Problem (7.1) Expressions for T and $\vec{L}$

$$T = \sum_I \frac{1}{2} m_I |\dot{\vec{r}}_I|^2 \quad , \quad \vec{r}_I = \vec{R} + \vec{r}_I{}'$$

$$= \sum_I \frac{1}{2} m_I |\dot{\vec{R}} + \dot{\vec{r}}_I{}'|^2$$

$$= \sum_I \frac{1}{2} m_I \left(|\dot{\vec{R}}|^2 + |\dot{\vec{r}}_I{}'|^2 + 2\dot{\vec{r}}_I{}' \cdot \dot{\vec{R}} \right)$$

$$= \frac{1}{2} M |\dot{\vec{R}}|^2 + \sum_I \frac{1}{2} m_I |\dot{\vec{r}}_I{}'|^2 + \left(\sum_I m_I \dot{\vec{r}}_I{}' \right) \cdot \dot{\vec{R}}$$

Now,

$$\sum_I m_I \dot{\vec{r}}_I{}' = \sum_I m_I \left(\dot{\vec{r}}_I - \dot{\vec{R}} \right)$$

$$= M \dot{\vec{R}}_{com} - M \dot{\vec{R}}$$

$$= M \left(\dot{\vec{R}}_{com} - \dot{\vec{R}} \right)$$

Thus,

$$\boxed{\begin{aligned} T &= \frac{1}{2} M |\dot{\vec{R}}|^2 + \sum_I \frac{1}{2} m_I |\dot{\vec{r}}_I{}'|^2 + M \dot{\vec{R}} \cdot \left(\dot{\vec{R}}_{com} - \dot{\vec{R}} \right) \\ &= T' + \frac{1}{2} M |\dot{\vec{R}}|^2 + M \dot{\vec{R}} \cdot \left(\dot{\vec{R}}_{com} - \dot{\vec{R}} \right) \end{aligned}}$$

Similarly,

$$\vec{L} = \sum_I m_I \vec{r}_I \times \dot{\vec{r}}_I$$

$$= \sum_I m_I \left(\vec{R} + \vec{r}_I{}' \right) \times \left(\dot{\vec{R}} + \dot{\vec{r}}_I{}' \right)$$

$$= \sum_I m_I \left(\vec{R} \times \dot{\vec{R}} + \vec{r}_I{}' \times \dot{\vec{r}}_I{}' + \vec{r}_I{}' \times \dot{\vec{R}} + \vec{R} \times \dot{\vec{r}}_I{}' \right)$$

$$\vec{L} = \sum_I m_I \, \vec{r}_I{}' \times \dot{\vec{r}}_I{}' + M \vec{R} \times \dot{\vec{R}}$$
$$+ \left(\sum_I m_I \vec{r}_I{}' \right) \times \dot{\vec{R}} + \vec{R} \times \left(\sum_I m_I \dot{\vec{r}}_I{}' \right)$$

Now, $\left(\sum_I m_I \vec{r}_I{}' \right) \times \dot{\vec{R}} = \left(\sum_I m_I (\vec{r}_I - \vec{R}) \right) \times \dot{\vec{R}}$

$$= M \vec{R}_{com} \times \dot{\vec{R}} - M \vec{R} \times \dot{\vec{R}}$$

$$\vec{R} \times \left(\sum_I m_I \dot{\vec{r}}_I{}' \right) = \vec{R} \times \left(\sum_I m_I (\dot{\vec{r}}_I - \dot{\vec{R}}) \right)$$
$$= \vec{R} \times \left(M \dot{\vec{R}}_{com} - M \dot{\vec{R}} \right)$$
$$= \vec{R} \times M \dot{\vec{R}}_{com} - M \vec{R} \times \dot{\vec{R}}$$

Thus,

$$\boxed{\vec{L} = \vec{L}' + M \vec{R} \times \dot{\vec{R}} + M \left(\vec{R}_{com} - \vec{R} \right) \times \dot{\vec{R}} + M \vec{R} \times \left(\dot{\vec{R}}_{com} - \dot{\vec{R}} \right)}$$

Problem (7.2) Kinetic energy in terms of I and $\vec{\omega}$

$$T = \sum_I \frac{1}{2} m_I |\dot{\vec{r}}_I|^2$$

$$= \sum_I \frac{1}{2} m_I |\vec{\omega} \times \vec{r}_I|^2$$

$$= \sum_I \frac{1}{2} m_I (\vec{\omega} \times \vec{r}_I) \cdot (\vec{\omega} \times \vec{r}_I)$$

$$= \sum_I \frac{1}{2} m_I \vec{\omega} \cdot (\vec{r}_I \times (\vec{\omega} \times \vec{r}_I))$$

$$= \frac{1}{2} \vec{\omega} \cdot \sum_I m_I \vec{r}_I \times (\vec{\omega} \times \vec{r}_I)$$

$$= \frac{1}{2} \vec{\omega} \cdot \sum_I m_I \vec{r}_I \times \dot{\vec{r}}_I$$

$$= \frac{1}{2} \vec{\omega} \cdot \vec{L}$$

Now $L_i = \sum_j I_{ij} \omega_j$

Thus

$$T = \frac{1}{2} \vec{\omega} \cdot \vec{L}$$

$$= \frac{1}{2} \sum_i \omega_i L_i$$

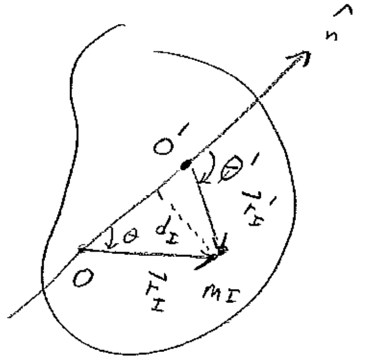
$$= \frac{1}{2} \sum_{i,j} \omega_i I_{ij} \omega_j$$

$$= \frac{1}{2} \omega^2 \sum_{i,j} n_i I_{ij} n_j$$

$$= \frac{1}{2} I(\hat{n}) \omega^2 \quad \text{where} \quad \vec{\omega} = \omega \hat{n}$$

Section 7.2

Show that $I(\hat{n})$ is independent of origin on $\hat{n}$



$$\sin(\pi-\theta) = \cancel{\sin\pi}^{\,0}\cos\theta - \cos\pi\sin\theta$$
$$= \sin\theta$$

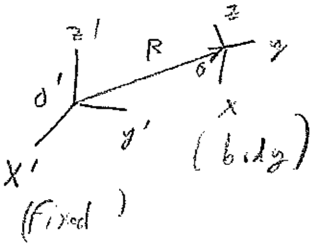
$$I(\hat{n}) = \sum_I m_I\, r_I^2 \sin^2\theta_I = \sum_I m_I d_I^2$$

$$I'(\hat{n}) = \sum_I m_I\, r_I'^2 \sin^2\theta_I' = \sum_I m_I d_I^2$$

$$\sin(2\pi-\theta) = \cancel{\sin(2\pi)}^{\,0}\cos\theta - \cos(2\pi)\sin\theta$$
$$= -\sin\theta$$

7.3

Exercise: Extend Ex. 7.2 to allow for translational motion.
Assume Origin of rigid body is located at COM
specified by $\vec{R}$ wrt fixed inertial frame



$$T = \frac{1}{2} \sum_I m_I |\dot{\vec{r}}_I|^2$$

$$\underbrace{\dot{\vec{r}}_I}_{\text{wrt fixed frame}} = \dot{\vec{R}} + \vec{\omega} \times \vec{r}_I \equiv \vec{V} + \vec{\omega} \times \vec{r}_I$$

$$T = \frac{1}{2} \sum_I m_I \left(\vec{V} + \vec{\omega} \times \vec{r}_I\right) \cdot \left(\vec{V} + \vec{\omega} \times \vec{r}_I\right)$$

$$= \frac{1}{2} \sum_I m_I \left[V^2 + 2\vec{V} \cdot (\vec{\omega} \times \vec{r}_I) + (\vec{\omega} \times \vec{r}_I) \cdot (\vec{\omega} \times \vec{r}_I)\right]$$

$$= \frac{1}{2} \underbrace{\sum_I m_I}_{M} V^2 + \underbrace{\vec{V} \cdot \sum_I m_I (\vec{\omega} \times \vec{r}_I)}_{\vec{V} \cdot \left[\vec{\omega} \times \left(\sum_I m_I \vec{r}_I\right)\right] = 0} + \frac{1}{2} \sum_I m_I (\vec{\omega} \times \vec{r}_I) \cdot (\vec{\omega} \times \vec{r}_I)$$

$$= \frac{1}{2} M V^2 + \frac{1}{2} \sum_I m_I \, \vec{\omega} \cdot \big(\underbrace{\vec{r}_I \times (\vec{\omega} \times \vec{r}_I)}_{\left[\vec{\omega} r_I^2 - \vec{r}_I (\vec{\omega} \cdot \vec{r}_I)\right]}\big)$$

