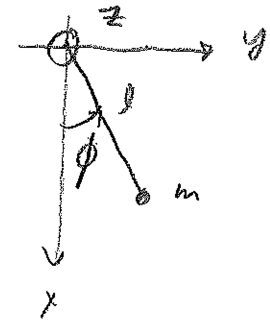
Problem: 8.1 M for simple and physical pendula

a)



$x = l\cos\phi$, $\quad q = \phi$ (generalized coord)
$y = l\sin\phi$

$$\vec{r} = l\cos\phi\,\hat{x} + l\sin\phi\,\hat{y}$$

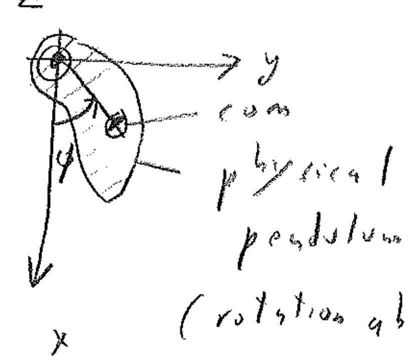
$$\frac{\partial\vec{r}}{\partial q} = -l\sin\phi\,\hat{x} + l\cos\phi\,\hat{y}$$

$$M = m\,\frac{\partial\vec{r}}{\partial q}\cdot\frac{\partial\vec{r}}{\partial q}$$

$$= m\left[l^2\sin^2\phi + l^2\cos^2\phi\right]$$

$$= \boxed{ml^2}$$

b)



(rotation about z-axis)

Mass point m_I $(I = 1, 2, \cdots N)$

ϕ : azimuthal angle of com
$= q$ (generalized coord)

$$\vec{r}_I(\phi) = \left[\sin\theta_I\cos(\phi_I - \phi)\,\hat{x} + \sin\theta_I\sin(\phi_I - \phi)\,\hat{y} + \cos\theta_I\,\hat{z}\right] r_I$$

$$\frac{\partial\vec{r}_I}{\partial\phi} = \left[\sin\theta_I\sin(\phi_I - \phi)\,\hat{x} - \sin\theta_I\cos(\phi_I - \phi)\,\hat{y}\right] r_I$$

Thus,

$$M = \sum_I m_I \frac{\partial \vec{r}_I}{\partial \phi} \cdot \frac{\partial \vec{r}_I}{\partial \phi}$$

$$= \sum_I m_I \left[\sin^2\theta_I \sin^2(\phi_I - \phi) + \sin^2\theta_I \cos^2(\phi_I - \phi) \right] r_I^2$$

$$= \sum_I m_I r_I^2 \sin^2\theta_I \underbrace{\left(\sin^2(\phi_I - \phi) + \cos^2(\phi_I - \phi) \right)}_{=1}$$

$$= \sum_I m_I r_I^2 \sin^2\theta_I$$

$$= I(\hat{z}) \qquad \text{[rotational inertia about } \hat{z}\text{]}$$

Problem: (8.2) Damped oscillator, real solution

$$\eta(t) = \mathrm{Re}\left[Z_+ e^{i(\sqrt{\omega_0^2-\varphi^2}+i\varphi)t} + Z_- e^{i(-\sqrt{\omega_0^2-\varphi^2}+i\varphi)t} \right]$$

$$= \mathrm{Re}\left[e^{-\varphi t} \left(Z_+ e^{i\sqrt{\omega_0^2-\varphi^2}\,t} + Z_- e^{-i\sqrt{\omega_0^2-\varphi^2}\,t} \right) \right]$$

$$= \frac{1}{2} e^{-\varphi t} \left\{ \left(Z_+ e^{i\sqrt{\ }t} + Z_- e^{-i\sqrt{\ }t} \right) + \left(Z_+^* e^{-i\sqrt{\ }t} + Z_-^* e^{i\sqrt{\ }t} \right) \right\}$$

$$= \frac{1}{2} e^{-\varphi t} \left\{ (Z_+ + Z_-^*) e^{i\sqrt{\ }t} + (Z_+^* + Z_-) e^{-i\sqrt{\ }t} \right\}$$

$$= \frac{1}{2} e^{-\varphi t} \left[(Z_+ + Z_-^*) e^{i\sqrt{\ }t} + c.c. \right]$$

$$= \frac{1}{2} e^{-\varphi t} \left[|Z_+ + Z_-^*| e^{i\phi} e^{i\sqrt{\ }t} + c.c. \right]$$

$$= \frac{1}{2} e^{-\varphi t} \left[A e^{i(\sqrt{\ }t+\phi)} + A e^{-i(\sqrt{\ }t+\phi)} \right]$$

$$= A e^{-\varphi t} \cos(\sqrt{\ }t + \phi)$$

$$= A e^{-\varphi t} \cos\left(\sqrt{\omega_0^2-\varphi^2}\; t + \phi \right)$$

$$= A e^{-\varphi t} \cos\left(\omega_0 t \sqrt{1 - \frac{\varphi^2}{\omega_0^2}} + \phi \right)$$

Problem 8.3 Damped, driven harmonic oscillator

$$\ddot{\eta} + 2\varphi\dot{\eta} + \omega_0^2\eta = F_0\cos(\omega t+\delta)$$

a) Particular solution

$$\eta_p = \mathrm{Re}[\zeta]\,,$$ where $\zeta(t)$ satisfies differential equation with $F_0 e^{i(\omega t+\delta)}$ as RHS.

Assume: $$\zeta = E\, e^{i(\omega t+\delta)}$$

$$\rightarrow \quad \dot{\zeta} = i\omega\zeta$$
$$\ddot{\zeta} = -\omega^2\zeta$$

Thus,
$$-\omega^2\zeta + 2i\omega\varphi\zeta + \omega_0^2\zeta = F_0 e^{i(\omega t+\delta)}$$

$$(-\omega^2 + 2i\omega\varphi + \omega_0^2)E\cancel{e^{i(\omega t+\delta)}} = F_0\cancel{e^{i(\omega t+\delta)}}$$

$$\left[(\omega_0^2-\omega^2) + 2i\omega\varphi\right]E = F_0$$

$$\rightarrow \quad E = \frac{F_0}{(\omega_0^2-\omega^2)+2i\omega\varphi}\,,\quad \frac{(\omega_0^2-\omega^2)-2i\omega\varphi}{(\omega_0^2-\omega^2)-2i\omega\varphi}$$

$$= \frac{F_0\left[(\omega_0^2-\omega^2)-2i\omega\varphi\right]}{(\omega_0^2-\omega^2)^2+4\omega^2\varphi^2}$$

$$= \frac{F_0}{(\omega_0^2-\omega^2)^2+4\omega^2\varphi^2}\, A e^{i\alpha}$$

$$A = \sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\varphi^2}$$

$$\tan\alpha = \frac{-2\omega\varphi}{(\omega_0^2 - \omega^2)}, \quad \boxed{\alpha = \arctan\left[\frac{-2\omega\varphi}{\omega_0^2 - \omega^2}\right]}$$

Thus,

$$E = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\varphi^2}} e^{i\alpha}$$

so $$\zeta = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\varphi^2\omega^2}} e^{i[\omega t + \delta + \alpha]}$$

$$\rightarrow \boxed{\eta_p = \mathrm{Re}(\zeta) = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\varphi^2\omega^2}} \cos(\omega t + \delta + \alpha)}$$

b) Find max amplitude (resonant freqs):

$$\text{Maximize} \quad \frac{F_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\phi^2\omega^2}} = A(\omega)$$

① Vary ω:

$$0 = \left.\frac{dA}{d\omega}\right|_{\omega_0 \text{ fixed}} = \frac{-\frac{1}{2}F_0}{[\quad]^{3/2}}\left[2(\omega_0^2-\omega^2)(-2\omega) + 8\phi^2\omega\right]$$

$$\to \quad -4\omega(\omega_0^2-\omega^2) + 8\phi^2\omega = 0$$

$$-4\omega\left[\omega_0^2-\omega^2 - 2\phi^2\right] = 0$$

$$\to \boxed{\omega^2 = \omega_0^2 - 2\phi^2}$$

② Vary ω_0:

$$0 = \left.\frac{dA}{d\omega_0}\right|_{\omega \text{ fixed}} = \frac{-\frac{1}{2}F_0}{[\quad]^{3/2}}\, 2(\omega_0^2-\omega^2)\,2\omega_0$$

$$\to \quad \omega_0(\omega_0^2-\omega^2) = 0$$

$$\omega_0 = 0, \quad \boxed{\omega^2 = \omega_0^2}$$

Exercise 8.4

$$\eta^a = Re\left[\sum_b Z_{ab}\, C_b\, e^{i\omega_b t}\right]$$

$$\eta_0 = \eta(t=0)$$
$$= Re[\,Z\,C\,]$$
$$= Z\,Re(C)$$

$$\Omega_{ab} = \omega_a^2\,\delta_{ab}$$

$$\Omega = \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}$$

$$\dot\eta_0 = \dot\eta(t=0)$$
$$= Re\left[\sum_b i Z_{ab}\,\omega_b\, C_b\right]$$
$$= Re\left[\sum_b i Z_{ab}\sqrt{\Omega_{bc}}\, C_c\right]$$
$$= -Z\,\Omega^{1/2}\, Im(C)$$

$$C = Re\,C + i\,Im\,C$$
$$iC = i\,Re\,C - Im\,C$$

Thus
$$\boxed{\begin{aligned}\eta_0 &= Z\,Re(C)\\ \dot\eta_0 &= -Z\Omega^{1/2}\,Im(C)\end{aligned}} \quad (8.64)$$

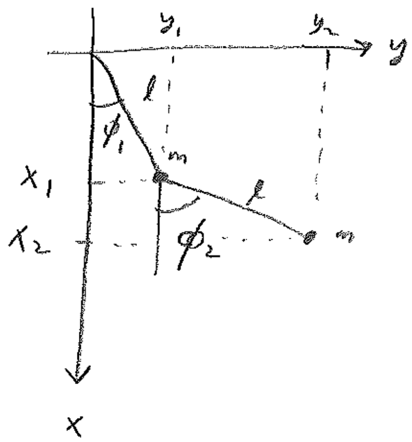
Multiply on left by $Z^T T$

$$Z^T T\eta_0 = Z^T T Z\; Re(C) = Re(C)$$
$$-Z^T T\dot\eta_0 = Z^T T Z\,\Omega^{1/2}\; Im(C) = \Omega^{1/2}\, Im(C)$$

so $$Im(C) = -\Omega^{-1/2} Z^T T\dot\eta_0$$

$$\rightarrow \boxed{C = Re(C) + i\,Im(C) = Z^T T\eta_0 - i\,\Omega^{-1/2} Z^T T\dot\eta_0}$$

