Fourier transform and normal form solution for the general case $g(x) \neq 0$.

Proof: $$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \frac{1}{2} \left[C(k) e^{-i\omega t} + C^*(-k) e^{i\omega t} \right] e^{ikx}$$

1st check reality: $y(x,t) = (A) + (B)$,

$$(B) = \frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{-\infty}^{\infty} dk \; C^*(-k) e^{i\omega t} e^{ikx} \qquad (k \to -p)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{\infty}^{-\infty} (-dp) \; C^*(p) \, e^{i\omega t} e^{-ipx} \qquad \left(\begin{matrix}\text{where} \\ \omega = |k| v\end{matrix}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{-\infty}^{\infty} dp \; C^*(p) \, e^{i\omega t} e^{-ipx}$$

$$= \left(\frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{-\infty}^{\infty} dp \; C(p) \, e^{-i\omega t} e^{ipx} \right)^*$$

$$= (A)^*$$

Relate $C(k)$ to initial conditions

$$y(x,0) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \frac{1}{2} \left[C(k) + C^*(-k) \right] e^{ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, F(k) e^{ikx}$$

so $$\boxed{\frac{1}{2} \left(C(k) + C^*(-k) \right) = F(k)} \qquad (1)$$

$$\dot{y}(x,0) = g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \frac{1}{2} \left[-i\omega \, C(k) + i\omega \, C^*(-k) \right] e^{ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \frac{-i\omega}{2} \left(C(k) - C^*(-k) \right) e^{ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, G(k) \, e^{ikx}$$

$$\rightarrow \quad \frac{-i\omega}{2} \left(C(k) - C^*(-k) \right) = G(k)$$

$$\text{or} \quad \boxed{\frac{1}{2} \left(C(k) - C^*(-k) \right) = \frac{i}{\omega} G(k)} \quad (2)$$

Thus, $\boxed{C(k) = F(k) + \frac{i}{\omega} G(k)}$ (by adding (1) and (2))

<u>check!</u>

$$C^*(-k) = F(k) - \frac{i}{\omega} G(k) \quad \text{(by subtracting (1) and (2))}$$

$$= F^*(-k) - \frac{i}{\omega} G^*(-k)$$

$$= \left(F(-k) + \frac{i}{\omega} G(-k) \right)^*$$

$$= \left[C(-k) \right]^*$$

$\omega = |k| v$

Thus,

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \frac{1}{2} \left[C(k) e^{-i\omega t} + C^*(-k) e^{i\omega t} \right] e^{ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2} \Big\{ \int_0^{\infty} dk \, C(k) e^{ik(x-vt)} + \int_{-\infty}^{0} dk \, C(k) e^{ik(x+vt)}$$

$$+ \int_0^{\infty} dk \, C^*(-k) e^{ik(x+vt)} + \int_{-\infty}^{0} dk \, C^*(-k) e^{ik(x-vt)} \Big\}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \Big\{ \int_0^{\infty} dk \left(F(k) + \frac{i}{\omega} G(k) \right) e^{ik\xi}$$

$$+ \int_{-\infty}^{0} dk \left(F(k) + \frac{i}{\omega} G(k) \right) e^{ik\eta}$$

$$+ \int_0^{\infty} dk \left(F^*(-k) - \frac{i}{\omega} G^*(-k) \right) e^{ik\eta}$$

$$+ \int_{-\infty}^{0} dk \left(F^*(-k) - \frac{i}{\omega} G^*(-k) \right) e^{ik\xi} \Big\}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \Big\{ \int_0^{\infty} dk \left(F(k) + \frac{i}{kv} G(k) \right) e^{ik\xi}$$

$$+ \int_{-\infty}^{0} dk \left(F(k) - \frac{i}{kv} G(k) \right) e^{ik\eta}$$

$$+ \int_0^{\infty} dk \left(F(k) - \frac{i}{kv} G(k) \right) e^{ik\eta}$$

$$+ \int_{-\infty}^{0} dk \left(F(k) + \frac{i}{kv} G(k) \right) e^{ik\xi} \Big\}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{2}\left\{ \int_{-\infty}^{\infty} dk \left(F(k) + \frac{i}{kv} G(k)\right) e^{ik\xi} \right.$$

$$\left. + \int_{-\infty}^{\infty} dk \left(F(k) - \frac{i}{kv} G(k)\right) e^{ik\eta} \right\}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{2}\left\{ \int_{-\infty}^{\infty} dk\, F(k) \left(e^{ik\xi} + e^{ik\eta}\right) \right.$$

$$\left. + \frac{1}{v}\int_{-\infty}^{\infty} dk\, \frac{i}{k}\, G(k) \left(e^{ik\xi} - e^{ik\eta}\right) \right\}$$

$$= \frac{1}{2}\left\{ \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, F(k)\, e^{ik\xi}}_{f(\xi)} + \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, F(k) e^{ik\eta}}_{f(\eta)} \right.$$

$$+ \frac{1}{v}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \frac{i}{k} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du\, g(u)\, e^{-iku}\, e^{ik\xi}$$

$$\left. - \frac{1}{v}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \frac{i}{k} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du\, g(u)\, e^{-iku}\, e^{ik\eta} \right\}$$

<u>Now:</u> $$\mp \frac{i}{2\pi} \int_{-\infty}^{\infty} dk\, \frac{1}{k}\, e^{\pm ik(x-x')} = \Theta(x-x')$$

$$= \frac{1}{2}\left(f(\xi) + f(\eta)\right)$$

$$- \frac{1}{2v} \int_{-\infty}^{\infty} du\, g(u)\, \Theta(\xi - u) + \frac{1}{2v} \int_{-\infty}^{\infty} du\, g(u)\, \Theta(\eta - u)$$

$$= \frac{1}{2}\left(f(\xi) + f(\eta)\right) - \frac{1}{2v} \int_{-\infty}^{\xi} du\, g(u) + \frac{1}{2v} \int_{-\infty}^{\eta} du\, g(u)$$

$$= \underbrace{\frac{1}{2} f(\xi) - \frac{1}{2v}\int_{-\infty}^{\xi} du\, g(u)}_{\Psi(\xi)} + \underbrace{\frac{1}{2} f(\eta) + \frac{1}{2v}\int_{-\infty}^{\eta} du\, g(u)}_{\Phi(\eta)}$$

$$= \Psi(\xi) + \Phi(\eta)$$

(as desired)

Exercise: 9.10

$$\frac{1}{\sin\theta}\left[\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)\right] - \frac{m^2}{\sin^2\theta}\,\Theta = -\ell(\ell+1)\,\Theta$$

$$\Theta(\theta) = y(x)\Big|_{x=\cos\theta} \qquad \text{(change of variable)}$$

$$\frac{d\Theta}{d\theta} = y'(x)\frac{dx}{d\theta} = -y'\sin\theta = -y'\sqrt{1-x^2}$$

Thus, $$-\frac{1}{\sin\theta}\frac{d}{d\theta} = \frac{d}{dx}$$

$$\text{LHS} = \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d}{d\theta}\Theta\right) - \frac{m^2}{\sin^2\theta}\Theta$$

$$= -\frac{d}{dx}\left(-\sin^2\theta\,\frac{dy}{dx}\right) - \frac{m^2}{1-x^2}\,y$$

$$= +\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] - \frac{m^2}{1-x^2}\,y$$

$$= (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - \frac{m^2}{1-x^2}\,y$$

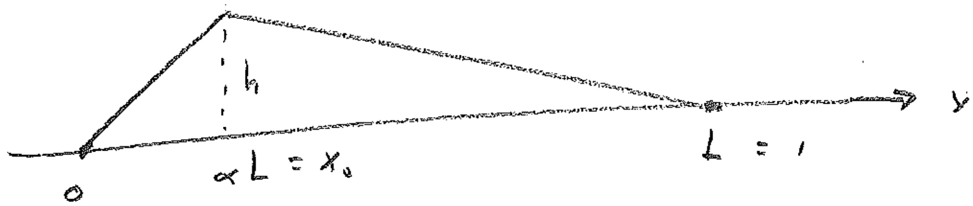
$$\text{RHS} = -\ell(\ell+1)\,y$$

Thus, $$\boxed{0 = (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[\ell(\ell+1) - \frac{m^2}{(1-x^2)}\right]y}$$

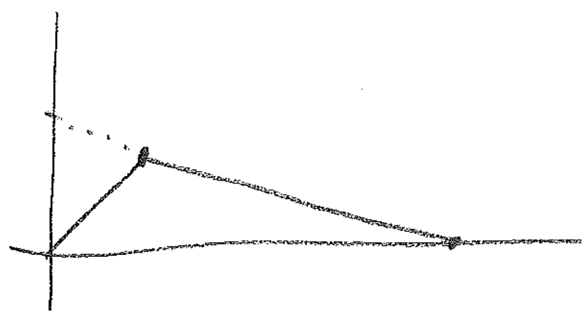
ε_e: assoc-legendre-equation 3

Plucked string – Fourier coefficients Prob (9.1) ①

Initial configuration $y_0(x) = y(x, t=0)$



$$y_0(x) = \begin{cases} \left(\frac{h}{\alpha L}\right) x & 0 < x < \alpha L \\ -\frac{h}{(1-\alpha)L}(x-L) & \alpha L < x < L \end{cases}$$