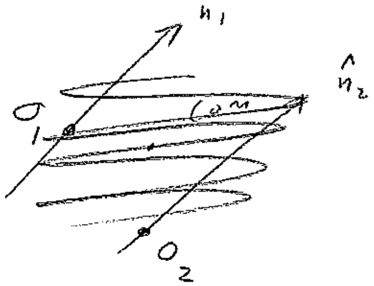
$$= \frac{1}{2} M V^2 + \frac{1}{2} \sum_I m_I \left(r_I^2 \omega^2 - (\vec{\omega} \cdot \vec{r}_I)^2\right)$$

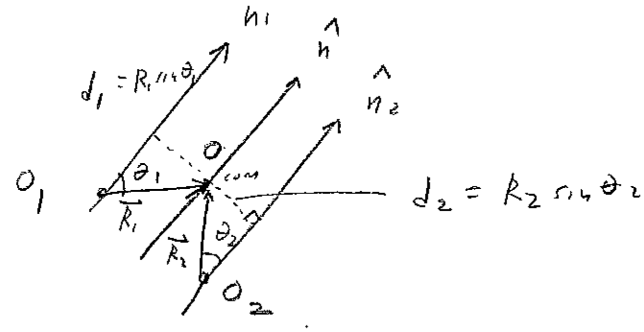
$$= \frac{1}{2} M V^2 + \frac{1}{2} \sum_{i,j} \omega_i \omega_j \Big(\underbrace{\sum_I m_I \left(r_I^2 \delta_{ij} - r_{Ii} r_{Ij}\right)}_{I_{ij}}\Big)$$

$$= \frac{1}{2} M V^2 + \frac{1}{2} \sum_{i,j} \omega_i I_{ij} \omega_j$$

Exercise 7.4

General parallel axis theorem:





$$I(\hat{n}_1) = I_{com}(\hat{n}) + M R_1^2 \sin^2\theta_1$$

$$I(\hat{n}_2) = I_{com}(\hat{n}) + M R_2^2 \sin^2\theta_2$$

subtract:

$$I(\hat{n}_2) - I(\hat{n}_1) = M R_2^2 \sin^2\theta_2 - M R_1^2 \sin^2\theta_1$$

$$= M(d_2^2 - d_1^2)$$

so $I(\hat{n}_2) = I(\hat{n}_1) + M(d_2^2 - d_1^2)$

①

~~A-L~~ B

AJP - 29562

"Generalization of Parallel Axis Theorem for the Rotational Inertia"

$$I_{ij} = \sum_I m_I \left(\delta_{ij} r_I^2 - r_{Ii} r_{Ij} \right)$$

$$\left(\epsilon_{ik\ell} r_\ell \right) \left(\epsilon_{kjm} r_m \right)$$

$$= - \epsilon_{i\ell k} \epsilon_{jmk} r_\ell r_m$$

$$= - \left(\delta_{ij} \delta_{\ell m} - \delta_{im} \delta_{\ell j} \right) r_\ell r_m$$

$$= - \delta_{ij} r^2 + r_i r_j$$

Thus, $I_{ij} = - \sum_I m_I (R_I)_i{}^k (R_I)_{kj}$

where $(R_I)_{ij} = \epsilon_{ijk} r_k$

$$L_i = \sum_I m_I \left(\vec{r}_I \times \dot{\vec{r}}_I \right)_i$$

$$= \sum_I m_I \epsilon_{ijk} r_{Ij} \dot{r}_{Ik}$$

$$= \sum_I m_I \epsilon_{ijk} r_{Ij} \left(\vec{\omega} \times \vec{r}_I \right)_k$$

$$= \sum_I m_I \epsilon_{ijk} r_{Ij} \epsilon_{k\ell m} \omega_\ell r_{Im}$$

$$= \sum_I m_I \epsilon_{ijk} r_{Ij} \epsilon_{k\ell m} r_{Im} \omega_\ell$$

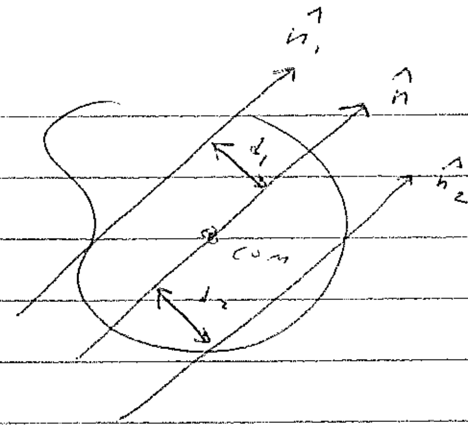
$$= I_{i\ell} \omega_\ell$$

where

$$I_{i\ell} = \sum_I m_I - \epsilon_{ikj} r_{Ij} \epsilon_{k\ell m} r_{Im} = - \sum_I (R_I)_i{}^k (R_I)^k{}_\ell$$

$I_1 = I_c + M d_1^2$

$I_2 = I_c + M d_2^2$



so subtracting

$I_2 = I_1 + M(d_2^2 - d_1^2)$

New equation

$I' = I + M[(R,R)] - 2m[(R,c)]$

$[(A,B)] = -\frac{1}{2}([\vec{r_A}][\vec{r_B}] + [\vec{r_B}][\vec{r_A}])$

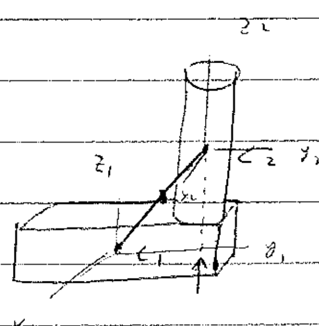
$$= \begin{bmatrix} (y_A y_B + z_A z_B) & -\frac{1}{2}(x_A y_B + y_A x_B) & -\frac{1}{2}(x_A z_B + z_A x_B) \\ \cdot & (x_A x_B + z_A z_B) & -\frac{1}{2}(y_A z_B + z_A y_B) \\ \cdot & \cdot & (x_A x_B + y_A y_B) \end{bmatrix}$$

(symmetric)

$M = 8\rho r^3$

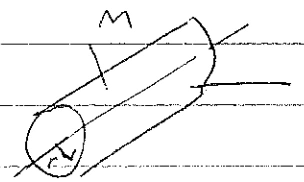
$M_1 = 4M = 32\rho r^3$

$M_2 = \pi M = 8\pi\rho r^3$

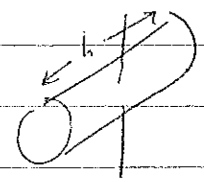


$x_1 = 0$
$y_1 = 3$
$z_1 = 5$

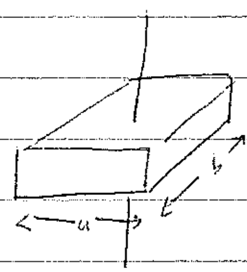
$\frac{8\pi\cancel{\rho r^3}\, r}{(32+8\pi)\cancel{\rho r^3}} = \left(\frac{\pi}{4+\pi}\right) r$



$$I = \frac{Mr^2}{2}$$



$$I = \frac{Mr^2}{4} + \frac{Mh^2}{12} = \frac{M}{12}\left(3r^2 + h^2\right)$$



$$I = \frac{M(a^2+b^2)}{12}$$

$\frac{M_1}{12}(4r^2+4r^2)$

$a = 2r$
$b = 8r$

cuboid:

$$\rightarrow I_1 = \begin{bmatrix} \frac{M_1}{12}(4r^2+64r^2) & 0 & 0 \\ 0 & \frac{M_1}{12}(4r^2+4r^2) & 0 \\ 0 & 0 & \frac{M_1}{12}(4r^2+64r^2) \end{bmatrix}$$