Problem (8.5) T and U for double pendulum



$x_1 = l\cos\phi_1\ ,\ y_1 = l\sin\phi_1$

$x_2 = l\cos\phi_1 + l\cos\phi_2$

$y_2 = l\sin\phi_1 + l\sin\phi_2$

$$T_{ab} = \sum_I m_I \frac{\partial \vec{r}_I}{\partial q^a}\cdot\frac{\partial \vec{r}_I}{\partial q^b}$$

$$= m\left[\frac{\partial \vec{r}_1}{\partial \phi_a}\cdot\frac{\partial \vec{r}_1}{\partial \phi_b} + \frac{\partial \vec{r}_2}{\partial \phi_a}\cdot\frac{\partial \vec{r}_2}{\partial \phi_b}\right]$$

$$\vec{r}_1 = x_1\hat{x} + y_1\hat{y}$$

$$= l\,(\cos\phi_1\hat{x} + \sin\phi_1\hat{y})$$

$$\vec{r}_2 = x_2\hat{x} + y_2\hat{y}$$

$$= l\left[(\cos\phi_1 + \cos\phi_2)\hat{x} + (\sin\phi_1 + \sin\phi_2)\hat{y}\right]$$

$$T_{11} = m\left[\frac{\partial \vec{r}_1}{\partial \phi_1}\cdot\frac{\partial \vec{r}_1}{\partial \phi_1} + \frac{\partial \vec{r}_2}{\partial \phi_1}\cdot\frac{\partial \vec{r}_2}{\partial \phi_1}\right]$$

$$= ml^2\left[(-\sin\phi_1\hat{x} + \cos\phi_1\hat{y})\cdot(-\sin\phi_1\hat{x} + \cos\phi_1\hat{y})\right.$$

$$\left. + \text{ same}\right]$$

$$= 2ml^2\left[\sin^2\phi_1 + \cos^2\phi_1\right]$$

$$= \boxed{2ml^2}$$

$$T_{22} = m\left[\cancelto{0}{\frac{\partial \vec{r}_1}{\partial \phi_2}}\cdot\cancelto{0}{\frac{\partial \vec{r}_1}{\partial \phi_2}} + \frac{\partial \vec{r}_2}{\partial \phi_2}\cdot\frac{\partial \vec{r}_2}{\partial \phi_2}\right]$$

$$= ml^2\left[(-\sin\phi_2\hat{x} + \cos\phi_2\hat{y})\cdot(-\sin\phi_2\hat{x} + \cos\phi_2\hat{y})\right]$$

$$= \boxed{ml^2}$$

$$T_{12} = m\left[\cancel{\frac{\partial \vec{r}_1}{\partial \phi_1}\cdot\frac{\partial \vec{r}_1}{\partial \phi_2}}^{\,0} + \frac{\partial \vec{r}_2}{\partial \phi_1}\cdot\frac{\partial \vec{r}_2}{\partial \phi_2}\right]$$

$$= ml^2\left[\left(-\sin\phi_1\,\hat{x} + \cos\phi_1\,\hat{y}\right)\cdot\left(-\sin\phi_2\,\hat{x} + \cos\phi_2\,\hat{y}\right)\right]$$

$$= ml^2\left[\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2\right]$$

$$= \boxed{ml^2\cos(\phi_1-\phi_2)}$$

$$T_{21} = T_{12} \quad \text{(same)}$$

Thus

$$T = \frac{1}{2}\sum_{a,b} T_{ab}\,\dot{q}^a\dot{q}^b$$

$$= \frac{1}{2}ml^2 \begin{pmatrix}\dot{\phi}_1 & \dot{\phi}_2\end{pmatrix}\begin{pmatrix}2 & \cos(\phi_1-\phi_2)\\ \cos(\phi_1-\phi_2) & 1\end{pmatrix}\begin{pmatrix}\dot{\phi}_1\\ \dot{\phi}_2\end{pmatrix}$$

$$= \frac{1}{2}ml^2 \begin{pmatrix}\dot{\phi}_1 & \dot{\phi}_2\end{pmatrix}\begin{pmatrix}2\dot{\phi}_1 + \cos(\phi_1-\phi_2)\dot{\phi}_2\\ \cos(\phi_1-\phi_2)\dot{\phi}_1 + \dot{\phi}_2\end{pmatrix}$$

$$= \frac{1}{2}ml^2\left(2\dot{\phi}_1^2 + \cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2 + \cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2 + \dot{\phi}_2^2\right)$$

$$= \frac{1}{2}ml^2\left(2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\right)$$

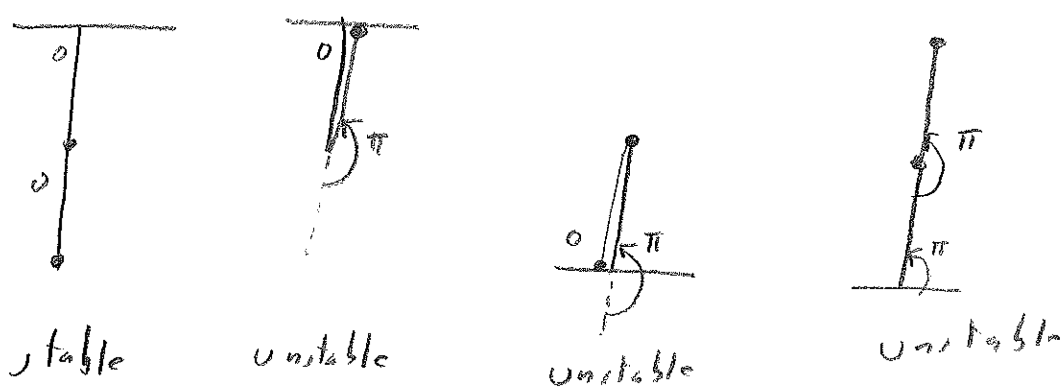
$$U = -mgx_1 - mgx_2$$
$$= -mg[l\cos\phi_1 + l\cos\phi_1 + l\cos\phi_2]$$
$$= -mgl[2\cos\phi_1 + \cos\phi_2]$$

Equilibrium configuration:

$$\frac{\partial U}{\partial \phi_1} = 2mgl\sin\phi_1 = 0 \rightarrow \phi_1 = 0, \pi$$

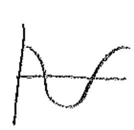
$$\frac{\partial U}{\partial \phi_2} = mgl\sin\phi_2 = 0 \rightarrow \phi_2 = 0, \pi$$

Thus $(\phi_1, \phi_2) = (0,0), (0,\pi), (\pi,0), (\pi,\pi)$



NOTE: $\frac{\partial^2 U}{\partial\phi_1 \partial\phi_2} = 0$, $\frac{\partial^2 U}{\partial \phi_1^2} = 2mgl\cos\phi_1$ > 0 if $\phi_1 = 0$, < 0 if $\phi_1 = \pi$

$$\frac{\partial^2 U}{\partial \phi_2^2} = mgl\cos\phi_2 \quad > 0 \text{ if } \phi_2 = 0, \quad < 0 \text{ if } \phi_2 = \pi$$



Required

$$\sum_{a,b} \frac{\partial^2 U}{\partial q^a \partial q^b}\bigg|_0 \eta^a \eta^b \geq 0$$

Thus, $\begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} \; mgl \begin{bmatrix} 2\cos\phi_1 & 0 \\ 0 & \cos\phi_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$

$$= mgl\left(2\cos\phi_1\,\eta_1^2 + \cos\phi_2\,\eta_2^2\right)$$

Want RHS $\geq 0 \quad \forall \; \eta_1, \eta_2$

For $(0,0) \rightarrow mgl\,(2\eta_1^2 + \eta_2^2) \geq 0$

$(0,\pi) \rightarrow mgl\,(2\eta_1^2 - \eta_2^2)$

$(\pi,0) \rightarrow mgl\,(-2\eta_1^2 + \eta_2^2)$

can be < 0 for sufficiently large η_2 or η_1

$(\pi,\pi) \rightarrow mgl\,(-2\eta_1^2 - \eta_2^2) \leq 0$

Thus, $(0,0)$ is only stable equilibrium point

Evaluate at equilibrium

$$T_{ab}\Big|_0 = ml^2 \begin{bmatrix} 2 & \cos(0-0) \\ \cos(0-0) & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} ml^2$$

$$\cancel{U_{ab} = \left(\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right)\bigg|_0}$$

$$U = -mgl\left[2\cos\phi_1 + \cos\phi_2\right]$$

$$\frac{\partial^2 U}{\partial\phi_a\partial\phi_b} = mgl\begin{bmatrix} 2\cos\phi_1 & 0 \\ 0 & \cos\phi_2 \end{bmatrix}$$

$$\rightarrow \quad U_{ab} = \left.\frac{\partial^2 U}{\partial\phi_a\partial\phi_b}\right|_0 = mgl\begin{bmatrix} 2\cos 0 & 0 \\ 0 & \cos 0 \end{bmatrix}$$

$$= mgl\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem: 8.6 Eigenvalues / eigenvectors for double pendulum

Given $T = m\ell^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $U = mg\ell \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Eigenvalues:

$$0 = \det(U - \omega^2 T)$$

$$= \det \begin{bmatrix} 2mg\ell - \omega^2 2m\ell^2 & -\omega^2 m\ell^2 \\ -\omega^2 m\ell^2 & mg\ell - \omega^2 m\ell^2 \end{bmatrix}$$

$$= \det m\ell^2 \begin{bmatrix} 2\frac{g}{\ell} - 2\omega^2 & -\omega^2 \\ -\omega^2 & \frac{g}{\ell} - \omega^2 \end{bmatrix}$$

$$= (m\ell^2)^2 \left(2\left(\frac{g}{\ell} - \omega^2\right)\left(\frac{g}{\ell} - \omega^2\right) - \omega^4 \right)$$

$$= (m\ell^2)^2 \left[2\left(\frac{g}{\ell}\right)^2 + 2\omega^4 - 4\left(\frac{g}{\ell}\right)\omega^2 - \omega^4 \right]$$

$$= (m\ell^2)^2 \left[\omega^4 - 4\left(\frac{g}{\ell}\right)\omega^2 + 2\left(\frac{g}{\ell}\right)^2 \right]$$

Quadratic equation for ω^2:

$$2\sqrt{2}\,\frac{g}{\ell} = \sqrt{8}\,\frac{g}{\ell} = \sqrt{16\left(\frac{g}{\ell}\right)^2 - 4\cdot 1\cdot 2\left(\frac{g}{\ell}\right)^2}$$

$$\omega^2 = \frac{4\left(\frac{g}{\ell}\right) \pm \sqrt{16\left(\frac{g}{\ell}\right)^2 - 4\cdot 1\cdot 2\left(\frac{g}{\ell}\right)^2}}{2}$$

$$= \left(\frac{g}{\ell}\right)\left[2 \pm \sqrt{2}\right]$$

so $\boxed{\omega_\pm^2 = \left(\frac{g}{\ell}\right)\left[2 \pm \sqrt{2}\right]}$

Eigenvectors:

$$\left(U - \omega_\pm^2 T\right) Z_\pm = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \left(\cancel{m} g l \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \omega_\pm^2 \cancel{m} l^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right) A_\pm$$

$$= l^2 \left(\frac{g}{l} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \omega_\pm^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right) A_\pm$$

$$= l^2 \begin{bmatrix} 2\left(\frac{g}{l} - \omega_\pm^2\right) & -\omega_\pm^2 \\ -\omega_\pm^2 & \frac{g}{l} - \omega_\pm^2 \end{bmatrix} A_\pm$$

Consider 1st ω_+^2:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\left(\frac{g}{l} - \omega_+^2\right) & -\omega_+^2 \\ -\omega_+^2 & \frac{g}{l} - \omega_+^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\left(\frac{g}{l} - \frac{g}{l}(2+\sqrt{2})\right) & -\frac{g}{l}(2+\sqrt{2}) \\ -\frac{g}{l}(2+\sqrt{2}) & \frac{g}{l} - \frac{g}{l}(2+\sqrt{2}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \frac{g}{l} \begin{bmatrix} 2\left(1-2-\sqrt{2}\right) & -(2+\sqrt{2}) \\ -(2+\sqrt{2}) & 1-2-\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \frac{g}{l}\begin{bmatrix} -2(1+\sqrt{2}) & -(2+\sqrt{2}) \\ -(2+\sqrt{2}) & -(1+\sqrt{2}) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Thus, $-2(1+\sqrt{2})v_1 - (2+\sqrt{2})v_2 = 0$

$$v_2 = \frac{-2(1+\sqrt{2})}{(2+\sqrt{2})}\frac{(2-\sqrt{2})}{(2-\sqrt{2})}v_1$$

$$= \frac{-2(\cancel{2}-\cancel{2}+2\sqrt{2}-\sqrt{2})}{4-2}v_1$$

$$= -\sqrt{2}\,v_1$$

$$\rightarrow \boxed{Z_+ = N_+\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}}$$

Consider ω_-^2:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\left(\frac{g}{l} - \frac{g}{l}(2-\sqrt{2})\right) & -\frac{g}{l}(2-\sqrt{2}) \\ -\frac{g}{l}(2-\sqrt{2}) & \frac{g}{l} - \frac{g}{l}(2-\sqrt{2}) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \frac{g}{l}\begin{bmatrix} 2(1-2+\sqrt{2}) & -(2-\sqrt{2}) \\ -(2-\sqrt{2}) & 1-2+\sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \frac{g}{l} \begin{bmatrix} -2(1-\sqrt{2}) & -(2-\sqrt{2}) \\ -(2-\sqrt{2}) & -(1-\sqrt{2}) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Thus, $\quad -2(1-\sqrt{2})V_1 - (2-\sqrt{2})V_2 = 0$

$$V_2 = \frac{-2(1-\sqrt{2})}{(2-\sqrt{2})} \frac{(2+\sqrt{2})}{(2+\sqrt{2})} V_1$$

$$= \frac{-2(\cancel{2} - \cancel{2} - 2\sqrt{2} + \sqrt{2})}{4-2} V_1$$

$$= +\sqrt{2}\, V_1$$

$$\rightarrow \boxed{Z_- = N_- \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}}$$