$(y - y_1) = m(x - x_1)$

$y = m(x - L)$

$m = \frac{-h}{L(1-\alpha)}$

check: $y_0(0) = 0$

$y_0(L) = 0$

$y_0(\alpha L) = h$

$y_0(\alpha L) = \frac{-h}{(1-\alpha)L}(\alpha L - L) = \frac{-hL(\alpha-1)}{(1-\alpha)L} = h$

$$y_0(x) = \begin{cases} \left(\frac{h}{\alpha}\right)\frac{x}{L} \\ -\frac{h}{(1-\alpha)}\left(\frac{x}{L} - 1\right) \end{cases} = \begin{cases} \frac{h}{\alpha} u & 0 < u < \alpha \\ \frac{h}{(1-\alpha)}(1-u) & \alpha < u < 1 \end{cases}$$

$$y_0(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

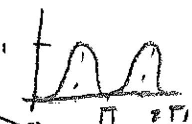
$$b_n = \frac{2}{L}\int_0^L y_0(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= 2\int_0^1 y_0(u)\sin(n\pi u)\,du$$

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx \qquad u = \frac{\pi x}{L}$$

$$= \frac{L}{\pi}\int_0^{\pi} \sin^2(nu)\,du$$

$$= \frac{L}{2}$$



$$= 2\left\{\int_0^{\alpha} \frac{h}{\alpha}u\sin(n\pi u)\,du + \int_{\alpha}^{1}\frac{h}{1-\alpha}(1-u)\sin(n\pi u)\,du\right\}$$

$$= 2h\left\{\frac{1}{\alpha}\int_0^{\alpha} u\sin(n\pi u)\,du\right.$$

$$+\frac{1}{1-\alpha}\int_{\alpha}^{1}\sin(n\pi u)\,du$$

$$\left.-\frac{1}{1-\alpha}\int_{\alpha}^{1}u\sin(n\pi u)\,du\right\}$$

$$= 2h\left\{\frac{1}{\alpha}\left(-\frac{1}{n\pi}\left[\alpha\cos(n\pi\alpha) - \cancel{0\cdot\cos(n\pi 0)}^{\,0}\right]\right.\right.$$

$$\left. + \frac{1}{(n\pi)^2}\left[\sin(n\pi\alpha) - \cancel{\sin(n\pi 0)}^{\,0}\right]\right)$$

$$+\frac{1}{1-\alpha}\left(-\frac{1}{n\pi}\right)\left(\underbrace{\cos(n\pi)}_{(-1)^n} - \cos(n\pi\alpha)\right)$$

$$-\frac{1}{1-\alpha}\left(-\frac{1}{n\pi}\left[1\cdot\overbrace{\cos(n\pi)}^{(-1)^n} - \alpha\cos(n\pi\alpha)\right]\right.$$

$$\left.\left. + \frac{1}{(n\pi)^2}\left[\cancel{\sin(n\pi)}^{\,0} - \sin(n\pi\alpha)\right]\right)\right\}$$

$$= \frac{2h}{n\pi}\left\{ \frac{1}{\alpha}\left(-\alpha\cos(n\pi\alpha) + \frac{1}{n\pi}\sin(n\pi\alpha)\right)\right.$$

$$- \frac{1}{1-\alpha}\left(\cancel{(-1)^n} - \cos(n\pi\alpha)\right)$$

$$\left. - \frac{1}{1-\alpha}\left(\cancel{-(-1)^n} + \alpha\cos(n\pi\alpha) - \frac{1}{n\pi}\sin(n\pi\alpha)\right)\right\}$$

$$= \frac{2h}{n\pi}\left\{ \cos(n\pi\alpha)\left(-1 + \frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha}\right)\right.$$

$$\left. + \sin(n\pi\alpha)\frac{1}{n\pi}\left(\frac{1}{\alpha} + \frac{1}{1-\alpha}\right)\right\}$$

$$= \frac{2h}{n\pi}\left\{ \cos(n\pi\alpha)\frac{1}{1-\alpha}\left(\cancel{-(1-\alpha)} + \cancel{(1-\alpha)}\right)\right.$$

$$\left. + \frac{1}{n\pi}\sin(n\pi\alpha)\left(\frac{(1-\cancel{\alpha}) + \cancel{\alpha}}{\alpha(1-\alpha)}\right)\right\}$$

$$= \frac{2h}{n^2\pi^2}\frac{1}{\alpha(1-\alpha)}\sin(n\pi\alpha)$$

thus $\boxed{b_n = \frac{2h\sin(n\pi\alpha)}{\pi^2\alpha(1-\alpha)}\frac{1}{n^2}}$

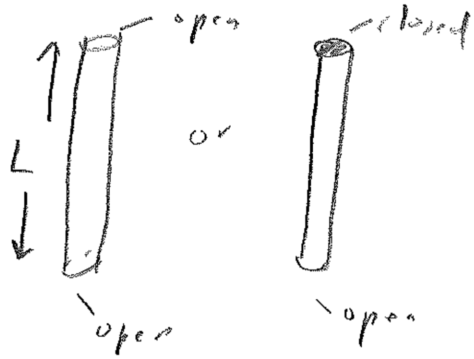
for plucked string release from rest

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} \frac{2h}{\pi^2\alpha(1-\alpha)}\frac{\sin(n\pi\alpha)}{n^2}\sin\left(\frac{n\pi x}{L}\right)\underbrace{\cos\left(\frac{n2\pi t}{T}\right)}_{\cos\left(\frac{n\pi v}{L}t\right)}}$$

Prob (9.2)

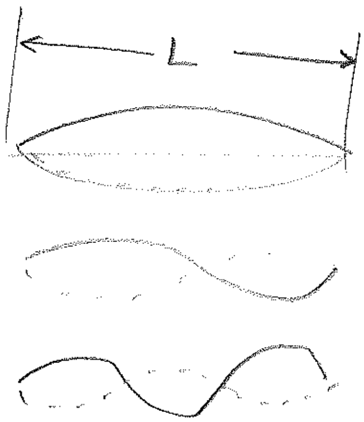
1-d wave equation:

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2} = 0$$



Pressure deviations are zero at open end of tube, maxima at closed ends.

Open at both ends



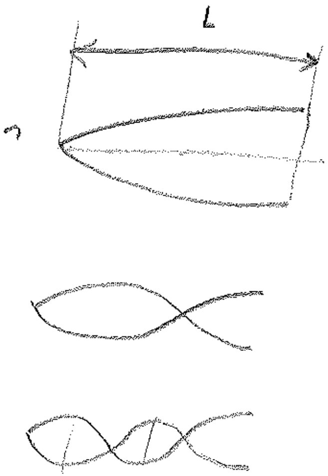
$n = 1$
$\lambda = 2L$

$n = 2$
$\lambda = L$

$n = 3$
$\lambda = \frac{2}{3}L$

Thus, $\lambda_n = \frac{2L}{n}$, $n = 1, 2, 3, \ldots$

Open



closed

$n = 1$
$\lambda = 4L$

$n = 3$
$\lambda = \frac{4}{3}L$

$n = 5$
$\lambda = \frac{4}{5}L$

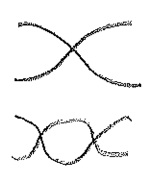
so $\lambda_n = \frac{4L}{n}$, $n = 1, 3, 5, \ldots$

$v = f\lambda \rightarrow f = \frac{v}{\lambda}$

$f_n = \frac{nv}{2L}$, $n = 1, 2, 3, \ldots$ open at both ends

$f_n = \frac{nv}{4L}$, $n = 1, 3, 5, \ldots$ closed at one end

Closed at both ends



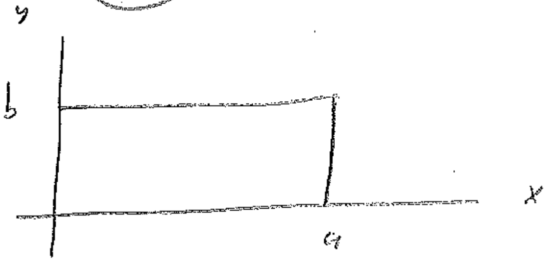
$\lambda = 2L$

$\lambda = L$

etc

Same as for tube open at both ends

Prob: 9.3 ~~Chapt 6.3~~



$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

$u = u(x,y,t)$, BCs: $u(0,y,t)=0$, $u(a,y,t)=0$, $u(x,0,t)=0$, $u(x,b,t)=0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

Let $u(x,y,t) = X(x)\,Y(y)\,T(t)$

$$\rightarrow \quad X''YT + Y''XT = \frac{1}{v^2} XYT''$$

$\div$ by XYT:

$$\underbrace{\frac{X''}{X}}_{-k_x^2} + \underbrace{\frac{Y''}{Y}}_{-k_y^2} = \underbrace{\frac{1}{v^2}\frac{T''}{T}}_{-\gamma^2} \qquad \text{where } \gamma^2 = \alpha^2+\beta^2$$

$$X'' = -k_x^2 X \quad \rightarrow \quad X(x) = \begin{cases} A\sin(k_x x) + B\cos(k_x x) \\ A_0 + B_0 x \end{cases}$$