$$= 4\mu r^2 \begin{bmatrix} \frac{68}{12} & 0 & 0 \\ 0 & \frac{8}{12} & 0 \\ 0 & 0 & \frac{68}{12} \end{bmatrix}$$

$$\vec{R}_1 = (X, Y, Z) = \frac{\pi r}{4+\pi}(0, 3, 5)$$

$$[(\vec{R}_1, \vec{R}_1)] = \left(\frac{\pi r}{4+\pi}\right)^2 \begin{bmatrix} 9+25 & 0 & 0 \\ 0 & 25 & -15 \\ 0 & -15 & 9 \end{bmatrix}$$

cuboid

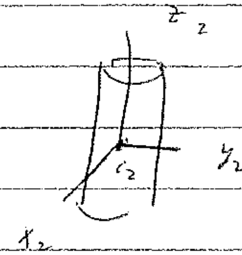
$$I_1' = I_1 + 4\mu[(\vec{R}_1, \vec{R}_1)] - 8\mu[(\vec{R}_1, \vec{R}_1)]$$

$$= 4\mu r^2 \begin{bmatrix} \frac{68}{12} & 0 & 0 \\ 0 & \frac{8}{12} & 0 \\ 0 & 0 & \frac{68}{12} \end{bmatrix} + 4\mu r^2 \left(\frac{\pi}{4+\pi}\right)^2 \begin{bmatrix} 34 & 0 & 0 \\ 0 & 25 & -15 \\ 0 & -15 & 9 \end{bmatrix}$$

cylinder: $C_2 = (0,0,0)$

$$R_2 = \frac{m_1 r}{m_1 + m_2}(0,-3,-5)$$

$$= \left(\frac{4r}{4+\pi}\right)(0,-3,-5)$$



$$I_2 = \begin{bmatrix} \frac{m_2}{12}(3r^2+64r^2) & 0 & 0 \\ 0 & \frac{m_2}{12}(3r^2+64r^2) & 0 \\ 0 & 0 & \frac{1}{2}m_2 r^2 \end{bmatrix}$$

$$= \underbrace{m_2}_{\pi\mu} r^2 \begin{bmatrix} \frac{67}{12} & 0 & 0 \\ 0 & \frac{67}{12} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

General calculation:

$$I' = I + M[(\vec{R},\vec{R})] - 2M[(\vec{R},\vec{C})]$$

$$I = \sum_i m_i \begin{bmatrix} (y_i^2+z_i^2) & -x_i y_i & -x_i z_i \\ -y_i x_i & (z_i^2+x_i^2) & -y_i z_i \\ -z_i x_i & -z_i y_i & (x_i^2+y_i^2) \end{bmatrix} = \begin{bmatrix} I_{xx} & \cdots & \cdots \\ \vdots & I_{yy} & \cdots \\ \vdots & \vdots & I_{zz} \end{bmatrix}$$

$$M[(\vec{R},\vec{R})] = M \begin{bmatrix} Y^2+Z^2 & -XY & -XZ \\ -YX & Z^2+X^2 & -YZ \\ -ZX & -ZY & X^2+Y^2 \end{bmatrix}$$

$$-2M[(\underset{\text{origin}}{\vec{R}},\underset{\text{com}}{\vec{C}})] = -2M \begin{bmatrix} Yy_c+Zz_c & -\frac{1}{2}(Xy_c+Yx_c) & -\frac{1}{2}(Xz_c+Zx_c) \\ -\frac{1}{2}(Yx_c+Xy_c) & Zz_c+Xx_c & -\frac{1}{2}(Yz_c+Zy_c) \\ -\frac{1}{2}(Zx_c+Xz_c) & -\frac{1}{2}(Zy_c+Yz_c) & Xx_c+Yy_c \end{bmatrix}$$

$$\rightarrow \quad I'_{zz} = I_{zz} + M(X^2+Y^2) - 2M(Xx_c+Yy_c)$$
$$= I_{zz} + M(X^2+Y^2) - 2MXx_c - 2MYy_c$$

$$I'_{xy} = I_{xy} - MXY + MXy_c + MYx_c$$

$\vdots$

etc

C

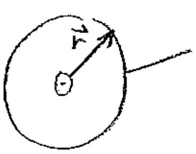
Problem 7.5 moments of inertia

$$I_{ij} = \int dV\, \rho(\vec r)\,(\delta_{ij} r^2 - r_i r_j)$$

$$I(\hat n) = \sum_{i,j} I_{ij}\, n^i n^j$$

$$= \int dV\, \rho(\vec r)\left(\sum_{i,j} \delta_{ij} n^i n^j r^2 - \sum_{i,j} r_i r_j n^i n^j\right)$$

$$= \int dV\, \rho(\vec r)\left(r^2 - (\vec r \cdot \hat n)^2\right)$$

a) 

Hoop, mass M, radius R

axis thru COM, out of page

$$\vec r \cdot \hat n = 0\ ,\quad \rho(\vec r) = \text{const} = \frac{M}{2\pi R \cdot A}$$

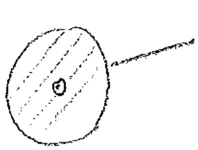
$$I \equiv I(\hat n)$$

$$= \int dV\, \rho(\vec r)\, r^2$$

$$= \cancel{A} \int_{\phi=0}^{2\pi} R\, d\phi \left(\frac{M}{2\pi R \cdot \cancel{A}}\right) R^2$$

$$= \frac{MR^2}{2\pi} \int_{\phi=0}^{2\pi} d\phi$$

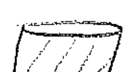
$$= \boxed{MR^2}$$

b) 

uniform solid disk, mass M, radius R

axis thru COM out of page

$$\vec r \cdot \hat n = 0\ ,\quad \rho(\vec r) = \frac{M}{\pi R^2 h}$$



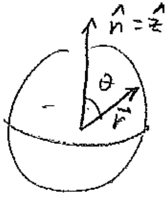
$$I = \int dV\, \rho(\vec r)\, r^2 = \frac{M}{\pi R^2 h} \int_0^h dz \int_{\phi=0}^{2\pi} \rho\, d\phi \int_0^R d\rho\, \rho^2$$

$$= \frac{M}{\pi R^2 \cancel{h}} \cdot \cancel{h} \cdot 2\pi \int_0^R d\rho\, \rho^3$$

$$= \frac{2M}{R^2} \left. \frac{\rho^4}{4} \right|_0^R$$

$$= \frac{2M}{R^2} \frac{R^4}{4}$$

$$= \boxed{\frac{1}{2} M R^2}$$

c) 

shell, mass M, radius R, axis through com

$$\rho = \frac{M}{4\pi R^2} \quad (\text{mass / area}) \qquad I = \int dA\, \rho(\vec{r}) \left(r^2 - (\vec{r}\cdot\hat{n})^2 \right)$$

$$\vec{r} \cdot \hat{n} = R \cos\theta$$

$$I = \int_{\theta=0}^{\pi} R^2 \sin\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi \left(R^2 - R^2 \cos^2\theta \right) \rho$$

$$= \frac{M}{4\pi R^2} R^4 \cdot 2\pi \int_{\theta=0}^{\pi} \sin\theta\, d\theta \left(1 - \cos^2\theta \right)$$

$$= \frac{1}{2} M R^2 \int_{-1}^{1} (1 - x^2)\, dx$$

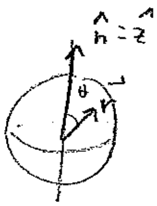
$x = \cos\theta$
$dx = -\sin\theta\, d\theta$
$\theta = 0, \pi \leftrightarrow x = 1, -1$

$$= \frac{1}{2} M R^2 \left. \left(x - \frac{x^3}{3} \right) \right|_{-1}^{1}$$

$$= \frac{1}{\cancel{2}} M R^2\, \cancel{2} \left(1 - \frac{1}{3} \right)$$

$$= \boxed{\frac{2}{3} M R^2}$$

d) uniform solid sphere, mass M, radius R, axis through COM



$$\vec{r}\cdot\hat{n} = r\cos\theta \quad , \quad \rho = \frac{M}{\frac{4}{3}\pi R^3}$$

$$I = \int dV \left(\frac{M}{\frac{4}{3}\pi R^3}\right)\left(r^2 - r^2\cos^2\theta\right)$$

$$= \frac{M}{\frac{4}{3}\pi R^3}\int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\, d\theta \int_0^R dr\; r^2\; r^2\left(1-\cos^2\theta\right)$$