Summary:

$$w_\pm^2 = \left(\frac{g}{l}\right)\left[2 \pm \sqrt{2}\right]$$

$$Z_\pm = N_\pm \begin{bmatrix} 1 \\ \mp\sqrt{2} \end{bmatrix}$$

Problem (8.7) orthonormality double pendulum

$$Z^T \, T \, Z = \mathbb{1}$$

where $T = ml^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$Z_\pm = N_\pm \begin{bmatrix} 1 \\ \mp\sqrt{2} \end{bmatrix}$$

Orthogonality: (should already be satisfied)

$$Z_+^T \, T \, Z_- = N_+ N_- \begin{bmatrix} 1 & -\sqrt{2} \end{bmatrix} ml^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

$$= N_+ N_- ml^2 \begin{bmatrix} 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 2+\sqrt{2} \\ 1+\sqrt{2} \end{bmatrix}$$

$$= N_+ N_- ml^2 \left(\cancel{2} + \cancel{\sqrt{2}} - \cancel{\sqrt{2}} - \cancel{2} \right)$$

$$= 0$$

similarly

$$Z_-^T \, T \, Z_+ = N_+ N_- \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} ml^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$$

$$= N_+ N_- ml^2 \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 2-\sqrt{2} \\ 1-\sqrt{2} \end{bmatrix}$$

$$= N_+ N_- ml^2 \left(\cancel{2} - \cancel{\sqrt{2}} + \cancel{\sqrt{2}} - \cancel{2} \right)$$

$$= 0.$$

Normalization:

$$1 = z_+^T\, T\, z_+ = N_+^2 \begin{bmatrix} 1 & -\sqrt{2} \end{bmatrix} m l^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$$

$$= N_+^2\, m l^2 \begin{bmatrix} 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 2-\sqrt{2} \\ 1-\sqrt{2} \end{bmatrix}$$

$$= N_+^2\, m l^2 \left(2-\sqrt{2} \; -\sqrt{2}+2\right)$$

$$= N_+^2\, m l^2\, 2\left(2-\sqrt{2}\right)$$

$$\rightarrow \boxed{N_+ = \frac{1}{\sqrt{2 m l^2 (2-\sqrt{2})}}}$$

$$1 = z_-^T\, T\, z_- = N_-^2 \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} m l^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

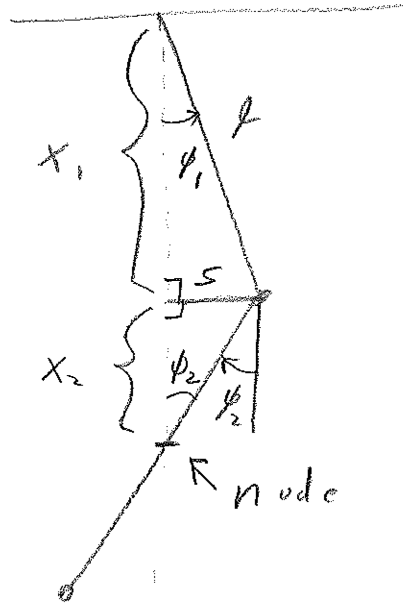
$$= N_-^2\, m l^2 \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 2+\sqrt{2} \\ 1+\sqrt{2} \end{bmatrix}$$

$$= N_-^2\, m l^2 \left(2+\sqrt{2} \; +\sqrt{2}+2\right)$$

$$= N_-^2\, m l^2\, 2\left(2+\sqrt{2}\right)$$

$$\rightarrow \boxed{N_- = \frac{1}{\sqrt{2 m l^2 (2+\sqrt{2})}}}$$

Problem: 8.8 Node for normal mode z_+



$$z_+ = N_+ \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

so $\phi_2 = -\sqrt{2}\,\phi_1$

Assume $\phi_1 \ll 1$
$\phi_2 \ll 1$

Location of node: $x_1 + x_2$

$$x_1 = \ell \cos\phi_1 \approx \ell$$

$$s = \ell \sin\phi_1 \approx \ell\phi_1$$

$$\tan\phi_2 = \frac{s}{x_2} \rightarrow x_2 = \frac{s}{\tan\phi_2}$$

$$\approx \frac{s}{\phi_2}$$

$$\approx \frac{\ell\phi_1}{\phi_2}$$

$$= \frac{\ell}{\sqrt{2}}$$

Thus, $x_1 + x_2 \approx \ell + \frac{\ell}{\sqrt{2}} = \ell\left(1 + \frac{1}{\sqrt{2}}\right) = \ell \cdot 1.707$

Problem: 8.9 Show $Z^T T Z = \mathbb{1}$ and $Z^T U Z = \Omega$

$$T = ml^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad U = mgl \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Z = \begin{bmatrix} N_+ & N_- \\ -\sqrt{2}N_+ & \sqrt{2}N_- \end{bmatrix} = \frac{1}{\sqrt{2ml^2}} \begin{bmatrix} \frac{1}{\sqrt{2-\sqrt{2}}} & \frac{1}{\sqrt{2+\sqrt{2}}} \\ \frac{-\sqrt{2}}{\sqrt{2-\sqrt{2}}} & \frac{\sqrt{2}}{\sqrt{2+\sqrt{2}}} \end{bmatrix}$$

$$Z^T T Z = \left(\frac{1}{\sqrt{2ml^2}}\right) \cdot ml^2 \; Z^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{-}} & \frac{1}{\sqrt{+}} \\ \frac{-\sqrt{2}}{\sqrt{-}} & \frac{\sqrt{2}}{\sqrt{+}} \end{bmatrix}$$

$$= \sqrt{\frac{ml^2}{2}}\, Z^T \begin{bmatrix} \frac{2-\sqrt{2}}{\sqrt{2-\sqrt{2}}} & \frac{2+\sqrt{2}}{\sqrt{2+\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{2-\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{2+\sqrt{2}}} \end{bmatrix}$$

$$= \sqrt{\frac{ml^2}{2}}\, Z^T \begin{bmatrix} \sqrt{2-\sqrt{2}} & \sqrt{2+\sqrt{2}} \\ -\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}} & \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}} \end{bmatrix}$$

$$(1-\sqrt{2})\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}-2}{\sqrt{2}} = \frac{-(2-\sqrt{2})}{\sqrt{2}}$$

$$= \sqrt{\frac{ml^2}{2}} \frac{1}{\sqrt{2ml^2}} \begin{bmatrix} \frac{1}{\sqrt{-}} & \frac{-\sqrt{2}}{\sqrt{-}} \\ \frac{1}{\sqrt{+}} & \frac{\sqrt{2}}{\sqrt{+}} \end{bmatrix} \begin{bmatrix} \sqrt{-} & \sqrt{+} \\ \frac{-\sqrt{-}}{\sqrt{2}} & \frac{\sqrt{+}}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{1}$$

$$\cancel{= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}$$

$$Z^T U Z = \left(\frac{1}{\sqrt{2ml^2}}\right) mgl \; Z^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{-}} & \frac{1}{\sqrt{+}} \\ -\frac{\sqrt{2}}{\sqrt{-}} & \frac{\sqrt{2}}{\sqrt{+}} \end{bmatrix}$$

$$= \sqrt{\frac{m}{2}}\, g \; Z^T \begin{bmatrix} \frac{2}{\sqrt{-}} & \frac{2}{\sqrt{+}} \\ -\frac{\sqrt{2}}{\sqrt{-}} & \frac{\sqrt{2}}{\sqrt{+}} \end{bmatrix}$$

$$= \sqrt{\frac{m}{2}}\, g \; \frac{1}{\sqrt{2ml^2}} \begin{bmatrix} \frac{1}{\sqrt{-}} & -\frac{\sqrt{2}}{\sqrt{-}} \\ \frac{1}{\sqrt{+}} & \frac{\sqrt{2}}{\sqrt{+}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{-}} & \frac{2}{\sqrt{+}} \\ -\frac{\sqrt{2}}{\sqrt{-}} & \frac{\sqrt{2}}{\sqrt{+}} \end{bmatrix}$$

$$= \frac{1}{2}\left(\frac{g}{l}\right) \begin{bmatrix} \frac{2+2}{2-\sqrt{2}} & 0 \\ 0 & \frac{2+2}{2+\sqrt{2}} \end{bmatrix}$$

$$= 2\left(\frac{g}{l}\right) \begin{bmatrix} \frac{1}{2-\sqrt{2}} & 0 \\ 0 & \frac{1}{2+\sqrt{2}} \end{bmatrix}$$

$$\left(\frac{1}{2-\sqrt{2}}\right)\left(\frac{2+\sqrt{2}}{2+\sqrt{2}}\right) = \frac{2+\sqrt{2}}{2}$$

$$\left(\frac{1}{2+\sqrt{2}}\right)\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right) = \frac{2-\sqrt{2}}{2}$$

$$= \left(\frac{g}{l}\right) \begin{bmatrix} 2+\sqrt{2} & 0 \\ 0 & 2-\sqrt{2} \end{bmatrix} = \begin{bmatrix} \omega_+^2 & 0 \\ 0 & \omega_-^2 \end{bmatrix} = \Omega$$

Problem: 8.10 Normal coords for double pendulum

$$\eta = Z\Phi \iff \Phi = Z^T T \eta$$

Thus, $\Phi = Z^T \; T \; \eta$

$$= \frac{1}{\sqrt{2ml^2}} \begin{bmatrix} \frac{1}{\sqrt{-}} & \frac{-\sqrt{2}}{\sqrt{-}} \\ \frac{1}{\sqrt{+}} & \frac{\sqrt{2}}{\sqrt{+}} \end{bmatrix} ml^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$= \sqrt{\frac{ml^2}{2}} \begin{bmatrix} \frac{1}{\sqrt{-}} & \frac{-\sqrt{2}}{\sqrt{-}} \\ \frac{1}{\sqrt{+}} & \frac{\sqrt{2}}{\sqrt{+}} \end{bmatrix} \begin{bmatrix} 2\phi_1 + \phi_2 \\ \phi_1 + \phi_2 \end{bmatrix}$$

$$= \sqrt{\frac{ml^2}{2}} \begin{bmatrix} \dfrac{2\phi_1 + \phi_2 - \sqrt{2}\phi_1 - \sqrt{2}\phi_2}{\sqrt{2-\sqrt{2}}} \\ \dfrac{2\phi_1 + \phi_2 + \sqrt{2}\phi_1 + \sqrt{2}\phi_2}{\sqrt{2+\sqrt{2}}} \end{bmatrix}$$

$$\phi_2(1-\sqrt{2}) = \frac{\phi_2}{\sqrt{2}}(\sqrt{2}-2) = -\frac{\phi_2}{\sqrt{2}}(2-\sqrt{2})$$

$$= \sqrt{\frac{ml^2}{2}} \begin{bmatrix} \sqrt{2-\sqrt{2}}\,\phi_1 - \frac{\phi_2}{\sqrt{2}}\sqrt{2-\sqrt{2}} \\ \sqrt{2+\sqrt{2}}\,\phi_1 + \frac{\phi_2}{\sqrt{2}}\sqrt{2+\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{\frac{ml^2}{2}}\sqrt{2-\sqrt{2}}\left(\phi_1 - \frac{\phi_2}{\sqrt{2}}\right) \\ \sqrt{\frac{ml^2}{2}}\sqrt{2+\sqrt{2}}\left(\phi_1 + \frac{\phi_2}{\sqrt{2}}\right) \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{2N_+}\left(\phi_1 - \frac{\phi_2}{\sqrt{2}}\right) \\ \frac{1}{2N_-}\left(\phi_1 + \frac{\phi_2}{\sqrt{2}}\right) \end{bmatrix}}$$