BC's: $X(0)=0$, $X(a)=0$ $\rightarrow$ $\boxed{X(x) = A\sin\left(\frac{n\pi x}{a}\right)}$

i.e., $k_x = \frac{n\pi}{a}$, $n=1,2,\ldots$

Similarly,

$$Y'' = -k_y^2 Y \quad \rightarrow \quad \boxed{Y(y) = C\sin\left(\frac{m\pi y}{b}\right)}$$

i.e., $k_y = \frac{m\pi}{b}$, $m=1,2,\ldots$

Thus, $T'' = -\gamma^2 v^2 T$

$$T(t) = D\sin(\gamma v t) + E\cos(\gamma v t)$$

where $\gamma = \sqrt{k_x^2+k_y^2} = \pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2} \equiv \gamma_{nm}$, $n=1,2,\ldots$; $m=1,2,\ldots$

General solution:

$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \Big[D_{mn} \sin(\gamma_{nm} v t) + E_{mn} \cos(\gamma_{nm} v t) \Big]}$$

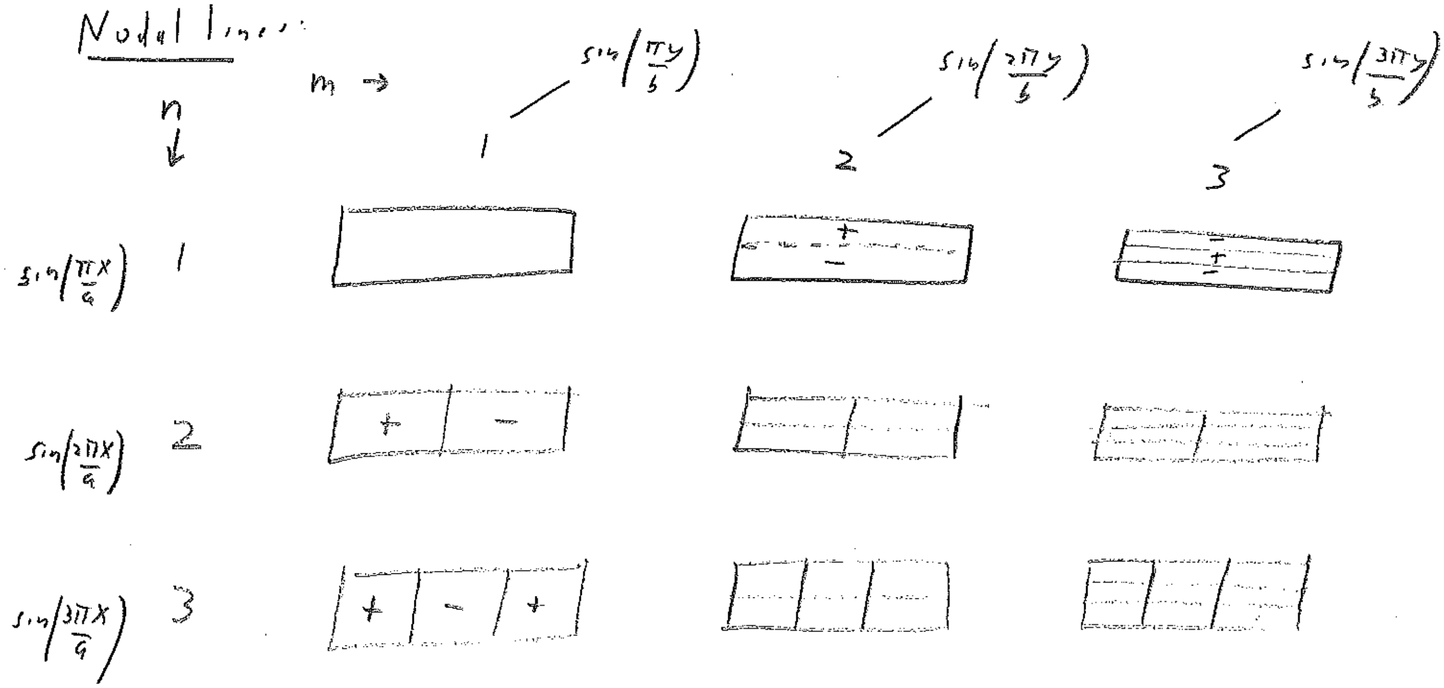
Frequencies: $\begin{matrix}\sin \\ \cos\end{matrix} (2\pi f t)$

Thus, $2\pi f_{nm} = \gamma_{nm} v$

$$f_{nm} = \frac{v}{2\cancel{\pi}} \cancel{\pi} \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

$$= \frac{v}{2} \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

Nodal lines:

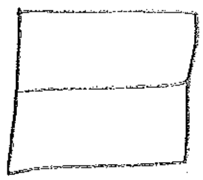


For a square $a = b$:

$$\rightarrow f_{nm} = \frac{v}{2a}\sqrt{n^2+m^2}$$

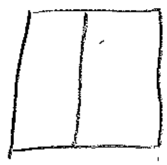
NOTE: $f_{nm} = f_{mn}$ so $f_{17} = f_{71} = f_{55}$

since $\sqrt{1^2+7^2} = \sqrt{7^2+1^2} = \sqrt{5^2+5^2}$



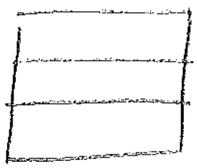
21

and



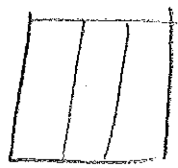
12

have same freq



31

and

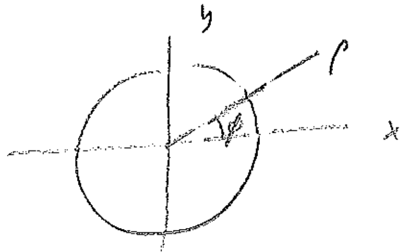


13

have same freq

etc.

Vibrating drum head: Prob 9.4



cylindrical polar coordinates
$(\rho, \phi, z=0)$

Solve wave equation for $u(t,x,y) = u(t,\rho,\phi)$
subject to BC: $u(t,\rho=a,\phi) = 0 \quad \forall\, t,\phi$
and IC's: $u(t=0,\rho,\phi) = F(\rho,\phi)$
$\dot{u}(t=0,\rho,\phi) = G(\rho,\phi)$

wave equation:

$$0 = -\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}$$

where v = wave velocity on stretched drum head
= function of Tension and mass/area.

Separation of variables:

$$u(t,\rho,\phi) = T(t)\,R(\rho)\,Q(\phi)$$

$$\rightarrow \quad 0 = -\frac{1}{v^2}T''RQ + TQ\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2}Q''TR$$

$$= \underbrace{-\frac{1}{v^2}\frac{T''}{T}}_{\text{function of } t} + \underbrace{\frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}(\rho R') + \frac{1}{\rho^2}\frac{Q''}{Q}}_{\text{function of } \rho,\phi}$$

$$\rightarrow \quad -\frac{1}{v^2}\frac{T''}{T} = k^2$$

$$T'' = -k^2v^2T, \qquad \omega^2 \equiv k^2v^2$$

$$T(t) = \begin{cases} A_0 + B_0 t, & k=0 \text{ — DC term } (\omega=0) \\ A\cos(kvt) + B\sin(kvt), & k\neq 0 \end{cases}$$

$$0 = k^2 + \frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}(\rho R') + \frac{1}{\rho^2}\frac{Q''}{Q}$$

$$0 = \underbrace{k^2\rho^2 + \frac{1}{R}\rho\frac{d}{d\rho}(\rho R')}_{\text{func of } \rho} + \underbrace{\frac{Q''}{Q}}_{\text{func of } \phi}$$

$$\frac{Q''}{Q} = -\alpha^2$$

$$Q'' = -\alpha^2 Q$$

$$Q(\phi) \begin{cases} C_0 + D_0\phi & \alpha = 0 \\ C\cos(\alpha\phi) + D\sin(\alpha\phi) & \alpha \neq 0 \end{cases}$$

Periodic BCs:

$$Q(\phi + 2\pi) = Q(\phi)$$

$$\rightarrow \quad C_0 + D_0(\phi + 2\pi) = C_0 + D_0\phi \quad \rightarrow D_0 = 0$$

$$\alpha = m = 1, 2, \ldots$$

thus,

$$Q(\phi) = \begin{cases} C_0 & \alpha = 0 \\ C_m\cos(m\phi) + D_m\sin(m\phi), & m = 1, 2, \ldots \end{cases}$$

$$= C_m\cos(m\phi) + D_m\sin(m\phi) \quad m = 0, 1, 2, \ldots$$

$$0 = k^2\rho^2 + \frac{1}{R}\rho\frac{d}{d\rho}(\rho R') - m^2$$

$$= \rho\frac{d}{d\rho}(\rho R') + (k^2\rho^2 - m^2)R$$

$$= \rho^2 R'' + \rho R' + (k^2\rho^2 - m^2)R$$

make a change of variables

$$x = k\rho$$

$$R' = \frac{dR}{d\rho} = \frac{dx}{d\rho}\frac{dR}{dx} = k\frac{dR}{dx}$$

so $$\rho R' = k\rho \frac{dR}{dx} = x\frac{dR}{dx}$$

$$R'' = \frac{d^2R}{d\rho^2} = k^2\frac{d^2R}{dx^2} \rightarrow \rho^2 R'' = x^2\frac{d^2R}{dx^2}$$