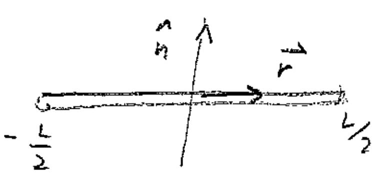
$$= \frac{M}{\frac{4}{3}\pi R^3}\; 2\pi \int_{-1}^{1} dx\left(1-x^2\right)\int_0^R dr\; r^4$$

$$= \frac{3}{2}\frac{M}{R^3}\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}\; \frac{r^5}{5}\Big|_0^R$$

$$= \frac{3}{\cancel{2}}\frac{M}{R^3}\;\cancel{2}\underbrace{\left(1-\frac{1}{3}\right)}_{\frac{2}{3}}\frac{R^5}{5}$$

$$= \boxed{\frac{2}{5}MR^2}$$

e) Uniform thin rod, mass M, length L, axis thru COM



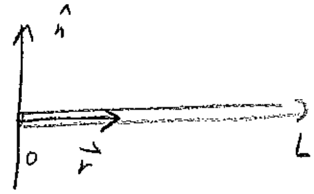
$$\vec{r}\cdot\hat{n} = 0 \qquad \rho = \frac{M}{L} \quad (\text{mass/length})$$

$$I = \int dx\; x^2 \rho$$

$$I = \frac{M}{L}\int_{-L/2}^{L/2} dx\; x^2 = \frac{M}{L}\frac{x^3}{3}\Big|_{-L/2}^{L/2} = \frac{M}{L}\frac{2}{3}\left(\frac{L^3}{8}\right) = \boxed{\frac{1}{12}ML^2}$$

④

f) uniform thin rod, mass M, length L, axis through end

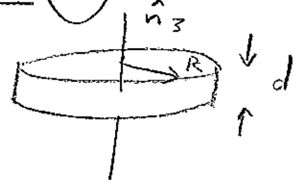


$\vec{r} \cdot \hat{n} = 0$, $\rho = \frac{M}{L}$ (mass/length)

$$I = \int dx \; x^2 \rho$$

$$I = \frac{M}{L} \int_0^L dx \; x^2 = \frac{M}{L} \frac{x^3}{3} \Big|_0^L = \boxed{\frac{1}{3} M L^2}$$

Problem 7.6 moments of inertia for this uniform circular disk!



$$\rho = \frac{M}{\pi R^2 d}$$

$$I_3 = I(\hat{n}_3) = \int dV\, \rho(\vec{r})\left(r^2 - (\vec{r}\cdot\hat{n}_3)^2\right)$$

$r^2 = \rho^2 + z^2$ cylindrical coords: (ρ, ϕ, z)

$\vec{r}\cdot\hat{n}_3 = z$ $\vec{r} = \rho\hat{\rho} + z\hat{z}$

Thus,
$$I_3 = \frac{M}{\pi R^2 d}\int_{z=-\frac{d}{2}}^{d/2} dz \int_{\phi=0}^{2\pi}\rho\, d\phi \int_{\rho=0}^{R} d\rho\left(\rho^2 + \cancel{z^2} - \cancel{z^2}\right)$$

$$= \frac{M}{\cancel{\pi} R^2 \cancel{d}}\cdot\cancel{d}\cdot 2\cancel{\pi}\int_{\rho=0}^{R} d\rho\, \rho^3$$

$$= \frac{2M}{R^2}\left.\frac{\rho^4}{4}\right|_0^R$$

$$= \boxed{\frac{1}{2}MR^2}$$

$$I_1 = I_2 = I(\hat{n}_1) = \int dV\, \rho(\vec{r})\left(r^2 - (\vec{r}\cdot\hat{n}_1)^2\right)$$

$$= \frac{M}{\pi R^2 d}\int_{z=-\frac{d}{2}}^{d/2} dz \int_{\phi=0}^{2\pi}\rho\, d\phi \int_{\rho=0}^{R} d\rho\left(\rho^2 + z^2 - \rho^2\cos^2\phi\right)$$

using $\hat{n}_1 = \hat{x}$, $\vec{r} = \rho\hat{\rho} + z\hat{z}$

$$= \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\hat{z}$$

$\rightarrow \vec{r}\cdot\hat{n}_1 = \rho\cos\phi$

(2

$$= \frac{M}{\pi R^2 d} \int_{z=-\frac{d}{2}}^{d/2} dz \int_{\phi=0}^{2\pi} \rho\, d\phi \int_{\rho=0}^{R} d\rho \left[\rho^2 (\underbrace{1-\cos^2\phi}_{\sin^2\phi}) + z^2\right]$$

$$= \frac{M}{\pi R^2 d} \left\{ \int_{z=-\frac{d}{2}}^{d/2} dz\, z^2 \int_{\phi=0}^{2\pi} d\phi \int_{\rho=0}^{R} \rho\, d\rho \right.$$

$$\left. + \int_{z=-\frac{d}{2}}^{d/2} dz \int_{\phi=0}^{2\pi} d\phi\, \sin^2\phi \int_{\rho=0}^{R} d\rho\, \rho^3 \right\}$$

$$= \frac{M}{\pi R^2 d} \left\{ \frac{z^3}{3}\Big|_{-d/2}^{d/2} \cdot 2\pi \cdot \frac{\rho^2}{2}\Big|_0^R \right.$$

$$\left. + d \cdot \frac{1}{2} \int_{\phi=0}^{2\pi} (1 - \underbrace{\cos 2\phi}_{0}) \; \frac{\rho^4}{4}\Big|_0^R \right\}$$

$$= \frac{M}{\cancel{\pi} R^2 d} \left\{ \frac{2}{3}\left(\frac{d^3}{8}\right) \cdot \cancel{2}\cancel{\pi} \cdot \frac{R^2}{\cancel{2}} + d \cdot \frac{1}{\cancel{2}} \cdot \cancel{2}\cancel{\pi}\, \frac{R^4}{4} \right\}$$

$$= M \left\{ \frac{d^2}{12} + \frac{R^2}{4} \right\}$$

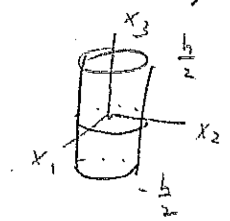
$$= \frac{1}{12} M d^2 + \frac{1}{4} M R^2$$

$$\cos 2\phi = \cos^2\phi - \sin^2\phi = 1 - 2\sin^2\phi$$

$$\sin^2\phi = \frac{1}{2}\left(1 - \cos(2\phi)\right)$$

(7.6)

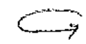
Exercise Calculate principal moments of inertia for a uniform circular cylinder of radius R, height h, total mass M

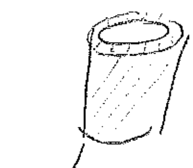


Com at center of cylinder

$dV = \rho\, d\rho\, d\phi\, dz$

$$I_3 \equiv I(\hat{x}_3)$$
$$= \int dm\,(x_1^2 + x_2^2)$$
$$= \int dm\, \rho^2$$
$$= \frac{2M}{R^2}\int_0^R \rho\, d\rho\, \rho^2$$
$$= \frac{2M}{R^2}\frac{\rho^4}{4}\Big|_0^R$$
$$= \frac{MR^2}{2}$$



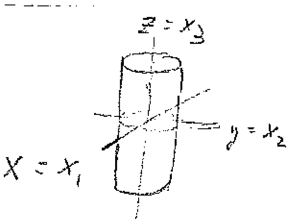


cylindrical shell

$$dm = \left(\frac{M}{\pi R^2 h}\right) dV$$
$$= \frac{M}{\pi R^2 h}\,\rho\, d\rho\, d\phi\, dz$$

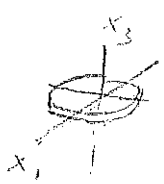
Integrate over ϕ, z:

$$dm = \frac{M}{\pi R^2 \cancel{h}}\,\rho\, d\rho \cdot 2\pi \cancel{h}$$
$$= \frac{2M}{R^2}\,\rho\, d\rho$$



$$I_1 = I(\hat{x}_1)$$
$$= \int \rho\, dV\,(r^2 - x^2)$$
$$= \int \rho\, dV\,(y^2 + z^2)$$

$$I_{ij} = \int \rho\, dV\,(r^2 \delta_{ij} - r_i r_j)$$
$$I(\hat{n}) = \sum_{i,j} I_{ij}\, n^i n^j$$