NOTE: $N_\pm = \dfrac{1}{\sqrt{2ml^2/(2\mp\sqrt{2})}} \rightarrow \dfrac{1}{2N_\pm} = \sqrt{\dfrac{ml^2}{2}}\sqrt{2\mp\sqrt{2}}$

Problem 8.11 Eigenvectors for linear triatomic molecule (1)

Given: $\omega_1 = 0$, $\omega_2 = \sqrt{\frac{k}{M}}$, $\omega_3 = \sqrt{\frac{k(2M+m)}{Mm}}$

Solve: $(U - \omega_a^2 T) Z_{(a)} = 0$

where $T = \begin{bmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{bmatrix}$, $U = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$

$\underline{\omega_1 = 0:}$

$$\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_1 - v_2 = 0 \rightarrow \boxed{v_2 = v_1}$

$-v_1 + 2v_2 - v_3 = 0$

$-v_2 + v_3 = 0 \rightarrow \boxed{v_3 = v_2 = v_1}$

$-v_1 + 2v_1 - v_1 = 0$ ✓ (automatically satisfied)

Thus, $\boxed{Z_1 = N_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$

$$\omega_2 = \sqrt{\frac{k}{m}} :$$

$$\begin{bmatrix} k - m\left(\frac{k}{m}\right) & -k & 0 \\ -k & 2k - m\left(\frac{k}{m}\right) & -k \\ 0 & -k & k - m\left(\frac{k}{m}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -k & 0 \\ -k & 2k - k\frac{m}{m} & -k \\ 0 & -k & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, $-kv_2 = 0 \rightarrow \boxed{v_2 = 0}$

$$-v_1 + \left(2 - \frac{m}{m}\right)\underbrace{\cancel{v_2}}_{0} - v_3 = 0 \rightarrow \boxed{v_3 = -v_1}$$

$-kv_2 = 0$ (already satisfied)

Thus, $\boxed{Z_2 = N_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$

$$\omega_3 = \sqrt{\frac{k\,(2M+m)}{Mm}}$$

$$\begin{bmatrix} k - \cancel{M}\dfrac{k(2M+m)}{\cancel{M}m} & -k & 0 \\ -k & 2k - \cancel{m}\dfrac{k(2M+m)}{M\cancel{m}} & -k \\ 0 & -k & k - \cancel{M}\dfrac{k(2M+m)}{\cancel{M}m} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k \begin{bmatrix} -2\frac{M}{m} & -1 & 0 \\ -1 & -\frac{m}{M} & -1 \\ 0 & -1 & -2\frac{M}{m} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, $-2\frac{M}{m} v_1 - v_2 = 0 \rightarrow \boxed{v_2 = -2\frac{M}{m} v_1}$

$-v_1 - \frac{m}{M} v_2 - v_3 = 0$

$-v_2 - 2\frac{M}{m} v_3 = 0 \rightarrow v_3 = \frac{-m}{2M} v_2 = v_1$

$-v_1 - \frac{m}{M} v_2 - v_1 = 0 \rightarrow v_2 = \frac{-2M}{m} v_1$ ✓

(already satisfied)

Thus $Z_3 = N_3 \begin{bmatrix} 1 \\ -2\frac{M}{m} \\ 1 \end{bmatrix} = N_3 \begin{bmatrix} m \\ -2M \\ m \end{bmatrix}$

Normalization:

$$Z^T T Z = 1$$

$$Z_{(b)}^T \, T \, Z_{(a)} = \delta_{ab}$$

Thus, for $T = \begin{bmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{bmatrix}$

we have $M\left(\left(Z^1_{(a)}\right)^2 + \left(Z^3_{(a)}\right)^2\right) + m\left(Z^2_{(a)}\right)^2 = 1$

Thus, $N_1^2\left[(1+1)M + m\right] = 1 \rightarrow N_1 = \frac{1}{\sqrt{2M+m}}$

$N_2^2\left[(1+1)M + 0\right] = 1 \rightarrow N_2 = \frac{1}{\sqrt{2M}}$

$N_3^2\left[\underbrace{(m^2+m^2)M + 4M^2 m}_{2m^2M + 4M^2 m}\right] = 1 \rightarrow N_3 = \frac{1}{\sqrt{2mM(2M+m)}}$

$= 2mM(m+2M)$

Problem (8.12) Center of mass for oscillatory normal modes for linear triatomic molecule

$$\eta_{com} = \frac{M\eta_1 + m\eta_2 + M\eta_3}{m+2M}$$

$$Z_{(2)} = \frac{1}{\sqrt{2M}} \boxed{\begin{matrix} 1 \\ 0 \\ -1 \end{matrix}} , \quad Z_{(3)} = \frac{1}{\sqrt{\frac{2M}{m}(2M+m)}} \boxed{\begin{matrix} 1 \\ -2M/m \\ 1 \end{matrix}}$$

For $Z_{(2)}$:

$$\eta_{com} = \frac{1}{\sqrt{2M}} \frac{\left(M\cdot 1 + \cancel{m\cdot 0}^{\,0} + M(-1)\right)}{m+2M}$$

$$= \boxed{0}$$

For $Z_{(3)}$:

$$\eta_{com} = \frac{1}{\sqrt{\frac{2M}{m}(2M+m)}} \frac{\left(M\cdot 1 + \overbrace{m\left(\frac{-2M}{m}\right)}^{-2M} + M\cdot 1\right)}{m+2M}$$

$$= \boxed{0}$$

So $\eta_{com} = 0$ for both $Z_{(2)}$, $Z_{(3)}$.

Problem (8.13) Recursion relation for det A

$$A = \begin{array}{c} \\ N \downarrow \end{array} \overset{\xrightarrow{\quad N \quad}}{\begin{vmatrix} -\lambda & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -\lambda & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -\lambda & -1 & \cdots & \cdots & \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & -\lambda \end{vmatrix}}$$

$A^{(N)}$: $N \times N$ matrix

$$\det A^{(N)} = -\lambda \det A^{(N-1)} + 1 \cdot \det \begin{vmatrix} -1 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -\lambda & -1 & \cdots & \cdots & 0 \\ & & \vdots & & & \\ 0 & 0 & \cdots & & -\lambda & -1 \\ 0 & 0 & \cdots & & -1 & -\lambda \end{vmatrix}$$

$$\underbrace{\qquad\qquad\qquad}_{-1 \cdot \det A^{(N-2)}}$$

$$+1 \cdot \det \begin{vmatrix} 0 & -1 & \\ 0 & & \\ \vdots & & \\ 0 & & \end{vmatrix} \rightarrow 0$$

$\uparrow$ all zeroes

$$= -\lambda \det\left(A^{(N-1)}\right) - \det\left(A^{N-2}\right)$$

Problem 8.14 Solving for C_+, C_- for loaded string

$$\det A^{(N)} = C_+ e^{iN\gamma} + C_- e^{-iN\gamma}$$

For $N=1$: $\det A^{(1)} = \det \begin{bmatrix} -\lambda \end{bmatrix} = -\lambda = 2\cos\gamma$

For $N=2$: $\det A^{(2)} = \det \begin{bmatrix} -\lambda & -1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = 4\cos^2\gamma - 1$

Thus,
$$-\lambda = C_+ e^{i\gamma} + C_- e^{-i\gamma}$$
$$\lambda^2 - 1 = C_+ e^{i2\gamma} + C_- e^{-i2\gamma}$$

$$\begin{bmatrix} 2\cos\gamma \\ 4\cos^2\gamma - 1 \end{bmatrix} = \begin{bmatrix} e^{i\gamma} & e^{-i\gamma} \\ e^{i2\gamma} & e^{-i2\gamma} \end{bmatrix} \begin{bmatrix} C_+ \\ C_- \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} C_+ \\ C_- \end{bmatrix} = \frac{1}{e^{-i\gamma} - e^{i\gamma}} \begin{bmatrix} e^{-i2\gamma} & -e^{-i\gamma} \\ -e^{i2\gamma} & e^{i\gamma} \end{bmatrix} \begin{bmatrix} 2\cos\gamma \\ 4\cos^2\gamma - 1 \end{bmatrix}$$

$$= \frac{1}{-2i\sin\gamma} \begin{bmatrix} 2\cos\gamma\, e^{-i2\gamma} - (4\cos^2\gamma - 1)e^{-i\gamma} \\ -2\cos\gamma\, e^{i2\gamma} + (4\cos^2\gamma - 1)e^{i\gamma} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Now:

$$C_+ = \frac{i}{2\sin\gamma}\left[\left(e^{i\gamma}+e^{-i\gamma}\right)e^{-i2\gamma} - \left(e^{i\gamma}+e^{-i\gamma}\right)^2 e^{-i\gamma} + e^{-i\gamma}\right]$$

$$= \frac{i}{2\sin\gamma}\left[e^{-i\gamma}+e^{-i3\gamma} - \left(e^{i2\gamma}+e^{-i2\gamma}+2\right)e^{-i\gamma}+e^{-i\gamma}\right]$$

$$= \frac{i}{2\sin\gamma}\left[\cancel{e^{-i\gamma}}+\cancel{e^{-i3\gamma}} - e^{i\gamma} - \cancel{e^{-i3\gamma}} - \cancel{2e^{-i\gamma}} + \cancel{e^{-i\gamma}}\right]$$

$$= \boxed{\frac{-ie^{i\gamma}}{2\sin\gamma}}$$

$$C_- = \frac{i}{2\sin\gamma}\left[-\left(e^{i\gamma}+e^{-i\gamma}\right)e^{i2\gamma} + \left(e^{i\gamma}+e^{-i\gamma}\right)^2 e^{i\gamma} - e^{i\gamma}\right]$$

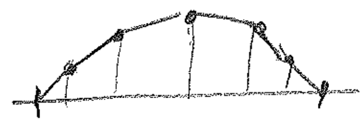
$$= C_+^* \quad \text{(by inspection)}$$

$$= \frac{ie^{-i\gamma}}{2\sin\gamma}$$

Problem (8.15) (N=5) normal modes for loaded string.

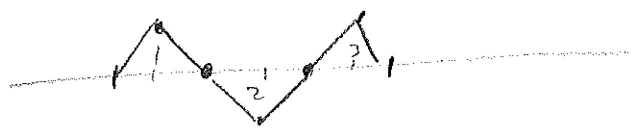
$$z_{(b)}^{a} = A_b \sin\left(\frac{ab\pi}{6}\right)$$

$b=1$: $\sin\left(\frac{a\pi}{6}\right)$: $\{\sin\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{2}\right), \sin\left(\frac{2\pi}{3}\right), \sin\left(\frac{5\pi}{6}\right)\}$



$b=2$: $\sin\left(\frac{a\pi}{3}\right)$: $\{\sin\left(\frac{\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right), \sin(\pi), \sin\left(\frac{4\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right)\}$

$b=3$: $\sin\left(\frac{a\pi}{2}\right)$: $\{\sin\left(\frac{\pi}{2}\right), \sin\pi, \sin\left(\frac{3\pi}{2}\right), \sin(2\pi), \sin\left(\frac{5\pi}{2}\right)$



$b=4$: $\sin\left(\frac{2a\pi}{3}\right)$: $\{\sin\left(\frac{2\pi}{3}\right), \sin\left(\frac{4\pi}{3}\right), \sin(2\pi), \sin\left(\frac{2\pi}{3}\right), \left(\frac{4\pi}{3}\right)\}$

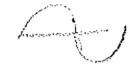
$b=5$: $\sin\left(\frac{a5\pi}{6}\right)$: $\{\sin\left(\frac{5\pi}{6}\right), \sin\left(\frac{5\pi}{3}\right), \sin\left(\frac{5\pi}{2}\right), \sin\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{6}\right)\}$

$\sin\left(\frac{\pi}{2}\right)$

$8\frac{\pi}{3} =$

$\frac{10\pi}{3}$

$\frac{25}{6}\pi = (4 + \frac{1}{6})\pi$



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Problem: Non-trivial solution of $Cx = 0$ for real C ①

$n \times n$ matrix

8.1

a) $Cx = 0$ (matrix equation)

If $\det C \neq 0$ then C^{-1} exists

$$\rightarrow \quad C^{-1} C x = 0$$
$$\mathbb{1} x = 0$$
$$x = 0 .$$

b) Non-trivial solutions therefore require $\det C = 0$.