Thus,

$$\boxed{0 = x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - m^2)R}$$

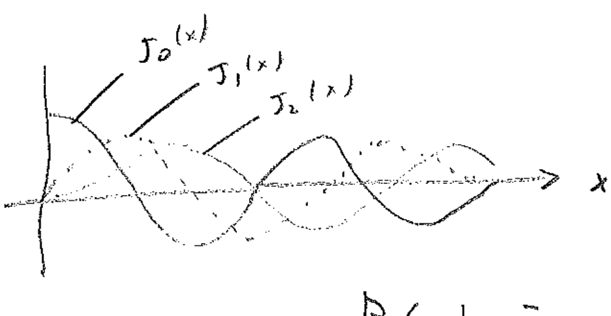
This is Bessel's equation of order $m = 0, 1, 2, \ldots$

<u>solns</u>: $R(x) = A_m J_m(x) + B_m N_m(x)$

<u>BC</u>: i) Finite at $\rho = 0$ $\rightarrow$ $B_m = 0$ since $N_m(x) \rightarrow \infty$ as $x \rightarrow 0$

ii) $R(\rho = a) = 0$ $\rightarrow$ $A_m J_m(ka) = 0$

ka = nth zero of J_m
$= x_{mn}$ $(n = 1, 2, \ldots)$



so $k = k_{mn} = \frac{x_{mn}}{a}$

<u>Therefore</u>:

$$R(x) = A_{mn} J_m(x_{mn}\rho/a)$$

$$Q(\phi) = C_m \cos m\phi + D_m \sin m\phi, \quad m = 0, 1, 2, \ldots$$

$$T(t) = A\cos(\omega_{mn}t) + B\sin(\omega_{mn}t)$$

where $\omega_{mn} = k_{mn}v = \frac{x_{mn}v}{a}$, $\quad m = 0, 1, 2, \ldots$
$\quad n = 1, 2, \ldots$

General solution: (ignoring DC)

$$u(t,\rho,\phi) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(x_{mn}\rho/a)\left[C_{mn}\cos(m\phi) + D_{mn}\sin(m\phi)\right]$$

$$\left[A_{mn}\cos(w_{mn}t) + B_{mn}\sin(w_{mn}t)\right]$$

subject to ICs: $u(t=0,\rho,\phi) = F(\rho,\phi)$

$\dot{u}(t=0,\rho,\phi) = G(\rho,\phi)$

NOTE:

i) No ϕ-dependence $\rightarrow$ $m=0$ so frequencies

$$w_{mn} = w_{0n} \equiv w_n = \frac{x_{0n}v}{a}, \quad n=1,2,\dots$$

ii) (ρ,ϕ) dependence $\rightarrow$

$$w_{mn} = \frac{x_{mn}v}{a}, \quad x_{mn}: n^{th} \text{ zero of } J_m(x)$$

$$\boxed{f_{mn} = \frac{1}{2\pi}\left(\frac{v}{a}\right)x_{mn}}$$

wolfram.com

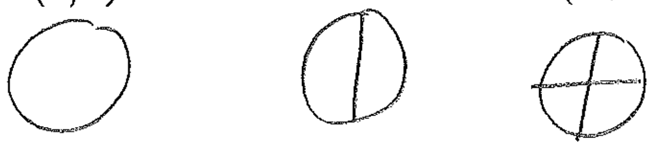
BesselFunctionZeros.html

m \ n	1	2	3	4	5
$J_0(x)$	2.4048	5.5201	8.6537	11.7915	14.9309
$J_1(x)$	3.8317	7.0156	10.1735	13.3237	16.4706
$J_2(x)$	5.1356	8.4172	11.6198	14.7960	17.9598
$J_3(x)$	6.3802	9.7610	13.0152	16.2235	19.4094
$J_4(x)$	7.5883	11.0647	14.3725	17.6160	20.8269
$J_5(x)$	8.7715	12.3386	15.7002	18.9501	22.2178

2.4048
3.8317
5.1356
5.5201
6.3802
7.0156

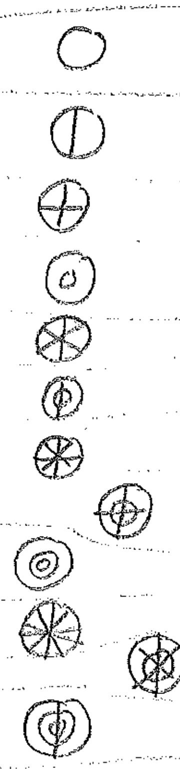
· modes: (m, n)

(0,1) (1,1) (2,1)

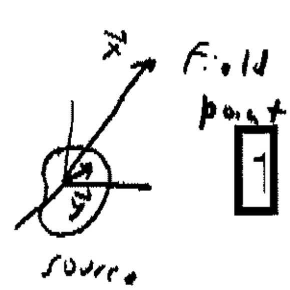


, etc.

	x_{mn}	ratio	(m, n)
1	2.4048	1	0, 1
2	3.8317	1.59	1, 1
3	5.1356	2.14	2, 1
4	5.5201	2.30	0, 2
5	6.3802	2.65	3, 1
6	7.0156	2.92	1, 2
7	7.5883	3.16	4, 1
8	8.4172	3.50	2, 2
9	8.6537	3.60	0, 3
10	8.7715	3.65	5, 1
11	9.7610	4.06	3, 2
12	10.1735		1, 3



Prob 9.5



1

$$f(t,\vec{x}) = -\frac{1}{4\pi}\int d^3y\, \frac{g(t-|\vec{x}-\vec{y}|, \vec{y})}{|\vec{x}-\vec{y}|}$$

$$\Box f = -\frac{\partial^2 f}{\partial t^2} + \nabla^2 f$$

Need to take $\frac{\partial^2}{\partial t^2}$ and ∇^2 of integrand.

$$\frac{\partial}{\partial t}\left(\frac{g(t-|\vec{x}-\vec{y}|, \vec{y})}{|\vec{x}-\vec{y}|}\right) = \frac{g'(t-|\vec{x}-\vec{y}|, \vec{y})}{|\vec{x}-\vec{y}|}$$

$$\frac{\partial^2}{\partial t^2}\left(\quad '' \quad\right) = \frac{g''(t-|\vec{x}-\vec{y}|, \vec{y})}{|\vec{x}-\vec{y}|}$$

$$\nabla^2\left(\frac{g(t-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|}\right) = \vec{\nabla}\cdot\vec{\nabla}\left(\frac{g(t-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|}\right)$$

$$= \vec{\nabla}\cdot\left(g(t-|\vec{x}-\vec{y}|,\vec{y})\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{y}|}\right) + \frac{1}{|\vec{x}-\vec{y}|}\vec{\nabla} g(t-|\vec{x}-\vec{y}|,\vec{y})\right)$$

$$= \vec{\nabla} g(t-|\vec{x}-\vec{y}|,\vec{y})\cdot\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{y}|}\right) + g(t-|\vec{x}-\vec{y}|,\vec{y})\,\nabla^2\left(\frac{1}{|\vec{x}-\vec{y}|}\right)$$
$$+ \vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{y}|}\right)\cdot\vec{\nabla} g(t-|\vec{x}-\vec{y}|,\vec{y}) + \frac{1}{|\vec{x}-\vec{y}|}\nabla^2 g(t-|\vec{x}-\vec{y}|,\vec{y})$$

$$= g(t-|\vec{x}-\vec{y}|,\vec{y})\,\nabla^2\left(\frac{1}{|\vec{x}-\vec{y}|}\right) + \frac{1}{|\vec{x}-\vec{y}|}\nabla^2 g(t-|\vec{x}-\vec{y}|,\vec{y})$$
$$+ 2\,\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{y}|}\right)\cdot\vec{\nabla} g(t-|\vec{x}-\vec{y}|,\vec{y})$$

Now:

$$\vec{\nabla} g(t-|\vec{x}-\vec{y}|,\vec{y}) = g'(t-|\vec{x}-\vec{y}|,\vec{y})\,\vec{\nabla}(-|\vec{x}-\vec{y}|)$$

$$= -g'(t-|\vec{x}-\vec{y}|,\vec{y})\,\vec{\nabla}(|\vec{x}-\vec{y}|)$$

$$\vec{\nabla}(|\vec{x}-\vec{y}|) = \vec{\nabla}\left(\sqrt{\delta_{ij}(x^i-y^i)(x^j-y^j)}\right)$$

$$= \frac{1}{\cancel{2}}\frac{1}{\sqrt{\ }}\cancel{2}(\vec{x}-\vec{y})$$

$$= \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|} \quad \text{(unit vector)}$$

$$\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{y}|}\right) = -\frac{1}{|\vec{x}-\vec{y}|^2}\vec{\nabla}(|\vec{x}-\vec{y}|)$$

$$= -\frac{(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}$$

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{y}|}\right) = -4\pi\,\delta^3(\vec{x}-\vec{y})$$

$$\nabla^2 g(t-|\vec{x}-\vec{y}|,\vec{y}) = \vec{\nabla}\cdot\left(-g'(t-|\vec{x}-\vec{y}|,\vec{y})\,\vec{\nabla}(|\vec{x}-\vec{y}|)\right)$$

$$= -\vec{\nabla} g'(t-|\vec{x}-\vec{y}|,\vec{y})\cdot\vec{\nabla}(|\vec{x}-\vec{y}|)$$

$$-\, g'(t-|\vec{x}-\vec{y}|,\vec{y})\,\nabla^2(|\vec{x}-\vec{y}|)$$

$$= +\,g''(t-|\vec{x}-\vec{y}|,\vec{y})\,\vec{\nabla}(|\vec{x}-\vec{y}|)\cdot\vec{\nabla}(|\vec{x}-\vec{y}|)$$

$$-\, g'(t-|\vec{x}-\vec{y}|,\vec{y})\,\vec{\nabla}\cdot\left(\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|}\right)$$

$$= g''(t-|\vec{x}-\vec{y}|,\vec{y})\,\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|}\cdot\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|}$$

$$-\, g'(t-|\vec{x}-\vec{y}|,\vec{y})\left((\vec{x}-\vec{y})\cdot\vec{\nabla}\frac{1}{|\vec{x}-\vec{y}|} + \frac{1}{|\vec{x}-\vec{y}|}\vec{\nabla}\cdot(\vec{x}-\vec{y})\right)$$

$$= g''(t-|\vec{x}-\vec{y}|,\vec{y}) - g'(t-|\vec{x}-\vec{y}|,\vec{y})\left(-\frac{(\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3} + \frac{1}{|\vec{x}-\vec{y}|}\cdot 3\right)$$

$$= g''(t-|\vec{x}-\vec{y}|,\vec{y}) - g'(t-|\vec{x}-\vec{y}|,\vec{y})\frac{2}{|\vec{x}-\vec{y}|}$$