$\rightarrow$

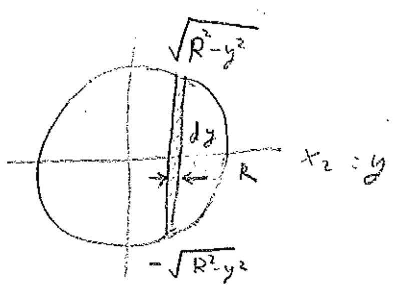
thin disk mass μ
(thickness dz)

$$I'_\mu = \int dm\, y^2$$
$$= 2\int_0^R \cancel{dz}\, dy\, 2\sqrt{R^2 - y^2}\,\frac{\mu}{\pi R^2 \cancel{dz}}\, y^2$$
$$= \frac{4\mu}{\pi R^2}\int_0^R dy\,\sqrt{R^2 - y^2}\; y^2$$
$$= \frac{4\mu}{\pi \cancel{R^2}}\int_0^{\pi/2} R\cos\theta\, d\theta\, R\cos\theta\, \cancel{R^2}\sin^2\theta$$
$$= \frac{4\mu R^2}{\pi}\int_0^{\pi/2} \cos^2\theta\,\sin^2\theta\, d\theta$$

$y = R\sin\theta$

$R^2 - y^2 = R^2(1 - \sin^2\theta)$
$= R^2\cos^2\theta$

$dy = R\cos\theta\, d\theta$



$$I'_\mu = \frac{4\mu R^2}{\pi}\int_0^{\pi/2} \cos^2\theta \sin^2\theta \, d\theta$$

$\cos^2\theta = 1-\sin^2\theta$

$$= \frac{4\mu R^2}{\pi}\int_0^{\pi/2} \left(1-\sin^2\theta\right)\sin^2\theta \, d\theta$$

$$= \frac{4\mu R^2}{\pi}\left[\int_0^{\pi/2} \sin^2\theta \, d\theta - \int_0^{\pi/2} \sin^4\theta \, d\theta\right]$$

<u>Now:</u> $$\int_0^{\pi/2} \sin^2\theta \, d\theta = \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/2}\right)$$

$$= \boxed{\frac{\pi}{4}}$$

$\cos 2x = \cos^2 x - \sin^2 x$
$= 1 - 2\sin^2 x$
$\sin^2 x = \frac{1-\cos 2x}{2}$

<u>And:</u> $$\int_0^{\pi/2} \sin^4\theta \, d\theta = \boxed{\frac{3}{16}\pi}$$ (see below)

<u>NOTE:</u> $$\int_0^{\pi/2} \sin^{2m} x \, dx = \frac{(2m-1)!!}{(2m)!!}\frac{\pi}{2}$$

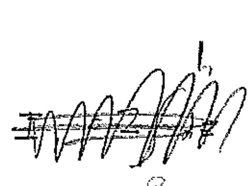
G&R
3.621, 3
p. 395

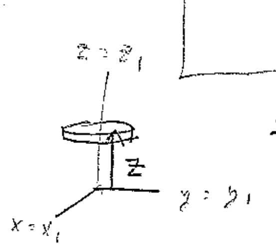
<u>m = 1:</u> $RHS = \frac{1!!}{2!!}\frac{\pi}{2} = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$

<u>m = 2:</u> $RHS = \frac{3!!}{4!!}\frac{\pi}{2} = \frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2} = \frac{3}{16}\pi$

Thus, $$\boxed{I'_\mu = \frac{4\mu R^2}{\pi}\left[\frac{\pi}{4} - \frac{3\pi}{16}\right] = \boxed{\frac{\mu R^2}{4}}}$$

$[\,] = \frac{\pi}{16}$





<u>Now:</u> $\mu = \frac{dz}{h} M$

└ infinitesimal mass

$$I_1 = 2\int_0^{\frac{h}{2}}\left(\frac{dz}{h}M\frac{R^2}{4} + \frac{dzM}{h}z^2\right)$$

$$\mu = \frac{dzM}{h}$$

$$I = I' + \mu a^2$$

$$a = z$$

$$= 2\frac{M}{h}\int_0^{\frac{h}{2}} dz\left(\frac{R^2}{4} + z^2\right)$$

$$= 2\frac{M}{h}\left[\frac{R^2}{4}\left(\frac{h}{2}\right) + \frac{z^3}{3}\Big|_0^{\frac{h}{2}}\right]$$

$$= 2\frac{M}{\cancel{h}}\frac{R^2\cancel{h}}{8} + \frac{2M}{h}\frac{1}{3}\left(\frac{h}{2}\right)^3$$

$$= \frac{MR^2}{4} + \frac{Mh^2}{12}$$

8

$$= \boxed{\frac{1}{4}M\left(R^2 + \frac{1}{3}h^2\right)}$$

(same for I_2)

Problem 7.7 Torque free motion with $\omega_2 = \text{const}$
for $I_1 < I_2 < I_3$

$$\dot{\omega}_1 = \omega_2 \omega_3 \frac{(I_2 - I_3)}{I_1}$$

$$\dot{\omega}_2 = \omega_3 \omega_1 \frac{(I_3 - I_1)}{I_2}$$

$$\dot{\omega}_3 = \omega_1 \omega_2 \frac{(I_1 - I_2)}{I_3}$$

NOTE: I will denote $\left(\frac{I_2 - I_3}{I_1}\right)$ as (23), etc.
to save some writing.

Take $\omega_2 = \text{const}$, $\omega_1 = \epsilon$, $\omega_3 = \epsilon$ (small but non-zero)
Check stability by taking 2nd time derivative.

$$\begin{aligned}
\ddot{\omega}_1 &= \dot{\omega}_2 \omega_3 (23) + \omega_2 \dot{\omega}_3 (23) \\
&= \omega_3^2 \omega_1 (31)(23) + \omega_2^2 \omega_1 (12)(23) \\
&= \epsilon^3 (31)(23) + \kappa^2 \omega_1 \\
&\approx \kappa^2 \omega_1 \quad (\text{ignoring } \epsilon^3)
\end{aligned}$$

$$\text{where } \kappa^2 = \omega_2^2 \left(\frac{I_1 - I_2}{I_3}\right)\left(\frac{I_2 - I_3}{I_1}\right) > 0$$

similarly

$$\begin{aligned}
\ddot{\omega}_3 &= \dot{\omega}_1 \omega_2 (12) + \omega_1 \dot{\omega}_2 (12) \\
&= \omega_2^2 \omega_3 (12)(13) + \omega_1^2 \omega_3 (12)(31) \\
&= \kappa^2 \omega_3 + \epsilon^3 (12)(31) \\
&\approx \kappa^2 \omega_3 \quad (\text{ignoring } \epsilon^3)
\end{aligned}$$

$$
\begin{aligned}
\ddot{\omega}_2 &= \dot{\omega}_3\,\omega_1\,(31) + \omega_3\,\dot{\omega}_1\,(31) \\
&= \omega_2\,\omega_1^2\,(31)(12) + \omega_2\,\omega_3^2\,(31)(23) \\
&= \omega_2\,\epsilon^2\,(31)(12) + \omega_2\,\epsilon^2\,(31)(23) \\
&= 0 \quad (\text{ignoring } \epsilon^2)
\end{aligned}
$$

Thus,
$$
\begin{aligned}
\ddot{\omega}_1 &\approx \kappa^2\,\omega_1 \\
\ddot{\omega}_3 &\approx \kappa^2\,\omega_3
\end{aligned}
$$

$$
\begin{aligned}
\rightarrow \quad \omega_1 &\approx A\,e^{\kappa t} + B\,e^{-\kappa t} \\
\omega_3 &\approx C\,e^{\kappa t} + D\,e^{-\kappa t}
\end{aligned}
$$

which grow exponentially with time.
So <u>unstable</u>

Problem (7.8) Showing $\dot{\psi} = -\Omega$

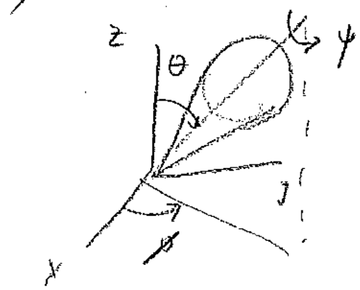
Use: $\omega_3 = \cos\theta\,\dot{\phi} + \dot{\psi}$ (7.34)