$\det C = 0$ iff at least one of the n equations in $Cx = 0$ is redundant.

Assume that the n^{th} equation is the only redundant equation.

Then:

$$\begin{aligned} C_{11} x^1 + C_{12} x^2 + \cdots + C_{1,n-1} x^{n-1} + C_{1n} x^n &= 0 \\ C_{21} x^1 + C_{22} x^2 + \cdots + C_{2,n-1} x^{n-1} + C_{2n} x^n &= 0 \\ &\vdots \\ C_{n-1,1} x^1 + C_{n-1,2} x^2 + \cdots + C_{n-1,n-1} x^{n-1} + C_{n-1,n} x^n &= 0 \end{aligned}$$

is equivalent to

$$D y = z$$

where $D = \begin{bmatrix} C_{11} & \cdots & C_{1,n-1} \\ & & \\ C_{n-1,1} & \cdots & C_{n-1,n-1} \end{bmatrix}$, $y = \left(\frac{1}{x^n}\right) \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^{n-1} \end{bmatrix}$

$(n-1) \times (n-1)$

$$z = \begin{bmatrix} -C_{1n} \\ -C_{2n} \\ \vdots \\ -C_{n-1,n} \end{bmatrix}$$

c) D^{-1} exist since all $(n-1)$-equations are linearly independent.

Thus, $Dy = z$

$$\rightarrow \quad y = D^{-1} z$$

<u>Now:</u> D^{-1} is real since C is real

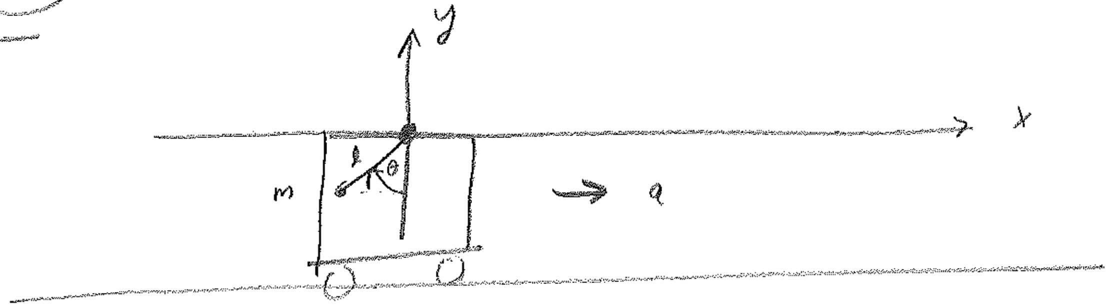
z is also real since $-C_{in}$ are real

Thus, y is real

$$\rightarrow \quad x^1 = y^1 x^n$$
$$x^2 = y^2 x^n$$
$$\vdots$$
$$x^{n-1} = y^{n-1} x^n$$

have at most a common complex phase factor x^n. [<u>NOTE</u>: we can assume that x^n has unit magnitude since $Cx = 0$ can always be rescaled appropriately]

Prob (8.2)



a) Position specified by (x,y)

$$x = x_{car} - l\sin\theta$$
$$= \frac{1}{2}at^2 \underbrace{+ x_0 + v_0 t}_{\text{set to zero for simplicity}} - l\sin\theta$$
$$= \frac{1}{2}at^2 - l\sin\theta$$

$$y = -l\cos\theta$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$
$$= \frac{1}{2}m\left[\left(at - l\cos\theta\,\dot{\theta}\right)^2 + \left(l\sin\theta\,\dot{\theta}\right)^2\right]$$
$$= \frac{1}{2}m\left[a^2t^2 - 2atl\cos\theta\,\dot{\theta} + l^2\cos^2\theta\,\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\theta}^2\right]$$
$$= \frac{1}{2}m\left[a^2t^2 - 2atl\cos\theta\,\dot{\theta} + l^2\dot{\theta}^2\right]$$

$$U = mgy = -mgl\cos\theta$$

$$L = \frac{1}{2}m\left[l^2\dot{\theta}^2 + a^2t^2 - 2atl\cos\theta\,\dot{\theta}\right] + mgl\cos\theta$$

$$\frac{\partial L}{\partial\theta} = matl\sin\theta\,\dot{\theta} - mgl\sin\theta$$

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} - matl\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = ml^2\ddot\theta + matl\sin\theta\,\dot\theta - mal\cos\theta$$

Thus,
$$0 = \cancel{matl\sin\theta\,\dot\theta} - mgl\sin\theta - ml^2\ddot\theta \cancel{- matl\sin\theta\,\dot\theta} + mal\cos\theta$$

$$= -\cancel{m}gl\sin\theta + \cancel{m}al\cos\theta - \cancel{m}l^2\ddot\theta$$

$$\rightarrow \boxed{\ddot\theta = -\frac{g}{l}\sin\theta + \frac{a}{l}\cos\theta}$$

$\theta = \theta_0$ solution:

$$\ddot\theta = 0 \rightarrow 0 = -\frac{g}{l}\sin\theta_0 + \frac{a}{l}\cos\theta_0$$

$$g\sin\theta_0 = a\cos\theta_0$$

$$\boxed{\tan\theta_0 = \frac{a}{g}}$$

b) small oscillations:

$$\ddot\theta = -\frac{g}{l}\sin\theta + \frac{a}{l}\cos\theta$$

Let: $\eta = \theta - \theta_0 \rightarrow \theta = \eta + \theta_0$

$$\sin\theta = \sin(\eta + \theta_0)$$
$$= \sin\eta\cos\theta_0 + \cos\eta\sin\theta_0$$
$$\approx \eta\cos\theta_0 + \sin\theta_0 \quad \text{(to 1}^{st}\text{ order)}$$

$$\cos\theta = \cos(\eta+\theta_0)$$

$$= \cos\eta\cos\theta_0 - \sin\eta\sin\theta_0$$

$$\approx \cos\theta_0 - \eta\sin\theta_0$$

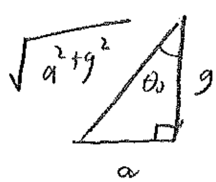
$$\ddot{\theta} = (\eta+\theta_0)^{\cdot\cdot}$$

$$= \ddot{\eta}$$

$$\rightarrow \quad \ddot{\eta} = -\frac{g}{l}\left[\eta\cos\theta_0 + \sin\theta_0\right] + \frac{a}{l}\left[\cos\theta_0 - \eta\sin\theta_0\right]$$

$$= \underbrace{-\frac{g}{l}\sin\theta_0 + \frac{a}{l}\cos\theta_0}_{0} + \eta\left[-\frac{g}{l}\cos\theta_0 - \frac{a}{l}\sin\theta_0\right]$$

Now: $\tan\theta_0 = \frac{a}{g}$



Thus, $\sin\theta_0 = \frac{a}{\sqrt{a^2+g^2}}$

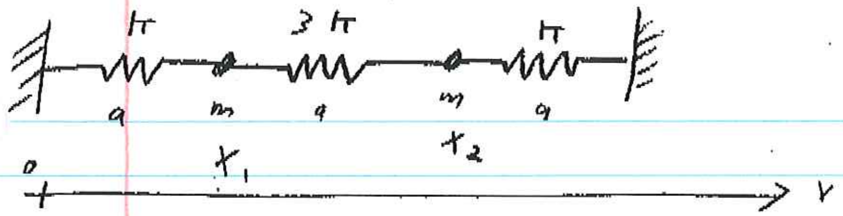
$$\cos\theta_0 = \frac{g}{\sqrt{a^2+g^2}}$$

So $$\ddot{\eta} = -\eta\left[\frac{g}{l}\cos\theta_0 + \frac{a}{l}\sin\theta_0\right]$$

$$= -\eta\left[\frac{g^2}{l\sqrt{a^2+g^2}} + \frac{a^2}{l\sqrt{a^2+g^2}}\right]$$

$$= -\eta\frac{\sqrt{a^2+g^2}}{l} \rightarrow \boxed{\omega^2 = \frac{\sqrt{a^2+g^2}}{l}}$$

Prob 8.3



$\eta_1 = x_1 - a$ (displacement from equilibrium)

$\eta_2 = x_2 - 2a$ "

a) $T = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2)$

$= \frac{1}{2} m(\dot{\eta}_1^2 + \dot{\eta}_2^2)$ since $\dot{\eta}_1 = \dot{x}_1$, $\dot{\eta}_2 = \dot{x}_2$

$= \frac{1}{2} \begin{bmatrix} \dot{\eta}_1 & \dot{\eta}_2 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix}$

so $T_{ab} = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$U = \frac{1}{2} k (x_1 - a)^2 + \frac{1}{2}(3k)((x_2 - x_1) - a)^2 + \frac{1}{2} k (x_2 - 2a)^2$

$= \frac{1}{2} k \eta_1^2 + \frac{3}{2} k (\eta_2 + \cancel{2a} - \eta_1 - \cancel{a} - \cancel{a})^2 + \frac{1}{2} k \eta_2^2$

$= \frac{1}{2} k \eta_1^2 + \frac{3}{2} k (\eta_2 - \eta_1)^2 + \frac{1}{2} k \eta_2^2$

$= \frac{1}{2} k \eta_1^2 + \frac{3}{2} k (\eta_2^2 + \eta_1^2 - 2\eta_1\eta_2) + \frac{1}{2} k \eta_2^2$

$= \frac{1}{2} k [4\eta_1^2 + 4\eta_2^2 - 6\eta_1\eta_2]$

$= \frac{1}{2} k \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$

$\begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} \begin{bmatrix} 4\eta_1 - 3\eta_2 \\ -3\eta_1 + 4\eta_2 \end{bmatrix}$

$4\eta_1^2 - 3\eta_1\eta_2 - 3\eta_1\eta_2 + 4\eta_2^2$

so $U_{ab} = k \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}$

b) $\det\left(V - \omega^2 T\right) = 0$ (to determine eigen frequencies)

$$0 = \det\left(k \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix} - \omega^2 m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} 4k - m\omega^2 & -3k \\ -3k & 4k - m\omega^2 \end{bmatrix}$$

$$= (4k - m\omega^2)^2 - 9k^2$$

$$= 16k^2 + m^2\omega^4 - 8mk\omega^2 - 9k^2$$

$$= m^2\left[\omega^4 - 8\left(\frac{k}{m}\right)\omega^2 + 7\left(\frac{k}{m}\right)^2\right]$$

Let $\boxed{\frac{k}{m} = \omega_0^2}$

Then $0 = \omega^4 - 8\omega^2\omega_0^2 + 7\omega_0^4$

$$= (\omega^2 - 7\omega_0^2)(\omega^2 - \omega_0^2)$$

Thus, $\boxed{\omega_1^2 = \omega_0^2}$, $\boxed{\omega_2^2 = 7\omega_0^2}$

(or $\omega_1 = \omega_0$, $\omega_2 = \sqrt{7}\omega_0$) choosing $+\sqrt{\ }$

c) Eigen Vectors

$\underline{\omega_1}$:

$$\begin{bmatrix} 4k - m\omega_0^2 & -3k \\ -3k & 4k - m\omega_0^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3k & -3k \\ -3k & 3k \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rightarrow v_1 - v_2 = 0 \rightarrow v_2 = v_1$

Thus $z_{(1)} = N_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{\omega_2}$:

$$\begin{bmatrix} 4k - 7m\omega_0^2 & -3k \\ -3k & 4k - 7m\omega_0^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3k & -3k \\ -3k & -3k \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \quad v_1 + v_2 = 0 \quad \rightarrow \quad v_2 = -v_1$$

Thus, $z_{(2)} = N_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

<u>Normalization:</u>

$$Z^T T Z = \mathbb{1} \quad , \quad T = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $1 = m N_1^2 \cdot 2 \rightarrow N_1 = \frac{1}{\sqrt{2m}}$

$1 = m N_2^2 \cdot 2 \rightarrow N_2 = \frac{1}{\sqrt{2m}}$

Thus, $\boxed{z_{(1)} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad z_{(2)} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

d) Check $z_{(1)}^T \, T \, z_{(2)} = 0$

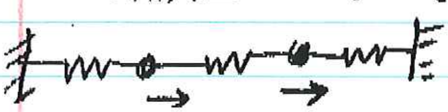
$$LHS = \frac{1}{2\cancel{m}} \begin{bmatrix} 1 & 1 \end{bmatrix} \cancel{m} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