Thus,

$$\Box f = -\frac{\partial^2 f}{\partial t^2} + \nabla^2 f$$

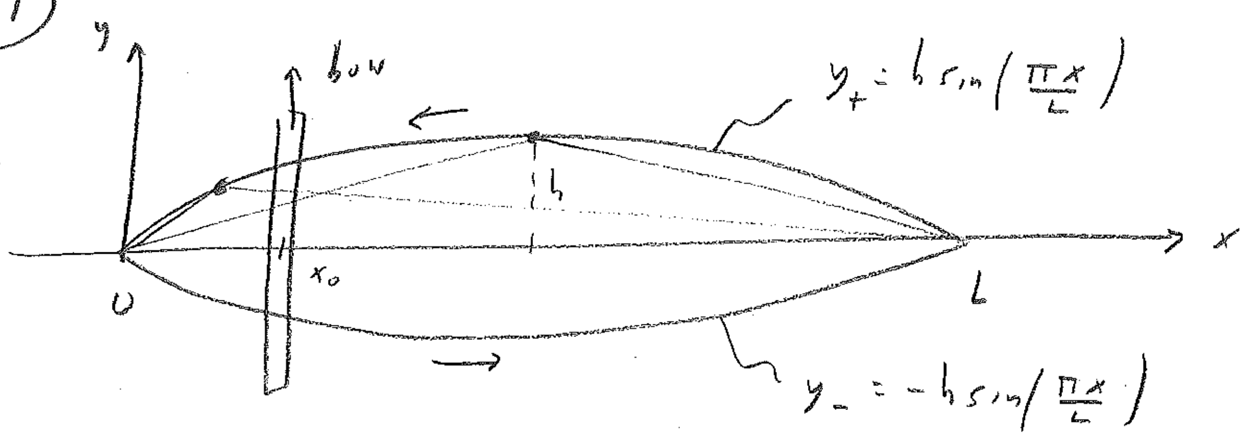
$$= -\frac{1}{4\pi}\int d^3y \Bigg[\cancel{\frac{-g''(t-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|}} + g(t-|\vec{x}-\vec{y}|,\vec{y})\left(-4\pi\,\delta^3(\vec{x}-\vec{y})\right) + \frac{1}{|\vec{x}-\vec{y}|}\left(\cancel{g''(t-|\vec{x}-\vec{y}|,\vec{y})} - \cancel{2g'(t-|\vec{x}-\vec{y}|,\vec{y})\frac{1}{|\vec{x}-\vec{y}|}}\right)$$

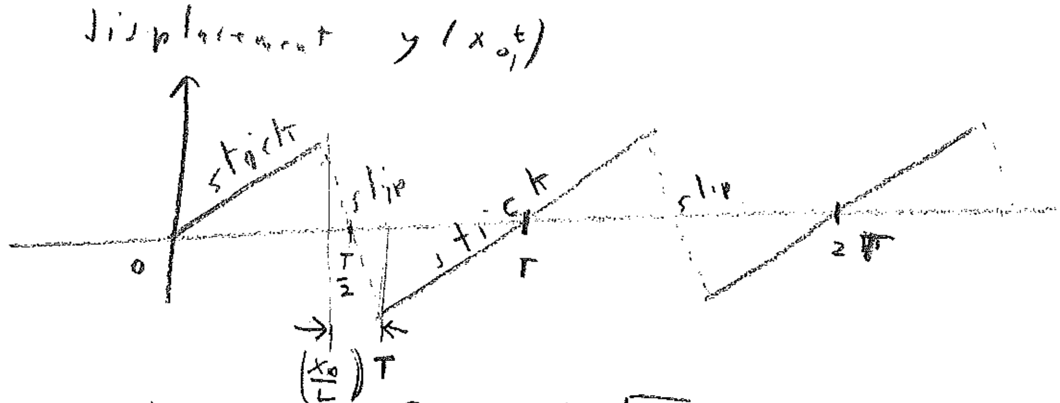
$$\underbrace{+\,2\,\frac{-(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}\cdot -g'(t-|\vec{x}-\vec{y}|,\vec{y})\frac{(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|}}_{\cancel{+2g'(t-|\vec{x}-\vec{y}|,\vec{y})\frac{1}{|\vec{x}-\vec{y}|^2}}} \Bigg]$$

$$= \int d^3y\; g(t-|\vec{x}-\vec{y}|,\vec{y})\;\delta^3(\vec{x}-\vec{y})$$

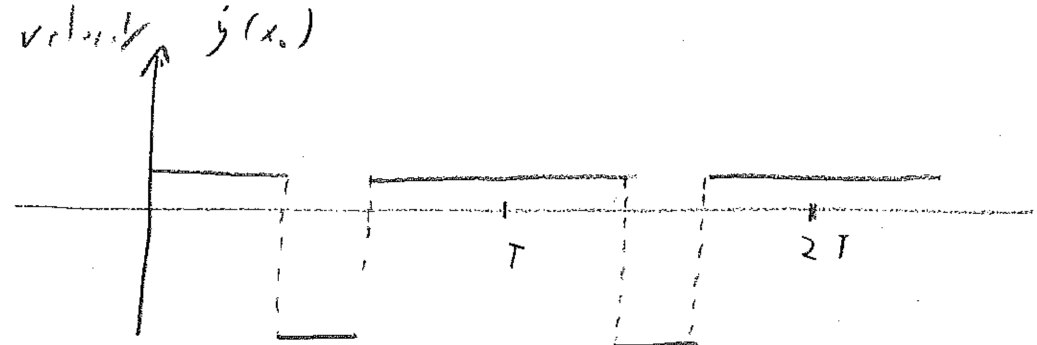
$$= g(t,\vec{x})$$

Prob 9.7





$T = \frac{1}{f_1}$, $\quad f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}} = \frac{v}{2L}$ — velocity of wave on string



$\frac{x_0}{L}$ = fraction of total length L = bow location

general solution to source-free wave equation: $-\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2} = 0$

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi v}{L}t\right) + B_n \sin\left(\frac{n\pi v t}{L}\right)\right]$$

Satisfies BC's: $y(0,t) = 0 = y(L,t) \quad \forall\, t$.

$$T = \frac{1}{f_1} = \frac{2L}{v} \quad \longrightarrow \quad \frac{n\pi v}{L} t = n 2\pi \left(\frac{v}{2L}\right) t$$

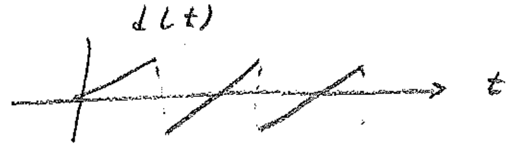
$$= 2\pi n \frac{t}{T}$$

Thus,

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(n\frac{2\pi t}{T}\right) + B_n \sin\left(n\frac{2\pi t}{T}\right)\right]$$

Displacement Curve:

$y(x_0, t) = D(t)$



$$D(t) = y(x_0, t)$$
$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x_0}{L}\right)\left[A_n \cos\left(\frac{n 2\pi t}{T}\right) + B_n \sin\left(\frac{n 2\pi t}{T}\right)\right]$$

$$\frac{T}{2} A_n \sin\left(\frac{n\pi x_0}{L}\right) = \int_0^T D(t) \cos\left(\frac{n 2\pi t}{T}\right) dt$$

$$\rightarrow A_n = \frac{2}{T} \frac{1}{\sin\left(\frac{n\pi x_0}{L}\right)} \int_0^T D(t) \cos\left(\frac{n 2\pi t}{T}\right) dt$$

$$B_n = \frac{2}{T} \frac{1}{\sin\left(\frac{n\pi x_0}{L}\right)} \int_0^T D(t) \sin\left(\frac{n 2\pi t}{T}\right) dt$$

Normalization:

$$\int_0^T \cos^2\left(\frac{n 2\pi t}{T}\right) dt = \int_0^{2\pi} \cos^2(nu) \frac{T}{2\pi} du$$