$\dot{\phi} = \dfrac{I_3\,\omega_3}{I_1\cos\theta}$ (7.58)

$\Omega = \dfrac{\omega_3(I_3 - I_1)}{I_1}$ (7.44)

Proof:
$$\dot{\psi} = \omega_3 - \cos\theta\,\dot{\phi}$$
$$= \omega_3 - \cancel{\cos\theta}\,\frac{I_3\,\omega_3}{I_1\cancel{\cos\theta}}$$
$$= \omega_3\left(\frac{I_1 - I_3}{I_1}\right)$$
$$= -\Omega$$

Symmetric top $(I_1 = I_2)$



$p_\psi = \text{const}$

$p_\phi = \text{const}$

$U(\theta)\ Mgh\cos\theta$

$$\frac{d\phi}{dt} = \dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}$$

$$\frac{d\psi}{dt} = \dot{\psi} = \frac{p_\psi}{I_3} - \cos\theta \left(\frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}\right)$$

$$= \frac{p_\psi}{I_3} - \cos\theta\,\dot{\phi}$$

$$V_{eff}(\theta) = \frac{1}{2}\frac{(p_\phi - p_\psi\cos\theta)^2}{I_1 \sin^2\theta} + Mgh\cos\theta$$

$$E' = E - \frac{1}{2}\frac{p_\psi^2}{I_3} = \frac{1}{2} I_1 \dot{\theta}^2 + V_{eff}(\theta)$$

$$\sqrt{2\left(E' - V_{eff}(\theta)\right)} = \frac{d\theta}{dt}$$

$$\rightarrow \quad dt = \frac{d\theta}{\sqrt{2\left(E' - V_{eff}(\theta)\right)}}$$

$$\sqrt{\frac{I_1}{Mgh}} \equiv T \quad \text{(time scale)}$$

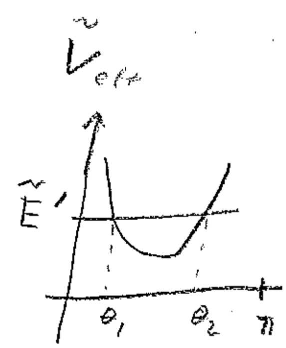
$$\sqrt{I_1 \cdot Mgh} \equiv L \quad \text{(ang. momentum scale)}$$

so $\quad T \frac{d}{dt} \to \frac{d}{d\tilde{t}}$

$$\frac{p_\phi,\ p_\psi}{L} \to \tilde{p}_\phi,\ \tilde{p}_\psi$$

$$\frac{V_{eff}'}{Mgh},\ \frac{E'}{Mgh} \to \tilde{V}_{eff},\ \tilde{E}$$

$$\boxed{\begin{aligned} \frac{d\phi}{d\tilde{t}} &= \frac{\tilde{p}_\phi - \tilde{p}_\psi \cos\theta}{\sin^2\theta} \\ \frac{d\psi}{d\tilde{t}} &= \left(\frac{I_1}{I_3}\right)\tilde{p}_\psi - \cos\theta\,\frac{d\phi}{d\tilde{t}} \\ \frac{d\theta}{d\tilde{t}} &= \sqrt{2\left(\tilde{E}' - \tilde{V}_{eff}(\theta)\right)} \end{aligned}}$$



where $\quad \tilde{V}_{eff}(\theta) = \frac{1}{2}\frac{(\tilde{p}_\phi - \tilde{p}_\psi\cos\theta)^2}{\sin^2\theta} + \cos\theta$

Adjustable parameters:

$I_1,\ I_3,\ Mgh,\ E',\ p_\psi,\ p_\phi,\ \phi(0),\ \psi(0),\ \theta(0)$

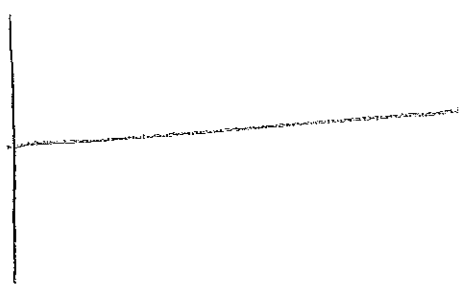
$E' = \tilde{V}_{eff}(\theta_1)$, $\quad \phi(0) = 0$, $\quad \psi(0) = 0$, $\quad \theta(0) = \theta_1$

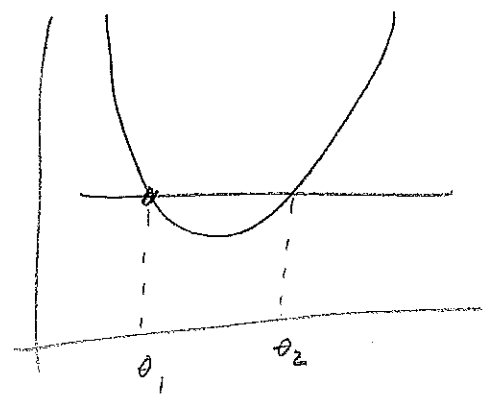
Three cases:

(i) $\left.\dfrac{d\phi}{d\tilde{t}}\right|_{\theta=\theta_1} = 0 \quad \rightarrow \quad \tilde{p}_\phi - \tilde{p}_\psi \cos\theta_1 = 0$ (cusp)

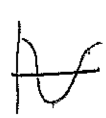
(ii) $\left.\dfrac{d\phi}{d\tilde{t}}\right|_{\theta=\theta_1} > 0 \quad \rightarrow \quad \tilde{p}_\phi - \tilde{p}_\psi \cos\theta_1 > 0$ (prograde)

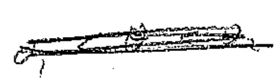
(iii) $\left.\dfrac{d\phi}{d\tilde{t}}\right|_{\theta=\theta_1} < 0 \quad \rightarrow \quad \tilde{p}_\phi - \tilde{p}_\psi \cos\theta_1 < 0$ (retrograde)





$$\frac{\tilde{p}_\phi - \tilde{p}_\psi \cos\theta_1}{\sin^2\theta_1} > 0$$





$$\frac{\tilde{p}_\phi - \tilde{p}_\psi \cos\theta_1}{\sin^2\theta_1} = 0$$

Problem (7.9) Kinetic energy for symmetric top with one point fixed $(I_1 = I_2,\ I_3)$

$$T = \frac{1}{2} \sum_i I_i \omega_i^2 \quad \text{where}$$

$$\omega_1 = -\sin\theta \cos\psi\, \dot{\phi} + \sin\psi\, \dot{\theta}$$
$$\omega_2 = \sin\theta \sin\psi\, \dot{\phi} + \cos\psi\, \dot{\theta}$$
$$\omega_3 = \cos\theta\, \dot{\phi} + \dot{\psi}$$

$$\rightarrow T = \frac{1}{2} I_1 \left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2} I_3 \omega_3^2$$

$$= \frac{1}{2} I_1 \Big(\sin^2\theta \cos^2\psi\, \dot{\phi}^2 + \sin^2\psi\, \dot{\theta}^2 \cancel{- 2\sin\theta \sin\psi \cos\psi\, \dot{\theta}\dot{\phi}}$$
$$+ \sin^2\theta \sin^2\psi\, \dot{\phi}^2 + \cos^2\psi\, \dot{\theta}^2 \cancel{+ 2\sin\theta \sin\psi \cos\psi\, \dot{\theta}\dot{\phi}} \Big)$$
$$+ \frac{1}{2} I_3 \left(\cos\theta\, \dot{\phi} + \dot{\psi}\right)^2$$

$$= \frac{1}{2} I_1 \left(\sin^2\theta\, \dot{\phi}^2 + \dot{\theta}^2\right) + \frac{1}{2} I_3 \left(\cos\theta\, \dot{\phi} + \dot{\psi}^2\right)$$