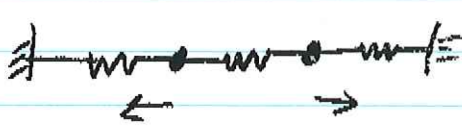
$$= \frac{1}{2}(1-1)$$

$$= \boxed{0} \checkmark$$

e) $z_{(1)} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (middle spring unstretched)



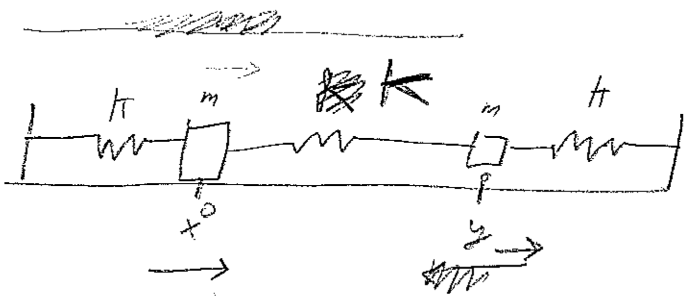
$z_2 = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

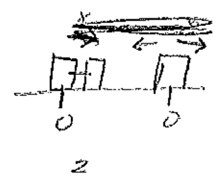


(8.4)

Problem: Determine the small oscillations of a system described by the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}\omega_0^2(x^2+y^2) + \alpha x y$$





$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

$$U = \frac{1}{2}kx^2 + \frac{1}{2}ky^2 + \frac{1}{2}K(x-y)^2$$

$$(x^2+y^2-2xy)$$

$$= \frac{1}{2}\underbrace{(k+K)}_{k_0}x^2 + \frac{1}{2}\underbrace{(k+K)}_{k_0}y^2 - Kxy$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k_0x^2 - \frac{1}{2}k_0y^2 + Kxy$$

$$= m\left[\frac{1}{2}(\dot{x}^2+\dot{y}^2) - \frac{1}{2}\left(\frac{k_0}{m}\right)(x^2+y^2) + \left(\frac{K}{m}\right)xy\right]$$

$$= \cancel{m}\left[\frac{1}{2}(\dot{x}^2+\dot{y}^2) - \frac{1}{2}\omega_0^2(x^2+y^2) + \alpha xy\right]$$

Equations of motion:

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}$$

$$= \ddot{x} + \omega_0^2 x - \alpha y$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y}$$

$$= \ddot{y} + \omega_0^2 y - \alpha x$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \qquad T_{ab} = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$U = \frac{1}{2}m\omega_0^2(x^2+y^2) - Kxy \;, \qquad U_{ab} = \left.\frac{\partial^2 U}{\partial q_a \partial q_b}\right|_{(0,0)}$$

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{(0,0)} = m\omega_0^2 \;, \quad \left.\frac{\partial^2 U}{\partial y^2}\right|_{(0,0)} = m\omega_0^2 \;, \quad \left.\frac{\partial^2 U}{\partial x \partial y}\right|_{(0,0)} = \frac{\partial}{\partial x}\left[m\omega_0^2 y - Kx\right]\Big|_{(0,0)}$$

$$= -K$$

$$U_{ab} = \begin{bmatrix} m\omega_0^2 & -K \\ -K & m\omega_0^2 \end{bmatrix}$$

$$U - \omega^2 T \qquad \det\begin{bmatrix} m(\omega_0^2-\omega^2) & -K \\ -K & m(\omega_0^2-\omega^2) \end{bmatrix}$$

$$m^2(\omega_0^2-\omega^2)^2 - K^2 = 0$$

$$(\omega_0^2-\omega^2)^2 - \alpha^2 = 0$$

$$(\omega_0^2-\omega^2)^2 = \alpha^2$$

$$\omega_0^2-\omega^2 = \pm\alpha$$

$$\rightarrow \boxed{\begin{aligned} \omega_+^2 &= \omega_0^2 + \alpha \\ \omega_-^2 &= \omega_0^2 - \alpha \end{aligned}}$$

~~Beat freq~~

$|\cancel{\omega_+ - \omega_-}| = 2\alpha$

Eigen vectors

$\underline{\omega^2 = \omega_+^2}$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m(\cancel{\omega_0^2} - \cancel{\omega_0^2} - \alpha) & -K \\ -K & m(\cancel{\omega_0^2} - \cancel{\omega_0^2} - \alpha) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} -m\alpha & -K \\ -K & -m\alpha \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\rightarrow \quad 0 = -m\alpha v_1 - K v_2$$

$$v_2 = -\frac{m\alpha}{K} v_1 = -v_1 \qquad \left(\text{since } \alpha = \frac{K}{m}\right)$$

Thus $v_2 = -v_1$

$$\boxed{Z_{(+)} = \frac{N_+}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$\underline{\omega^2 = \omega_-^2}$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m\alpha & -K \\ -K & m\alpha \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$v_2 = \frac{m\alpha}{K} v_1 = v_1$$

$$\rightarrow \quad \boxed{Z_{(-)} = \frac{N_-}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$\underline{\text{Normalization}}$:

$$\mathbb{1} = Z^T T Z = m Z^T Z$$

$$\rightarrow \quad 1 = m \frac{N_+^2}{2} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = m N_+^2$$

$$\rightarrow \boxed{N_+ = \frac{1}{\sqrt{m}}}$$

similarly $\boxed{N_- = \frac{1}{\sqrt{m}}}$ so $\boxed{Z_{(\pm)} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ \mp 1 \end{bmatrix}}$

~~Beat freq~~

~~$\omega_{beat} = |\omega_+ - \omega_-|$~~

~~ω~~

Small coupling $\quad K \ll k \quad \rightarrow \quad \alpha \ll \omega_0^2$

Then $\quad \omega_+^2 = \omega_0^2\left(1 + \frac{\alpha}{\omega_0^2}\right), \quad \omega_-^2 = \omega_0^2\left(1 - \frac{\alpha}{\omega_0^2}\right)$

$$\rightarrow \omega_+ \simeq \omega_0\left(1 + \frac{1}{2}\frac{\alpha}{\omega_0^2}\right)$$

$$\omega_- \simeq \omega_0\left(1 - \frac{1}{2}\frac{\alpha}{\omega_0^2}\right)$$

$$\omega_{beat} \equiv |\omega_+ - \omega_-|$$

$$= \frac{\alpha}{\omega_0}$$

$$= \frac{K/m}{\sqrt{k_0/m}}$$

$$= \frac{K/m}{\sqrt{(k+K)/m}}$$

now $K \ll k$

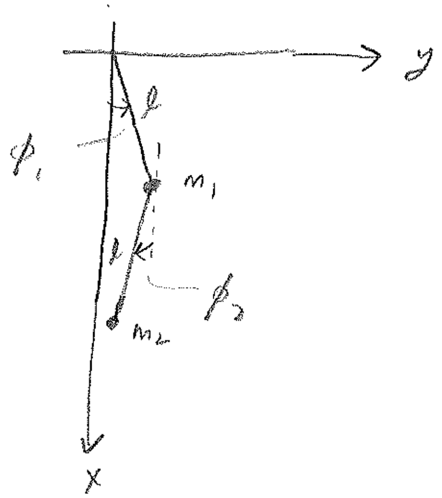
so $k + K \simeq k$

$$\simeq \frac{K/m}{\sqrt{k/m}}$$

$$= \sqrt{\frac{K^2}{km}}$$

Prob 8.5

a)



Equilibrium $\phi_1 = 0, \phi_2 = 0$

so $\eta^a = \boxed{\begin{matrix}\phi_1 \\ \phi_2\end{matrix}}$

$$x_1 = l \cos\phi_1$$
$$y_1 = l \sin\phi_1$$

$$x_2 = x_1 + l \cos\phi_2 = l(\cos\phi_1 + \cos\phi_2)$$

$$y_2 = y_1 + l \sin\phi_2 = l(\sin\phi_1 + \sin\phi_2)$$

$$T = \frac{1}{2}\left[m_1(\dot{x}_1^2 + \dot{y}_1^2) + m_2(\dot{x}_2^2 + \dot{y}_2^2)\right]$$

$$= \frac{1}{2} m_1 \underbrace{(l^2 \sin^2\phi_1 + l^2 \cos^2\phi_1)}_{l^2} \dot{\phi}_1^2$$

$$+ \frac{1}{2} m_2 l^2 \left[\sin^2\phi_1 \dot{\phi}_1^2 + \sin^2\phi_2 \dot{\phi}_2^2 + 2\sin\phi_1 \sin\phi_2 \dot{\phi}_1\dot{\phi}_2 + \cos^2\phi_1 \dot{\phi}_1^2 + \cos^2\phi_2 \dot{\phi}_2^2 + 2\cos\phi_1 \cos\phi_2 \dot{\phi}_1 \dot{\phi}_2\right]$$

$$= \frac{1}{2} m_1 l^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l^2 \left[\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\cos(\phi_1 - \phi_2)\dot{\phi}_1\dot{\phi}_2\right]$$

$$= \frac{1}{2} l^2 \left[(m_1 + m_2)\dot{\phi}_1^2 + m_2 \dot{\phi}_2^2 + 2m_2 \cos(\phi_1 - \phi_2)\dot{\phi}_1\dot{\phi}_2\right]$$

$$\approx \frac{1}{2} l^2 \left[(m_1 + m_2)\dot{\phi}_1^2 + m_2 \dot{\phi}_2^2 + 2m_2 \dot{\phi}_1 \dot{\phi}_2\right]$$

$$= \frac{1}{2} T_{ab} \dot{\eta}^a \dot{\eta}^b$$

$$\rightarrow \boxed{T_{ab} = l^2 \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}}$$

$1 - \frac{x^2}{2}$

$$U = -m_1 g x_1 - m_2 g x_2$$

$$= -m_1 g l \cos\phi_1 - m_2 g l (\cos\phi_1 + \cos\phi_2)$$

$$\simeq -gl \left[m_1 \left(1 - \frac{\phi_1^2}{2}\right) + m_2 \left(\left(1 - \frac{\phi_1^2}{2}\right) + \left(1 - \frac{\phi_2^2}{2}\right) \right) \right]$$

$$\cancel{U = \frac{1}{2} \sum U_{ab} \eta^a \eta^b}$$

$$= \underbrace{-(m_1 + 2m_2) g l}_{\text{const}} + \frac{1}{2}\left[m_1 g l \phi_1^2 + m_2 g l \phi_1^2 + m_2 g l \phi_2^2 \right]$$

$$= U_0 + \frac{1}{2} g l \left[(m_1 + m_2) \phi_1^2 + m_2 \phi_2^2 \right]$$

$$= U_0 + \frac{1}{2} U_{ab} \eta^a \eta^b$$

$$\rightarrow \boxed{U_{ab} = gl \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix}}$$

b) Normal mode frequencies

$$0 = \det(U - \omega^2 T)$$

$$= \det \begin{bmatrix} gl(m_1 + m_2) - \omega^2 (m_1 + m_2) l^2 & -\omega^2 m_2 l^2 \\ -\omega^2 m_2 l^2 & gl m_2 - \omega^2 m_2 l^2 \end{bmatrix}$$

$$\cancel{0 = g\ell(m_1+m_2) - \omega^2(m_1+m_2)\ell^2}$$

$$0 = \det \begin{vmatrix} (m_1+m_2)(g\ell - \omega^2\ell^2) & -\omega^2\ell^2 m_2 \\ -\omega^2\ell^2 m_2 & m_2(g\ell - \omega^2\ell^2) \end{vmatrix}$$

$$= m_2(m_1+m_2)(g\ell-\omega^2\ell^2)^2 - \omega^4\ell^4 m_2^2$$

$$= m_2(m_1+m_2)(g^2\ell^2 + \omega^4\ell^4 - 2g\omega^2\ell^3) - \omega^4\ell^4 m_2^2$$

$$= m_2(m_1+m_2)g^2\ell^2 + \omega^4\ell^4(m_1m_2 + \cancel{m_2^2}) - 2m_2(m_1+m_2)g\omega^2\ell^3$$

$$\cancel{-\omega^4\ell^4 m_2^2}$$

$$= \ell^4 m_1 m_2 \left[\omega^4 - 2\left(\frac{m_1+m_2}{m_1}\right)\frac{g}{\ell}\omega^2 + \left(\frac{m_1+m_2}{m_1}\right)\frac{g^2}{\ell^2}\right]$$