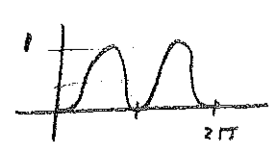
$$= \frac{T}{2\pi} \int_0^{2\pi} \cos^2(nu)\, du$$

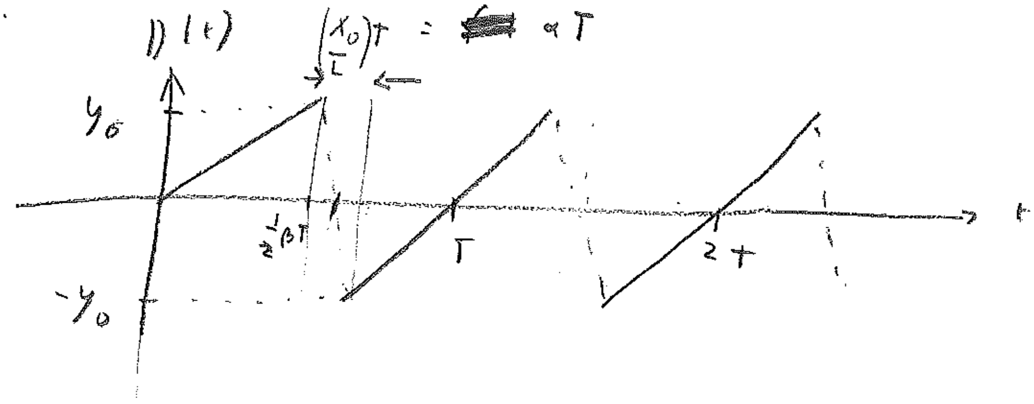
$$= \frac{T}{2}$$

let $u = \frac{2\pi t}{T}$

$du = \frac{2\pi}{T} dt$

$\cos^2$





$$y_0 = h\sin\left(\frac{\pi x_0}{L}\right)$$

$$\alpha = \frac{x_0}{L} = \text{Fraction}$$

$$\beta = 1-\alpha \quad , \quad \alpha = 1-\beta$$

$$\frac{1}{2}\beta + \alpha = \frac{1}{2}(1-\alpha) + \alpha = \frac{1}{2}(1+\alpha) = 1 - \frac{\beta}{2}$$

$$\frac{1}{2}\left(T - \left(\frac{x_0}{L}\right)T\right) = \frac{1}{2}T(1-\alpha) = \frac{1}{2}\beta T$$

<u>Check</u>: $\frac{1}{2}\beta T + \alpha T + \frac{1}{2}\beta T = (\beta + \alpha)T = T$ ✓

Three parts to $D(t)$:

(i) $t \in [0, \frac{1}{2}\beta T]$

$$\boxed{D_1(t) = \left(\frac{y_0}{\frac{1}{2}\beta}\right)\frac{t}{T}}$$

$$\frac{y_0}{\frac{1}{2}\beta T}(\cancel{t}^{\,0} - T) = -\frac{T y_0}{\frac{1}{2}\beta T} = -\frac{y_0}{\frac{1}{2}\beta}$$

(ii) $t \in \left[\frac{1}{2}\beta T, \left(\frac{1}{2}\beta + \alpha\right)T\right]$, slope $= \frac{-2y_0}{\alpha T}$

$$\boxed{D_2(t) = \left(\frac{-2y_0}{\alpha}\right)\frac{t}{T} + \frac{y_0}{\alpha}}$$

<u>check</u>: $D_2\left(\frac{1}{2}T\right) = \frac{-2y_0}{\alpha}\frac{1}{2} + \frac{y_0}{\alpha} = 0$

$$\left(\frac{-2y_0}{\alpha T}\right)\left(\cancel{t}^{\,0} - \frac{1}{2}(\alpha+\beta)T\right) = \frac{y_0}{\alpha}(\alpha+\beta) = \frac{y_0}{\alpha}$$

(iii) $t \in \left[\left(\frac{1}{2}\beta + \alpha\right)T, T\right] = \left[\left(1-\frac{\beta}{2}\right)T, T\right]$

$$D_3(t) = \frac{y_0}{\frac{1}{2}\beta T}t - \frac{y_0}{\frac{1}{2}\beta} = \boxed{\left(\frac{y_0}{\frac{1}{2}\beta}\right)\left(\frac{t}{T} - 1\right)} = D_3(t)$$

<u>check</u>: $D_3\left(\frac{1}{2}(1+\alpha)T\right) = \frac{y_0}{\frac{1}{2}\beta T}\left(\left(1-\frac{\beta}{2}\right)T - T\right) = \frac{-y_0 \cancel{\frac{\beta}{2}T}}{\cancel{\frac{\beta}{2}T}} = -y_0$

Summary:

$$D_1(t) = \frac{2y_0}{\beta}\left(\left(\frac{t}{T}\right) - 0\right), \qquad t \in [0, \tfrac{1}{2}\beta T]$$

$$D_2(t) = -\frac{2y_0}{\alpha}\left(\left(\frac{t}{T}\right) - \frac{1}{2}\right), \qquad t \in [\tfrac{1}{2}\beta T, (1-\tfrac{\beta}{2})T]$$

$$D_3(t) = \frac{2y_0}{\beta}\left(\left(\frac{t}{T}\right) - 1\right), \qquad t \in [(1-\tfrac{\beta}{2})T, T]$$

---

check: $D_1(0) = 0$, $D_2(\frac{T}{2}) = 0$, $D_3(T) = 0$

equal

$$D_1\left(\frac{\beta T}{2}\right) = \frac{\cancel{2}y_0}{\cancel{\beta}}\frac{\cancel{\beta}}{\cancel{2}} = y_0$$

$$D_2\left(\frac{\beta T}{2}\right) = -\frac{\cancel{2}y_0}{\alpha}\frac{\beta}{\cancel{2}} + \frac{y_0}{\alpha} = -\frac{y_0}{\alpha}\underbrace{(\beta-1)}_{-\alpha} = y_0$$

equal

$$D_2\left((1-\tfrac{\beta}{2})T\right) = -\frac{2y_0}{\alpha}\left(1-\frac{\beta}{2}\right) + \frac{y_0}{\alpha}$$

$$= \frac{y_0}{\alpha}\left[-2\left(1-\frac{\beta}{2}\right)+1\right]$$

$$= \frac{y_0}{\alpha}\left[-1+\beta\right]$$

$$= -y_0$$

$$D_3\left((1-\tfrac{\beta}{2})T\right) = \frac{2y_0}{\beta}\left(\left(\cancel{1}-\frac{\beta}{2}\right)-\cancel{1}\right)$$

$$= -y_0$$

NOTE: $1-\frac{\beta}{2} = \frac{1}{2}(1+\alpha)$, $\quad \frac{1}{2}\beta = \frac{1}{2}(1-\alpha)$

Let $u = t/T$

$$D_1(u) = \frac{2y_0}{(1-\alpha)}\left(u - 0\right) \qquad u \in \left[0, \tfrac{1}{2}(1-\alpha)\right]$$

$$D_2(u) = -\frac{2y_0}{\alpha}\left(u - \tfrac{1}{2}\right) \qquad u \in \left[\tfrac{1}{2}(1-\alpha), \tfrac{1}{2}(1+\alpha)\right]$$

$$D_3(u) = \frac{2y_0}{(1-\alpha)}\left(u - 1\right) \qquad u \in \left[\tfrac{1}{2}(1+\alpha), 1\right]$$

$$A_n = \frac{2}{T}\,\frac{1}{\sin\left(\frac{n\pi y_0}{L}\right)}\int_0^T D(t)\cos\left(\frac{n2\pi t}{T}\right)dt$$

$$= \frac{2}{\sin\left(\frac{n\pi x_0}{L}\right)}\int_0^1 D(u)\cos(n2\pi u)\,du$$

$$B_n = \frac{2}{\sin\left(\frac{n\pi x_0}{L}\right)}\int_0^1 D(u)\sin(n2\pi u)\,du$$

NOTE:
$n = 1, 2, \ldots$

$$\int_0^1 C\cdot\cos(n2\pi u) = \frac{C}{n2\pi}\sin(2\pi n u)\Big|_0^1$$

$$= \frac{C}{n2\pi}\Big(\underbrace{\sin(2\pi n)}_{0} - \underbrace{\sin 0}_{0}\Big)$$

$$= 0$$

$$\int_0^1 C\cdot\sin(2n\pi u) = \frac{-C}{n2\pi}\cos(2\pi n u)\Big|_0^1$$

$$= \frac{-C}{n2\pi}\Big(\underbrace{\cos(2\pi n)}_{1} - \underbrace{\cos 0}_{1}\Big)$$

$$= 0$$

$$\int_a^b \cos(2\pi n u)\,du = \frac{1}{2\pi n}\sin(2\pi n u)\Big|_a^b$$

cos

$$= \boxed{\frac{1}{2\pi n}\left[\sin(2\pi n b) - \sin(2\pi n a)\right]}$$

$$\int_a^b \sin(2\pi n u)\,du = \frac{-1}{2\pi n}\cos(2\pi n u)\Big|_a^b$$

sin

$$= \boxed{-\frac{1}{2\pi n}\left[\cos(2\pi n b) - \cos(2\pi n a)\right]}$$

$$\int_a^b u\,\underbrace{\cos(2\pi n u)\,du}_{dv = \cos(2\pi n u)du} = \frac{1}{2\pi n}u\sin(2\pi n u)\Big|_a^b - \frac{1}{2\pi n}\int_a^b \sin(2\pi n u)\,du$$