①

Problem 7.10 Effective potential for symmetric top with one point fixed

$$U_{eff}(\theta) = \frac{1}{2}\left(\frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}\right)^2 + Mgh\cos\theta$$

$$\frac{dU_{eff}}{d\theta} = \frac{(p_\phi - p_\psi\cos\theta)\, p_\psi \sin\theta}{I_1 \sin^2\theta} - \frac{(p_\phi - p_\psi\cos\theta)^2}{I_1 \sin^3\theta}\cos\theta$$

$$- Mgh\sin\theta$$

$$= \frac{1}{I_1 \sin^3\theta}(p_\phi - p_\psi\cos\theta)\left[p_\psi \sin^2\theta - (p_\phi - p_\psi\cos\theta)\cos\theta\right]$$

$$- Mgh\sin\theta$$

Define: $\beta \equiv p_\phi - p_\psi\cos\theta$

$$\left.\frac{dU_{eff}}{d\theta}\right|_{\theta=\theta_0} = 0$$

$$\rightarrow \quad 0 = \frac{1}{I_1 \sin^3\theta_0}\left\{\beta\, p_\psi \sin^2\theta_0 - \beta^2\cos\theta_0 - MghI_1\sin^4\theta_0\right\}$$

$$= \frac{-1}{I_1 \sin^3\theta_0}\left\{\cos\theta_0\,\beta^2 - p_\psi \sin^2\theta_0\,\beta + MghI_1\sin^4\theta_0\right.$$

$\rightarrow$ quadratic equation for β

$$\cos\theta_0\,\beta^2 - p_\psi\sin^2\theta_0\,\beta + MghI_1\sin^4\theta_0 = 0$$

Solution:

$$\beta_\pm = \frac{p_\psi \sin^2\theta_0 \pm \sqrt{p_\psi^2 \sin^4\theta_0 - 4\cos\theta_0 Mgh I_1 \sin^4\theta_0}}{2\cos\theta_0}$$

$$= \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left[1 \pm \sqrt{1 - \frac{4MghI_1\cos\theta_0}{p_\psi^2}}\right]$$

Problem 7.11 Calculation of potential for precession of equinoxes (1)

$$U = -\frac{GM'}{r} \sum_I \frac{m_I}{\sqrt{1 + \left(\frac{r_I'}{r}\right)^2 - 2\left(\frac{r_I'}{r}\right)\cos\gamma_I}}$$

$$= -\frac{GM'}{r} \sum_I m_I \sum_{n=0} \left(\frac{r_I'}{r}\right)^n P_n(\cos\gamma_I)$$

using generating function for Legendre polynomials

Calculate 1st three terms:

$$U_0 = -\frac{GM'}{r} \sum_I m_I \underbrace{\left(\frac{r_I'}{r}\right)^0}_{=1} \underbrace{P_0(\cos\gamma_I)}_{=1}$$

$$= -\frac{GM'M}{r}$$

$$U_1 = -\frac{GM'}{r} \sum_I m_I \left(\frac{r_I'}{r}\right) P_1(\cos\gamma_I)$$

$$= -\frac{GM'}{r^2} \sum_I m_I r_I' \cos\gamma_I$$

$$= -\frac{GM'}{r^2} M \underbrace{x_{com}}_{=0}$$

$$= 0$$

$$U_2 = -\frac{GM'}{r} \sum_I m_I \left(\frac{r_I'}{r}\right)^2 \frac{1}{2}\left(3\cos^2\gamma_I - 1\right)$$

$$= -\frac{3}{2}\frac{Gm'}{r^3}\sum_I m_I r_I'^2\left(\cos^2\gamma_I - \frac{1}{3}\right)$$

$$= \frac{3}{2}\frac{Gm'}{r^3}\sum_I m_I r_I'^2\left(\frac{1}{3} - \cos^2\gamma_I\right)$$

<u>Recall:</u>

$$I_{ij} \equiv \sum_I m_I\left(\delta_{ij}\, r_I'^2 - r_{Ii}'\, r_{Ij}'\right)$$

$$Q_{ij} \equiv I_{ij} - \frac{1}{3}\,Tr(I)\,\delta_{ij}$$

$$Tr(I) = \sum_i I_{ii}$$

$$= \sum_I m_I\left(3r_I'^2 - r_I'^2\right) = \sum_I 2m_I r_I'^2$$

So $$Q_{ij} = \sum_I m_I\left(\delta_{ij}\, r_I'^2 - r_{Ii}'\, r_{Ij}' - \frac{2}{3} r_I'^2\delta_{ij}\right)$$

$$= \sum_I m_I\left(\frac{1}{3} r_I'^2\delta_{ij} - r_{Ii}'\, r_{Ij}'\right)$$

<u>Now:</u> $\cos\gamma_I = \hat{r}_I' \cdot \hat{r}$, $\hat{u} = \frac{\vec{r}}{r} = \hat{r}$

Thus, $$\sum_{i,j} Q_{ij}\, u_i u_j = \sum_{i,j} Q_{ij}\frac{x^i}{r}\frac{x^j}{r}$$

$$= \sum_I m_I\left(\frac{1}{3} r_I'^2 - \sum_{i,j}\frac{r_{Ii}' x_i}{r}\frac{r_{Ij}' x_j}{r}\right)$$

$$= \sum_I m_I\left(\frac{1}{3} r_I'^2 - r_I'^2\cos^2\gamma_I\right)$$

$$= \sum_I m_I r_I'^2\left(\frac{1}{3} - \cos^2\gamma_I\right)$$

so $U_2 = \frac{3}{2} \frac{GM'}{r^3} \sum_I r_I'^2 \left(\frac{1}{3} - \cos^2 \gamma_I \right)$

$= \frac{3}{2} \frac{GM'}{r^3} \sum_{i,j} Q_{ij} u_i u_j$

Work in principal axes basis:

$I_{ij} = I_i \delta_{ij}$

$\rightarrow \sum_{i,j} Q_{ij} u_i u_j = \sum_{i,j} \left(I_{ij} - \frac{1}{3} Tr(I) \delta_{ij} \right) u_i u_j$

$= \sum_{i,j} \left(I_i \delta_{ij} - \frac{1}{3} Tr(I) \delta_{ij} \right) u_i u_j$

$= I_1 u_1^2 + I_2 u_2^2 + I_3 u_3^2 - \frac{1}{3} Tr(I)$

$= I_1 u_1^2 + I_2 u_2^2 + I_3 u_3^2 - \frac{1}{3} (I_1 + I_2 + I_3)$

symmetric rigid body: $I_1 = I_2$

$\rightarrow \sum_{i,j} Q_{ij} u_i u_j = I_1 (u_1^2 + u_2^2) + I_3 u_3^2 - \frac{1}{3} (2 I_1 + I_3)$

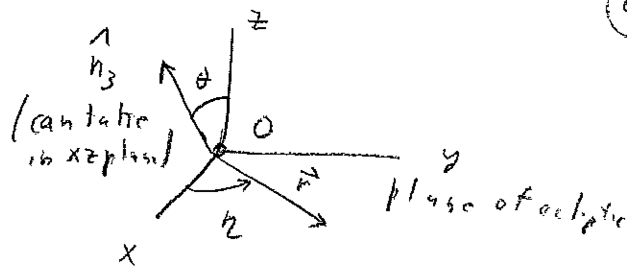
$= I_1 (1 - u_3^2) + I_3 u_3^2 - \frac{2}{3} I_1 - \frac{1}{3} I_3$

$= (I_3 - I_1) u_3^2 + \frac{1}{3} I_1 - \frac{1}{3} I_3$

$= (I_3 - I_1) \left[u_3^2 - \frac{1}{3} \right]$

$= \frac{2}{3} (I_3 - I_1) \frac{1}{2} (3 u_3^2 - 1)$

$= \frac{2}{3} (I_3 - I_1) P_2(u_3)$



Now,

$$u_3 = \hat{n}_3 \cdot \hat{r}$$

$$= \hat{n}_3 \cdot \frac{\vec{r}}{r}$$

$$= \left(\sin\theta\,\hat{x} + \cos\theta\,\hat{z}\right)\cdot\left(\cos\eta\,\hat{x} + \sin\eta\,\hat{y}\right) \qquad \theta: \text{ tilt angle}$$

$$= \sin\theta\cos\eta$$

Thus,

$$U_2 = \frac{3}{2}\frac{GM'}{r^3}\sum_{i,j} Q_{ij}\,u_i u_j$$

$$= \cancel{\frac{3}{2}}\frac{GM'}{r^3}\,\cancel{\frac{2}{3}}\left(I_3 - I_1\right) P_2\left(\sin\theta\cos\eta\right)$$

$$= \frac{1}{2}\frac{GM'}{r^3}\left(I_3 - I_1\right)\left(3\sin^2\theta\cos^2\eta - 1\right)$$