Thus,

$$\boxed{\omega_\pm^2 = \frac{2\left(\frac{g}{\ell}\right)\left(\frac{m_1+m_2}{m_1}\right) \pm \sqrt{4\left(\frac{m_1+m_2}{m_1}\right)^2\frac{g^2}{\ell^2} - 4\left(\frac{m_1+m_2}{m_1}\right)\frac{g^2}{\ell^2}}}{2}}$$

$$= \frac{g}{\ell}\left(\frac{m_1+m_2}{m_1}\right) \pm \frac{g}{\ell}\left(\frac{m_1+m_2}{m_1}\right)\sqrt{1 - \left(\frac{m_1}{m_1+m_2}\right)}$$

$$= \frac{g}{\ell}\left(\frac{m_1+m_2}{m_1}\right)\left[1 \pm \sqrt{1 - \frac{m_1}{m_1+m_2}}\right]$$

$$= \frac{g}{\ell}\left(\frac{m_1+m_2}{m_1}\right)\left[1 \pm \sqrt{\frac{m_2}{m_1+m_2}}\right]$$

c) Assume $m_1 >> m_2$

Then $$\frac{m_1+m_2}{m_1} = 1+\frac{m_2}{m_1} = 1+\epsilon^2 \quad \text{where } \epsilon = \sqrt{\frac{m_2}{m_1}}$$

$$\sqrt{\frac{m_2}{m_1+m_2}} = \sqrt{\frac{m_2}{m_1\left(1+\frac{m_2}{m_1}\right)}} = \sqrt{\frac{\epsilon^2}{1+\epsilon^2}} \approx \epsilon$$

Thus $$\boxed{\omega_\pm^2 \approx \frac{g}{l}\left(1+\epsilon^2\right)\left[1\pm\epsilon\right] \approx \frac{g}{l}\left(1\pm\epsilon\right) + O(\epsilon^2)}$$

d) Beat freq

$$|\omega_+ - \omega_-| = \left|\sqrt{\frac{g}{l}(1+\epsilon)} - \sqrt{\frac{g}{l}(1-\epsilon)}\right|$$

$$\approx \left|\sqrt{\frac{g}{l}}\left(\cancel{1}+\frac{\epsilon}{2}\right) - \sqrt{\frac{g}{l}}\left(\cancel{1}-\frac{\epsilon}{2}\right)\right|$$

$$= \boxed{\sqrt{\frac{g}{l}}\,\epsilon}$$

e) Eigen vectors:

$$\begin{bmatrix} (m_1+m_2)(gl-\omega^2 l^2) & -\omega^2 l^2 m_2 \\ -\omega^2 l^2 m_2 & m_2(gl-\omega^2 l^2) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

only one indep equation.

$\omega_+^2 = \frac{g}{l}(1+\epsilon)$.

$$\begin{bmatrix} (m_1+m_2)gl(1-(1+\epsilon)) & -gl(1+\epsilon)m_2 \\ -gl(1+\epsilon)m_2 & m_2 gl(1-(1+\epsilon)) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-(m_1+m_2)\epsilon V_1 \quad -(1+\epsilon)m_2 V_2 = 0$$

$$-(m_1+m_2)\Big[\epsilon V_1 + (1+\epsilon)\underbrace{\left(\frac{m_2}{m_1+m_2}\right)}_{\approx \epsilon^2} V_2\Big] = 0$$

$$-(m_1+m_2)\epsilon\,[V_1 + \underbrace{(\epsilon+\epsilon^2)}_{\approx \epsilon} V_2] = 0$$

Thus, $\boxed{V_1 = -\epsilon V_2}$

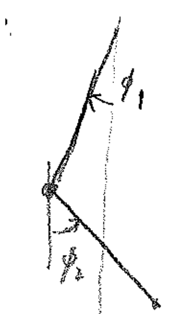
$\omega_-^2 = \frac{g}{l}(1-\epsilon)$:

Same with $\epsilon \to -\epsilon$: $\boxed{V_1 = +\epsilon V_2}$

Thus: $\vec{V}_+ \propto \begin{bmatrix} -\epsilon \\ 1 \end{bmatrix}$, $\vec{V}_- \propto \begin{bmatrix} \epsilon \\ 1 \end{bmatrix}$

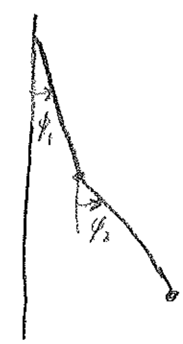
High freq:

ω_+:



180° out of phase

ω_-:



In phase

Normalize:

$$V_+ = \frac{1}{\sqrt{2}\,\ell\,\epsilon\,\sqrt{m_1}} \begin{bmatrix} -\epsilon \\ 1 \end{bmatrix}$$

$$V_- = \frac{1}{\sqrt{2}\,\ell\,\epsilon\,\sqrt{m_1}} \begin{bmatrix} \epsilon \\ 1 \end{bmatrix}$$

$$T_{ab} = \ell^2 \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}$$

$$= m_1 \ell^2 \begin{bmatrix} 1 + \frac{m_2}{m_1} & \frac{m_2}{m_1} \\ \frac{m_2}{m_1} & \frac{m_2}{m_1} \end{bmatrix}$$

$$= m_1 \ell^2 \begin{bmatrix} 1+\epsilon^2 & \epsilon^2 \\ \epsilon^2 & \epsilon^2 \end{bmatrix} \approx m_1 \ell^2 \begin{bmatrix} 1 & \epsilon^2 \\ \epsilon^2 & \epsilon^2 \end{bmatrix}$$

$$V_+^a T_{ab} V_+^b = \begin{bmatrix} -\epsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & \epsilon^2 \\ \epsilon^2 & \epsilon^2 \end{bmatrix} \begin{bmatrix} -\epsilon \\ 1 \end{bmatrix} \frac{1}{\cancel{\ell^2}\epsilon^2 \cancel{m_1}} \frac{\cancel{m_1 \ell^2}}{2}$$

$$= \frac{1}{2\epsilon^2} \begin{bmatrix} -\epsilon & 1 \end{bmatrix} \begin{bmatrix} -\epsilon \\ \epsilon^2 \end{bmatrix}$$

$-\epsilon + \epsilon^2 \approx -\epsilon$

$-\epsilon^3 + \epsilon^2 \approx \epsilon^2$

$$= 1$$

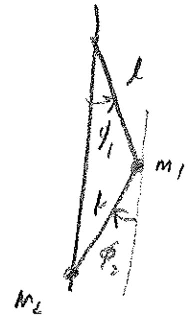
Similarly $V_-^a T_{ab} V_-^b = \boxed{1}$

$$V_+^a T_{ab} V_-^b = \begin{bmatrix} -\epsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & \epsilon^2 \\ \epsilon^2 & \epsilon^2 \end{bmatrix} \begin{bmatrix} +\epsilon \\ 1 \end{bmatrix} \frac{1}{2\epsilon^2}$$

$$= \frac{1}{2\epsilon^2} \begin{bmatrix} -\epsilon & 1 \end{bmatrix} \begin{bmatrix} \epsilon \\ \epsilon^2 \end{bmatrix}$$

$$= \frac{1}{2\epsilon^2}\left(-\epsilon^2 + \epsilon^2\right) = \boxed{0}$$

f) Initial conditions



$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} \phi_0 \\ -\phi_0 \end{bmatrix}$$

$$\begin{bmatrix} \dot\phi_1 \\ \dot\phi_2 \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

matrix of eigenvectors

$$\cancel{= C_+ v_+ e^{i\omega_+ t} + C_- v_- e^{-i\omega_- t}}$$

$$\eta^a = \mathrm{Re}\Big(\sum_b \underbrace{Z_{ab}}_{A} C_b e^{i\omega_b t}\Big)$$

$$\dot\eta^a = \mathrm{Re}\Big(\sum_b \underbrace{i\omega_b Z_{ab}}_{B} C_b e^{i\omega_b t}\Big)$$

$$\begin{bmatrix} \phi_0 \\ -\phi_0 \end{bmatrix} = \mathrm{Re}\big(\overset{\text{real}}{A}\ \underset{(\mathrm{Re}\,C + i\,\mathrm{Im}\,C)}{C}\big) \rightarrow \mathrm{Re}\,C = A^{-1}\begin{bmatrix} \phi_0 \\ -\phi_0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathrm{Re}\big(i\ \underset{\text{real}}{B}\ \underset{(\mathrm{Re}\,C + i\,\mathrm{Im}\,C)}{C}\big) \rightarrow -\mathrm{Im}\,C = B^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so $\boxed{\mathrm{Im}\,C = 0}$

$$C = A^{-1}\begin{bmatrix} \phi_0 \\ -\phi_0 \end{bmatrix} = \left(\frac{1}{\ell\epsilon\sqrt{2m_1}}\begin{bmatrix} -\epsilon & \epsilon \\ 1 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} \phi_0 \\ -\phi_0 \end{bmatrix}$$

$$= \frac{\ell\cancel{\epsilon}\sqrt{2m_1}}{-2\cancel{\epsilon}}\begin{bmatrix} 1 & -\epsilon \\ -1 & -\epsilon \end{bmatrix}\begin{bmatrix} \phi_0 \\ -\phi_0 \end{bmatrix}$$

$-\epsilon$

$-\epsilon$ ~~$-\epsilon$~~

$$= -\ell\sqrt{\frac{m_1}{2}} \begin{bmatrix} \phi_0 + \epsilon\phi_0 \\ -\phi_0 + \epsilon\phi_0 \end{bmatrix}$$

$$= -\ell\sqrt{\frac{m_1}{2}}\,\phi_0 \begin{bmatrix} 1+\epsilon \\ -1+\epsilon \end{bmatrix}$$

Then,

$$\begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = Re\left\{ \frac{1}{\ell\epsilon\sqrt{2m_1}} \begin{bmatrix} -\epsilon & \epsilon \\ 1 & 1 \end{bmatrix} \left(-\ell\sqrt{\frac{m_1}{2}}\,\phi_0\right) \begin{bmatrix} (1+\epsilon)e^{i\omega_+ t} \\ (-1+\epsilon)e^{i\omega_- t} \end{bmatrix} \right\}$$

$$= Re\left\{ \frac{-\phi_0}{2\epsilon} \begin{bmatrix} -\epsilon(1+\epsilon)e^{i\omega_+ t} + \epsilon(-1+\epsilon)e^{i\omega_- t} \\ (1+\epsilon)e^{i\omega_+ t} + (-1+\epsilon)e^{i\omega_- t} \end{bmatrix} \right\}$$

$$= \begin{cases} \dfrac{\phi_0}{2}\Big((1+\epsilon)\cos(\omega_+ t) + (1-\epsilon)\cos(\omega_- t)\Big) \\[2ex] -\dfrac{\phi_0}{2\epsilon}\Big((1+\epsilon)\cos(\omega_+ t) - (1-\epsilon)\cos(\omega_- t)\Big) \end{cases}$$

$$= \begin{cases} \dfrac{\phi_0}{2}\Big[\big(\cos(\omega_+ t) + \cos(\omega_- t)\big) + \epsilon\big(\cos(\omega_+ t) - \cos(\omega_- t)\big)\Big] \\[2ex] -\dfrac{\phi_0}{2\epsilon}\Big[\big(\cos(\omega_+ t) - \cos(\omega_- t)\big) + \epsilon\big(\cos(\omega_+ t) + \cos(\omega_- t)\big)\Big] \end{cases}$$

<u>NOTE</u>: when $t=0$ $\begin{bmatrix} \phi_1(0) \\ \phi_2(0) \end{bmatrix} = \begin{bmatrix} \phi_0 \\ -\phi_0 \end{bmatrix}$ and $\begin{bmatrix} \dot\phi_1(0) \\ \dot\phi_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ since $\sin(\omega_\pm 0) = 0$