$$dv = \cos(2\pi n u)du$$

$$v = \frac{1}{2\pi n}\sin(2\pi n u)$$

u cos

$$= \boxed{\frac{1}{2\pi n}\left(b\sin(2\pi n b) - a\sin(2\pi n a)\right) + \frac{1}{(2\pi n)^2}\left[\cos(2\pi n b) - \cos(2\pi n a)\right]}$$

$$\int_a^b u\,\underbrace{\sin(2\pi n u)\,du}_{dv} = \frac{-1}{2\pi n}u\cos(2\pi n u)\Big|_a^b + \frac{1}{2\pi n}\int_a^b \cos(2\pi n u)\,du$$

$$v = \frac{-1}{2\pi n}\cos(2\pi n u)$$

$$= \boxed{-\frac{1}{2\pi n}\left[b\cos(2\pi n b) - a\cos(2\pi n a)\right] + \frac{1}{(2\pi n)^2}\left[\sin(2\pi n b) - \sin(2\pi n a)\right]}$$

u sin

$$A_n = \frac{2}{\sin\left(\frac{n\pi x_0}{L}\right)} \left\{ \int_0^{\frac{1}{2}(1-\alpha)} \frac{2y_0}{(1-\alpha)}\, u \cos(2\pi n u)\, du \right.$$

$$+ \int_{\frac{1}{2}(1-\alpha)}^{\frac{1}{2}(1+\alpha)} -\frac{2y_0}{\alpha}\left(u-\frac{1}{2}\right)\cos(2\pi n u)\, du$$

$$\left. + \int_{\frac{1}{2}(1+\alpha)}^{1} \frac{2y_0}{(1-\alpha)}(u-1)\cos(2\pi n u)\, du \right\}$$

$$= \frac{4y_0}{\sin\left(\frac{n\pi x_0}{L}\right)} \left\{ \frac{1}{(1-\alpha)} \int_0^{\frac{1}{2}(1-\alpha)} u \cos(2\pi n u)\, du \right.$$

$$+ \frac{1}{(1-\alpha)} \int_{\frac{1}{2}(1+\alpha)}^{1} (u-1)\cos(2\pi n u)\, du$$

$$\left. - \frac{1}{\alpha}\int_{\frac{1}{2}(1-\alpha)}^{\frac{1}{2}(1+\alpha)} \left(u-\frac{1}{2}\right)\cos(2\pi n u)\, du \right\}$$

$$= \frac{4y_0}{\sin\left(\frac{n\pi x_0}{L}\right)} \left\{ \frac{1}{(1-\alpha)} \left[\frac{1}{2\pi n}\left(\left(\frac{1-\alpha}{2}\right)\sin(\pi n(1-\alpha)) - \cancel{0 \sin 0}\right) \right. \right.$$

$$\left. + \frac{1}{(2\pi n)^2}\left[\cos(\pi n(1-\alpha)) - 1\right] \right]$$

$$+ \frac{1}{(1-\alpha)}\left(\frac{1}{2\pi n}\left(1 \cdot \cancelto{0}{\sin(2\pi n)} - \left(\frac{1+\alpha}{2}\right)\sin(\pi n(1+\alpha))\right) \right.$$

$$+ \frac{1}{(2\pi n)^2}\left(1 - \cos(\pi n(1+\alpha))\right)$$

$$\left. - \frac{1}{2\pi n}\left(\cancelto{0}{\sin(2\pi n)} - \sin(\pi n(1+\alpha))\right) \right)$$

$$- \frac{1}{\alpha}\left(\frac{1}{2\pi n}\left(\left(\frac{1+\alpha}{2}\right)\sin(\pi n(1+\alpha)) - \left(\frac{1-\alpha}{2}\right)\sin(n\pi(1-\alpha))\right) + \frac{1}{(2\pi n)^2}\left(\cos(n\pi(1+\alpha)) - \cos(n\pi(1-\alpha))\right) \right.$$

$$\left. \left. - \frac{1}{2}\,\frac{1}{2\pi n}\left(\sin(n\pi(1+\alpha)) - \sin(n\pi(1-\alpha))\right) \right) \right\}$$

$$= \frac{4y_0}{\sin\left(\frac{n\pi x_0}{L}\right)} \Bigg\{ \frac{1}{(1-\alpha)}\frac{1}{2\pi n}\Bigg(\left(\frac{1-\alpha}{2}\right)\underbrace{\sin\left(n\pi(1-\alpha)\right)}_{\pm(-1)^n \sin(n\pi\alpha)} + \frac{1}{2\pi n}\Big(\underbrace{\cos\left(n\pi(1-\alpha)\right)}_{(-1)^n\cos(n\pi\alpha)} - 1\Big)\Bigg)$$

$$+ \frac{1}{(1-\alpha)}\frac{1}{2\pi n}\Bigg(-\left(\frac{1+\alpha}{2}\right)\underbrace{\sin\left(n\pi(1+\alpha)\right)}_{(-1)^n\sin(n\pi\alpha)} + \frac{1}{2\pi n}\Big(1 - \underbrace{\cos\left(n\pi(1+\alpha\right)}_{(-1)^n\cos(n\pi\alpha)}$$

$$+ \underbrace{\sin\left(n\pi(1+\alpha)\right)}_{(-1)^n\sin(n\pi\alpha)}\Bigg)$$

$$- \frac{1}{\alpha}\frac{1}{2\pi n}\Bigg(\left(\frac{1+\alpha}{2}\right)\overbrace{\sin\left(n\pi(1+\alpha)\right)}^{(-1)^n\sin(n\pi\alpha)} - \left(\frac{1-\alpha}{2}\right)\overbrace{\sin\left(n\pi(1-\alpha)\right)}^{-(-1)^n\sin(n\pi\alpha)}$$

$$+ \frac{1}{2\pi n}\Big(\underbrace{\cos\left(n\pi(1+\alpha)\right)}_{(-1)^n\cos(n\pi\alpha)} - \underbrace{\cos\left(n\pi(1-\alpha)\right)}_{(-1)^n\cos(n\pi\alpha)}\Big)$$

$$- \frac{1}{2}\Big(\underbrace{\sin\left(n\pi(1+\alpha)\right)}_{(-1)^n\sin(n\pi\alpha)} - \underbrace{\sin\left(n\pi(1-\alpha)\right)}_{-(-1)^n\sin(n\pi\alpha)}\Big)\Bigg)\Bigg\}$$

<u>Now:</u>

$$\sin\left(n\pi(1\pm\alpha)\right) = \cancel{\sin(n\pi)}^{\,0}\cos(n\pi\alpha) \pm \cos(n\pi)\sin(n\pi\alpha)$$

$$= \pm(-1)^n\sin(n\pi\alpha)$$

$$\cos\left(n\pi(1\pm\alpha)\right) = \cos(n\pi)\cos(n\pi\alpha) \mp \cancel{\sin(n\pi)}^{\,0}\sin(n\pi\alpha)$$

$$= (-1)^n\cos(n\pi\alpha)$$

$$= \frac{4y_0}{\sin\left(\frac{n\pi x_0}{L}\right)}\left(\frac{1}{2\pi n}\right)\Bigg\{ \frac{1}{(1-\alpha)}\Bigg(-\left(\frac{1-\alpha}{2}\right)(-1)^n\sin(n\pi\alpha) + \frac{1}{2\pi n}\Big((-1)^n\cos(n\pi\alpha) - 1\Big)\Bigg)$$

$$+ \frac{1}{(1-\alpha)}\Bigg(-\left(\frac{1+\alpha}{2}\right)(-1)^n\sin(\;) + \frac{1}{2\pi n}\Big(1 - (-1)^n\cos(\;)\Big)$$

$$+ (-1)^n\sin(\;)\Bigg)$$

$$- \frac{1}{\alpha}\Bigg(\left(\frac{1+\alpha}{2}\right)(-1)^n\sin(\;) + \left(\frac{1-\alpha}{2}\right)(-1)^n\sin(\;) - (-1)^n\sin(\;)\Bigg)\Bigg\}$$

$$= \frac{4y_0}{\sin\left(\frac{n\pi x_0}{L}\right)}\left(\frac{1}{2\pi n}\right)\Bigg\{$$

$$\sin(n\pi\alpha)\Bigg[-\frac{1}{2}(-1)^n - \left(\frac{1+\alpha}{1-\alpha}\right)\frac{1}{2}(-1)^n + \frac{(-1)^n}{(1-\alpha)} - \left(\frac{1+\cancel{\alpha}}{2\alpha}\right)(-1)^n - \left(\frac{1-\cancel{\alpha}}{2\alpha}\right)(-1)^n + \frac{1}{\alpha}(-1)^n\Bigg]$$

$$+\cos(n\pi\alpha)\Bigg[\cancel{\frac{1}{(1-\alpha)}\frac{1}{2\pi n}(-1)^n} \quad \cancel{-\frac{1}{(1-\alpha)}\frac{1}{2\pi n}(-1)^n}\Bigg]$$

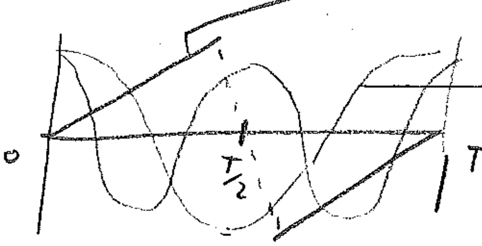
$$+\Bigg[\cancel{\frac{1}{(1-\alpha)}\left(\frac{-1}{2\pi n}\right)} + \cancel{\frac{1}{(1-\alpha)}\left(\frac{1}{2\pi n}\right)}\Bigg]\Bigg\}$$

$$= \frac{4y_0}{\sin\left(\frac{n\pi x_0}{L}\right)}\left(\frac{1}{2\pi n}\right)(-1)^n \sin(n\pi\alpha)\underbrace{\Bigg[-\frac{1}{2} - \frac{1}{2}\left(\frac{1+\alpha}{1-\alpha}\right) + \left(\frac{1}{1-\alpha}\right) \cancel{-\frac{1}{\alpha}} + \cancel{\frac{1}{\alpha}}\Bigg]}$$

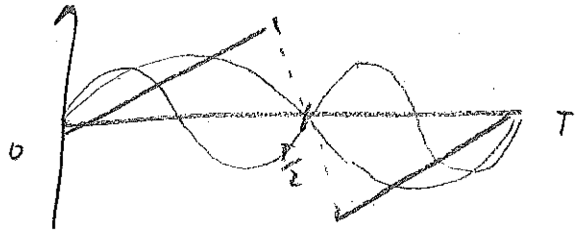
$$= \frac{1}{2}\left(\frac{1}{1-\alpha}\right)\Big[-(\cancel{1}-\cancel{\alpha}) - (\cancel{1}+\cancel{\alpha}) + \cancel{2}\Big]$$

$$= \boxed{0} \quad !$$

Thus, $\boxed{A_n = 0}$



$$B_n = \frac{2}{T}\,\frac{1}{\sin\left(\frac{n\pi x_0}{L}\right)}\int_0^T \underbrace{D(t)}_{\text{odd about } T/2}\,\underbrace{\sin\left(\frac{n2\pi t}{T}\right)}_{\text{odd about } T/2}\,dt$$



$$\rightarrow\quad B_n = \frac{4}{T}\,\frac{1}{\sin\left(\frac{n\pi x_0}{L}\right)}\int_0^{T/2} D(t)\sin\left(\frac{n2\pi t}{T}\right)dt$$

$$= \frac{4}{\sin\left(\frac{n\pi y_0}{L}\right)}\int_0^{1/2} D(u)\sin(n2\pi u)\,du$$

$$= \frac{4}{\sin\left(\frac{n\pi x_0}{L}\right)}\left\{\int_0^{\frac{1}{2}(1-\alpha)} \frac{2y_0}{(1-\alpha)}\,u\,\sin(n2\pi u)\,du - \int_{\frac{1}{2}(1-\alpha)}^{\frac{1}{2}} \frac{2y_0}{\alpha}\left(u-\frac{1}{2}\right)\sin(n2\pi u)\,du\right\}$$

$$= \frac{8y_0}{\sin\left(\frac{n\pi x_0}{L}\right)}\left\{\left(\frac{1}{1-\alpha}\right)\int_0^{\frac{1}{2}(1-\alpha)} u\,\sin(n2\pi u)\,du - \frac{1}{\alpha}\int_{\frac{1}{2}(1-\alpha)}^{\frac{1}{2}}\left(u-\frac{1}{2}\right)\sin(n2\pi u)\,du\right\}$$

$$= \frac{8y_0}{\sin\left(\frac{n\pi x_0}{L}\right)} \left\{ \left(\frac{1}{1-\alpha}\right) \left(\frac{-1}{2\pi n} \left[\left(\frac{1-\alpha}{2}\right) \underbrace{\cos\left(n\pi(1-\alpha)\right)}_{(-1)^n \cos(n\pi\alpha)} - \cancel{0 \cos 0}^{\,0} \right] \right.\right.$$

$$\left. + \frac{1}{(2\pi n)^2} \left[\underbrace{\sin\left(n\pi(1-\alpha)\right)}_{-(-1)^n \sin(n\pi\alpha)} - \cancel{\sin(0)}^{\,0} \right] \right)$$

$$- \frac{1}{\alpha} \left(-\frac{1}{2\pi n} \left[\frac{1}{2} \underbrace{\cos(n\pi)}_{(-1)^n} - \left(\frac{1-\alpha}{2}\right) \underbrace{\cos\left(n\pi(1-\alpha)\right)}_{(-1)^n \cos(n\pi\alpha)} \right] \right.$$

$$+ \frac{1}{(2\pi n)^2} \left[\underbrace{\cancel{\sin(n\pi)}}_{0} - \underbrace{\sin\left(n\pi(1-\alpha)\right)}_{-(-1)^n \sin(n\pi\alpha)} \right]$$

$$\left.\left. - \frac{1}{2} \left(-\frac{1}{2\pi n}\right) \left[\underbrace{\cos(n\pi)}_{(-1)^n} - \underbrace{\cos\left(n\pi(1-\alpha)\right)}_{(-1)^n \cos(n\pi\alpha)} \right] \right) \right\}$$

$$= \frac{8y_0}{\sin\left(\frac{n\pi x_0}{L}\right)} \left(\frac{1}{2\pi n}\right) (-1)^n \left\{ \left(\frac{1}{1-\alpha}\right) \left(-\left(\frac{1-\alpha}{2}\right) \cos(\,) - \frac{1}{(2\pi n)} \sin(\,) \right) \right.$$

$$- \frac{1}{\alpha} \left(-\left[\frac{1}{2} - \left(\frac{1-\alpha}{2}\right) \cos(\,) \right] + \frac{1}{(2\pi n)} \sin(\,) \right.$$

$$\left.\left. + \frac{1}{2} \left[1 - \cos(\,) \right] \right) \right\}$$

$$= \frac{8y_0}{\sin\left(\frac{n\pi x_0}{L}\right)} \left(\frac{1}{2\pi n}\right) (-1)^n \left\{ \sin(\,) \left[-\frac{1}{2\pi n} \left(\frac{1}{1-\alpha}\right) - \frac{1}{2\pi n} \frac{1}{\alpha} \right] \right.$$

$$\left. + \cos(\,) \left[-\frac{1}{2} - \frac{1}{\alpha} \left(\frac{1-\alpha}{2}\right) + \frac{1}{2\alpha} \right] - \frac{1}{\alpha} \cancel{\left(-\frac{1}{2} + \frac{1}{2} \right)}^{\,0} \right\}$$

$$= \frac{8 y_0}{\sin\left(\frac{n\pi x_0}{L}\right)} \left(\frac{1}{2\pi n}\right)(-1)^n \left\{ \sin(n\pi\alpha)\left(\frac{-1}{2\pi n}\right)\left(\frac{1}{1-\alpha}\right)\Big[\underbrace{\cancel{1} + \left(\frac{1-\alpha}{\alpha}\right)}_{\frac{1}{\alpha}}\Big] \right.$$

$$\left. + \cos(n\pi\alpha)\left(\frac{1}{2\alpha}\right)\Big[\underbrace{-\cancel{\alpha} - \cancel{(1-\alpha)} + \cancel{1}}_{=0}\Big] \right\}$$

$$= \frac{8 y_0}{\sin\left(\frac{n\pi x_0}{L}\right)} \left(\frac{1}{(2\pi n)^2}\right)(-1)^{n+1} \frac{\sin(n\pi\alpha)}{\alpha(1-\alpha)}$$

$$= \frac{8 y_0}{\cancel{\sin\left(\frac{n\pi x_0}{L}\right)}} \left(\frac{(-1)^{n+1}}{(2\pi n)^2}\right) \frac{\cancel{\sin(n\pi\alpha)}}{\alpha(1-\alpha)} \qquad \left(\text{since } \alpha = \frac{x_0}{L}\right)$$

$$= (-1)^{n+1} \left(\frac{2 y_0}{\pi^2 n^2}\right) \frac{1}{\alpha\beta}$$

$$= (-1)^{n+1} \left(\frac{2}{\pi^2 n^2}\right) \frac{y_0}{\alpha\beta}$$

where $\beta = 1-\alpha$

$y_0 = h\sin(\pi\alpha)$

$\boxed{0 < \alpha < 1}$

$$\boxed{B_n = (-1)^{n+1} \frac{2}{\pi^2 n^2} \frac{h\sin(\pi\alpha)}{\alpha(1-\alpha)}}$$

Thus,

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\cancelto{0}{A_n} \cos\left(\frac{n 2\pi t}{T}\right) + B_n \sin\left(\frac{n2\pi t}{T}\right)\right]$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{2}{\pi^2 n^2}\right)\frac{h \sin(\pi\alpha)}{\alpha(1-\alpha)} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n2\pi t}{T}\right)$$

$$= \left(\frac{2}{\pi^2}\right) h \frac{\sin(\pi\alpha)}{\alpha(1-\alpha)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n2\pi t}{T}\right)$$

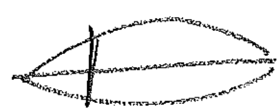
---

plot this as a function of x for each value of t

Take: $L = 1\,m$, $T = 1\,sec$, $h = 0.1\,m$ | $T = \frac{2L}{v}$, $v = \frac{2L}{T}$

$\rightarrow v = \frac{2L}{T} = 2\ m/s$

$\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots$



$n_{max} = 10$