Problem: 8.6

To show: (from Marion - Thornton)

$$\sum_{a=1}^{N} \sin\left(\frac{ab\pi}{N+1}\right) \sin\left(\frac{ac\pi}{N+1}\right) = \frac{N+1}{2}\,\delta_{bc}$$

$$LHS = \sum_{a=1}^{N} \left(\frac{1}{2i}\right)^2 \left(e^{\frac{iab\pi}{N+1}} - e^{-\frac{iab\pi}{N+1}}\right)$$

$$\left(e^{\frac{iac\pi}{N+1}} - e^{\frac{-iab\pi}{N+1}}\right)$$

$$= -\frac{1}{4} \sum_{a=1}^{N} \left[e^{\frac{ia(b+c)\pi}{N+1}} + e^{-\frac{ia(b+c)\pi}{N+1}} \right.$$

$$\left. - e^{-\frac{ia(b-c)\pi}{N+1}} - e^{\frac{ia(b-c)\pi}{N+1}} \right]$$

Thus, it suffices to know:

$$\sum_{a=1}^{N} e^{\frac{ian\pi}{N+1}} \qquad \text{where } n = \begin{cases} b+c \\ -(b+c) \\ -(b-c) \\ (b-c) \end{cases}$$

Now: $\cancel{\sum_{n=1}^{N} e^{inx} = \sum_{n=0}^{N} e^{inx} - 1}$

$\cancel{= \left(\frac{1-e^{i(N+1)x}}{1-e^{ix}}\right) - 1}$ ~~(see proof next page)~~

Note:
$(|r|<1)$

$$\sum_{n=0}^{N} r^n = 1 + r + r^2 + \cdots r^N$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Thus

$$\sum_{n=0}^{N} r^n = \sum_{n=0}^{\infty} r^n - \sum_{n=N+1}^{\infty} r^n$$

$$= \frac{1}{1-r} - \sum_{m=0}^{\infty} r^{m+N+1} \qquad \text{let } m = n-(N+1)$$

$$= \frac{1}{1-r} - r^{N+1} \sum_{m=0}^{\infty} r^m$$

$$= \left(\frac{1}{1-r}\right)\left[1 - r^{N+1}\right]$$

$$= \left(\frac{1-r^{N+1}}{1-r}\right)$$

$$\sum_{n=1}^{N} e^{inx} = \left(\frac{1-e^{i(N+1)x}}{1-e^{ix}}\right) - 1$$

$$= \frac{\cancel{1} - e^{i(N+1)x} - \cancel{1} + e^{ix}}{1-e^{ix}}$$

$$= \frac{e^{ix}\left[1 - e^{iNx}\right]}{1-e^{ix}}$$

$$= \frac{e^{ix}\, e^{i\frac{Nx}{2}}\left[e^{-i\frac{Nx}{2}} - e^{i\frac{Nx}{2}}\right]}{e^{i\frac{x}{2}}\left[e^{-ix/2} - e^{i\frac{x}{2}}\right]}$$

$$= e^{i\frac{x}{2}} e^{i\frac{Nx}{2}} \frac{\sin\left(\frac{Nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

$$= \frac{e^{\frac{i(N+1)x}{2}} \sin\left(\frac{Nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

Thus, $\boxed{\sum_{n=1}^{N} e^{inx} = \frac{e^{\frac{i(N+1)x}{2}} \sin\left(\frac{Nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}}$

Evaluate:

$$\sum_{a=1}^{N} \sin^2\left(\frac{ab\pi}{N+1}\right) = -\frac{1}{4} \sum_{a=1}^{N} \left[e^{ia\frac{2\pi b}{N+1}} + e^{-ia\frac{2\pi b}{N+1}} - 1 - 1 \right]$$

$(b = c)$

$$= \frac{1}{2} \sum_{a=1}^{N} 1 - \frac{1}{4} \sum_{a=1}^{N} \left(e^{ia\frac{2\pi b}{N+1}} + e^{-ia\frac{2\pi b}{N+1}} \right)$$

$$= \frac{1}{2} N - \frac{1}{4} e^{i\pi b} \frac{\sin\left(\frac{N\pi b}{N+1}\right)}{\sin\left(\frac{\pi b}{N+1}\right)} - \frac{1}{4} e^{-i\pi b} \frac{\sin\left(\frac{N\pi b}{N+1}\right)}{\sin\left(\frac{\pi b}{N+1}\right)}$$

$1 + 2 + 3$
$= 6$
$= N(N+1)$

$$= \frac{1}{2}N - \frac{1}{4}\left(e^{i\pi b} + e^{-i\pi b}\right)\frac{\sin\left(\frac{N\pi b}{N+1}\right)}{\sin\left(\frac{\pi b}{N+1}\right)}$$

$$= \frac{1}{2}N - \frac{1}{2}\underbrace{\cos(\pi b)}_{(-1)^b}\frac{1}{\sin\left(\frac{\pi b}{N+1}\right)}\sin\left(\left((N+1)-1\right)\frac{\pi b}{N+1}\right)$$

$$= \frac{N}{2} - \frac{1}{2}(-1)^b\frac{1}{\sin\left(\frac{\pi b}{N+1}\right)}\sin\left(\pi b - \frac{\pi b}{N+1}\right)$$

$$= \frac{N}{2} - \frac{1}{2}(-1)^b\frac{1}{\sin\left(\frac{\pi b}{N+1}\right)}\left[\underbrace{\sin(\pi b)}_{0}\cos\left(\frac{\pi b}{N+1}\right) - \underbrace{\cos(\pi b)}_{(-1)^b}\sin\left(\frac{\pi b}{N+1}\right)\right]$$

$$= \frac{N}{2} + \frac{1}{2} = \boxed{\frac{N+1}{2}}$$

Thus,

$$\boxed{\sum_{a=1}^{N}\sin^2\left(\frac{ab\pi}{N+1}\right) = \frac{N+1}{2}}$$

Suppose $b \neq c$:

$$\sum_{a=1}^{N} \sin\left(\frac{ab\pi}{N+1}\right)\sin\left(\frac{ac\pi}{N+1}\right)$$

$$= -\frac{1}{4}\sum_{a=1}^{N}\left[e^{\frac{ia(b+c)\pi}{N+1}} + e^{-\frac{ia(b+c)\pi}{N+1}} - e^{-\frac{ia(b-c)\pi}{N+1}} - e^{\frac{ia(b-c)\pi}{N+1}} \right]$$

$$= -\frac{1}{4}\left[\frac{e^{\frac{i(b+c)\pi}{2}} \sin\left(\frac{N\pi(b+c)}{2(N+1)}\right)}{\sin\left(\frac{\pi(b+c)}{2(N+1)}\right)} + \frac{e^{-\frac{i(b+c)\pi}{2}} \sin(\quad)}{\sin(\quad)} \right.$$

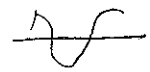
$$\left. - \frac{e^{-\frac{i(b-c)\pi}{2}} \sin\left(\frac{N\pi(b-c)}{2(N+1)}\right)}{\sin\left(\frac{\pi(b-c)}{2(N+1)}\right)} - \frac{e^{\frac{i(b-c)\pi}{2}} \sin(\quad)}{\sin(\quad)} \right]$$

$$= -\frac{1}{2}\left[\frac{\cos\left(\frac{(b+c)\pi}{2}\right) \sin\left(\frac{N\pi(b+c)}{2(N+1)}\right)}{\sin\left(\frac{\pi(b+c)}{2(N+1)}\right)} - \frac{\cos\left(\frac{(b-c)\pi}{2}\right) \sin\left(\frac{N\pi(b-c)}{2(N+1)}\right)}{\sin\left(\frac{\pi(b-c)}{2(N+1)}\right)} \right]$$

A $-$ B

$$\frac{\pi(b+c)}{2} - \frac{\pi(b+c)}{2(N+1)}$$

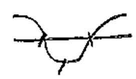
$$Ⓐ = \frac{\cos\left(\frac{(b+c)\pi}{2}\right) \sin\left(\overbrace{((N+1)-1)\frac{\pi(b+c)}{2(N+1)}}\right)}{\sin\left(\frac{\pi(b+c)}{2(N+1)}\right)}$$

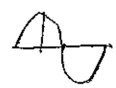


$$= \frac{\cos\left(\frac{(b+c)\pi}{2}\right)\left[\sin\left(\frac{\pi(b+c)}{2}\right)\cos\left(\frac{(b+c)\pi}{2(N+1)}\right) - \cos\left(\frac{\pi(b+c)}{2}\right)\sin\left(\frac{(b+c)\pi}{2(N+1)}\right)\right]}{\sin\left(\frac{(b+c)\pi}{2(N+1)}\right)}$$

<u>Now:</u>

$$\cos\left(\frac{(b+c)\pi}{2}\right) = \begin{cases} 0 & \pi/2 \\ -1 & \pi \\ 0 & 3\pi/2 \\ 1 & 2\pi \end{cases}$$





$$\sin\left((b+c)\frac{\pi}{2}\right) = \begin{cases} 1 & \pi/2 \\ 0 & \pi \\ -1 & 3\pi/2 \\ 0 & 2\pi \end{cases}$$

$\cancel{A \cos A}$

$= \sin \cancel{2A}$

$\sin(2x) = 2 \sin x \cos x$

Thus, $\cos\left((b+c)\frac{\pi}{2}\right)\sin\left((b+c)\frac{\pi}{2}\right) = \frac{1}{2}\sin\left((b+c)\pi\right)$

$$= \boxed{0}$$

$$Ⓐ = -\cos^2\left(\frac{\pi(b+c)}{2}\right) = \begin{cases} 0 & \pi/2 \\ 1 & \pi \\ 0 & 3\pi/2 \\ 1 & 2\pi \end{cases}$$

$$B = -\cos^2\left(\frac{\pi(b-c)}{2}\right)$$

Thus, $\displaystyle\sum_{q=1}^{N} \sin\left(\frac{qb\pi}{N+1}\right)\sin\left(\frac{qc\pi}{N+1}\right) = -\frac{1}{2}\left[A - B\right]$

$(b \neq c)$
$$= \frac{1}{2}\left[\cos^2\left(\frac{\pi(b+c)}{2}\right) - \cos^2\left(\frac{\pi(b-c)}{2}\right)\right]$$

$$= \frac{1}{2}\left(\cos\left(\frac{\pi(b+c)}{2}\right) - \cos\left(\frac{\pi(b-c)}{2}\right)\right)\left(\cos\left(\frac{\pi(b+c)}{2}\right) + \cos\left(\frac{\pi(b-c)}{2}\right)\right)$$

$$= \frac{1}{2}\left[\cancel{\cos\left(\frac{\pi b}{2}\right)\cos\left(\frac{\pi c}{2}\right)} - \sin\left(\frac{\pi b}{2}\right)\sin\left(\frac{\pi c}{2}\right) - \cancel{\cos\left(\frac{\pi b}{2}\right)\cos\left(\frac{\pi c}{2}\right)} - \sin\left(\frac{\pi b}{2}\right)\sin\left(\frac{\pi c}{2}\right)\right]$$

$$\left[\cos\left(\frac{\pi b}{2}\right)\cos\left(\frac{\pi c}{2}\right) \cancel{- \sin\left(\frac{\pi b}{2}\right)\sin\left(\frac{\pi c}{2}\right)} + \cos\left(\frac{\pi b}{2}\right)\cos\left(\frac{\pi c}{2}\right) \cancel{+ \sin\left(\frac{\pi b}{2}\right)\sin\left(\frac{\pi c}{c}\right)}\right]$$

$$= 2\sin\left(\frac{\pi b}{2}\right)\sin\left(\frac{\pi c}{2}\right)\cos\left(\frac{\pi b}{2}\right)\cos\left(\frac{\pi c}{2}\right)$$

$$= 2\left(\frac{1}{2}\sin(\pi b)\right)\left(\frac{1}{2}\sin(\pi c)\right)$$

$\sin(\pi b) \to 0$, $\sin(\pi c) \to 0$

$$= \boxed{0}$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$$

Thus, $\boxed{\displaystyle\sum_{q=1}^{N} \sin\left(\frac{qb\pi}{N+1}\right)\sin\left(\frac{qc\pi}{N+1}\right) = 0 \quad (b \neq c)}$