Average over one complete orbit

$$\frac{1}{2\pi}\int_{\eta=0}^{2\pi} d\eta \; ; \quad \rightarrow \quad \frac{1}{2\pi}\int_{\eta=0}^{2\pi}\cos^2\eta = \frac{1}{2}$$

$$\rightarrow \quad \bar{U}_2 = \frac{1}{2}\frac{GM'}{r^3}\left(I_3 - I_1\right)\left(\frac{3}{2}\sin^2\theta - 1\right) \qquad \sin^2\theta = 1 - \cos^2\theta$$

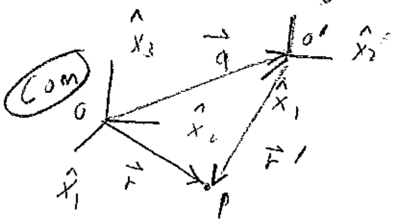
$$= \frac{1}{2}\frac{GM'}{r^3}\left(I_3 - I_1\right)\left(\frac{1}{2} - \frac{3}{2}\cos^2\theta\right)$$

$$= -\frac{1}{2}\frac{GM'}{r^3}\left(I_3 - I_1\right)\frac{1}{2}\left(3\cos^2\theta - 1\right)$$

$$= -\frac{1}{2}\frac{GM'}{r^3}\left(I_3 - I_1\right)P_2\left(\cos\theta\right)$$

(7.1)

Problem: Calculate the components of the ~~rotational~~ inertial tensor I'_{ij} w.r.t a coordinate frame (origin O' not at COM.)



$$I'_{ij} = \sum_I m_I \left(r_I'^2 \delta_{ij} - r'_{Ii} r'_{Ij} \right)$$

Now: $\vec{r}_I = \vec{r}_I\,' + \vec{a} \quad \rightarrow \quad r'_{Ii} = r_{Ii} - a_i$

$$r'_{Ij} = r_{Ij} - a_j$$

$$\begin{aligned} r_I'^2 &= \vec{r}_I\,' \cdot \vec{r}_I\,' \\ &= (\vec{r}_I - \vec{a}) \cdot (\vec{r}_I - \vec{a}) \\ &= r_I^2 + a^2 - 2\vec{a}\cdot\vec{r}_I \\ &= r_I^2 + a^2 - 2\sum_l a_l r_{Il} \end{aligned}$$

O' not at COM

$$\rightarrow \boxed{I'_{ij}} = \sum_I m_I \left[\left(r_I^2 + a^2 - 2\sum_l a_l r_{Il} \right) \delta_{ij} - (r_{Ii} - a_i)(r_{Ij} - a_j) \right]$$

$$\begin{aligned} &= \sum_I m_I \Big[\left(r_I^2 \delta_{ij} - r_{Ii} r_{Ij} \right) \\ &\qquad + \left(a^2 \delta_{ij} - a_i a_j \right) \\ &\qquad + \delta_{ij} \left(-2 \sum_l a_l r_{Il} \right) + a_i r_{Ij} + a_j r_{Ii} \Big] \end{aligned}$$

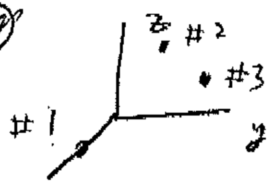
$$= I_{ij} + \left(\sum_I m_I \right) \left(a^2 \delta_{ij} - a_i a_j \right)$$

$$- 2\delta_{ij} \sum_l a_l \Big(\underbrace{\sum_I m_I r_{Il}}_{0} \Big)$$

O at COM

$$+ a_i \Big(\underbrace{\sum_I m_I r_{Ij}}_{0} \Big) + a_j \Big(\underbrace{\sum_I m_I r_{Ii}}_{0} \Big)$$

$$= \boxed{I_{ij} + M\left(a^2 \delta_{ij} - a_i a_j\right)}$$



$m_1 = m_2 = m_3 \equiv m$ Prob 7.2

$\vec{r}_1 = (a, 0, 0) \rightarrow r_1^2 = a^2$

$\vec{r}_2 = (0, a, 2a) \rightarrow r_2^2 = a^2 + 4a^2 = 5a^2$

$\vec{r}_3 = (0, 2a, a) \rightarrow r_3^2 = 4a^2 + a^2 = 5a^2$

a) $$I_{ij} = m\left[\left(r_1^2 + r_2^2 + r_3^2\right)\delta_{ij} - r_{1i} r_{1j} - r_{2i} r_{2j} - r_{3i} r_{3j}\right]$$

$$= m\left\{ 11a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - a^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} - a^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{pmatrix} \right\}$$

$$= ma^2 \left\{ \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix} - \begin{pmatrix} 1 & & \\ & 5 & 4 \\ & 4 & 5 \end{pmatrix} \right\}$$

$$= ma^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix}$$

$$= 2ma^2 \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$

b) Find eigenvectors and eigenvalues of I.

Eigenvalues are the principal moments of inertia

Eigenvectors are the principal axes for the body

$$0 = \det\left(I - \lambda \mathbb{1}\right)$$

$$= \det \begin{pmatrix} 10ma^2 - \lambda & 0 & \\ 0 & 6ma^2 - \lambda & -4ma^2 \\ & -4ma^2 & 6ma^2 - \lambda \end{pmatrix}$$

$$0 = (10\,ma^2 - \lambda)\left[(6\,ma^2 - \lambda)^2 - 16\,m^2a^4\right]$$

$$= (10\,ma^2 - \lambda)\left[36\,m^2a^4 + \lambda^2 - 12\,ma^2\lambda - 16\,m^2a^4\right]$$

$$= (10\,ma^2 - \lambda)\left[20\,m^2a^4 - 12\,ma^2\lambda + \lambda^2\right]$$

$$= (10\,ma^2 - \lambda)(\lambda - 10\,ma^2)(\lambda - 2\,ma^2)$$

Eigen values: $\lambda_1 = 2ma^2$, $\lambda_2 = \lambda_3 = 10\,ma^2$ (Double root)

Eigen vectors:

$\underline{\lambda_1 = 2ma^2}$:

$$\begin{bmatrix} 8ma^2 & 0 & 0 \\ 0 & 4ma^2 & -4ma^2 \\ 0 & -4ma^2 & 4ma^2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow V_1 = 0$, $V_2 - V_3 = 0 \rightarrow V_2 = V_3$

Thus $\boxed{\hat{n}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$

$\underline{\lambda_{2,3} = 10ma^2}$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4ma^2 & -4ma^2 \\ 0 & -4ma^2 & -4ma^2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

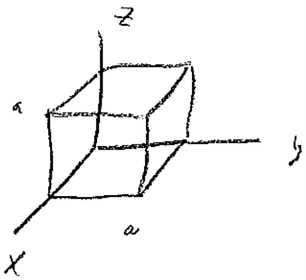
$\rightarrow V_1 =$ any thing, $V_2 + V_3 = 0 \rightarrow V_3 = -V_2$

So $\boxed{\hat{n}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \hat{n}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$

are legitimate choices.

Problem: 7.3

a) Calculate



$$I_{ij} = \int dV \rho(\vec{r}) \left(r^2 \delta_{ij} - r_i r_j \right)$$

$$= \rho \int_0^a dx \int_0^a dy \int_0^a dz \left[(x^2+y^2+z^2)\delta_{ij} - r_i r_j \right]$$

$$I_{xx} = \rho \int_0^a\int_0^a\int_0^a dx\,dy\,dz \left[(x^2+y^2+z^2) - x^2 \right]$$

$$= \rho \int_0^a\int_0^a\int_0^a dx\,dy\,dz \left[y^2 + z^2 \right]$$

$$= \rho a \left[a \frac{y^3}{3}\Big|_0^a + a \frac{z^3}{3}\Big|_0^a \right]$$

$$= \frac{1}{3} \rho a^2 \left[a^3 + a^3 \right]$$

$$= \frac{2}{3} \rho a^5$$

Now: $M = \rho \cdot \text{volume} = \rho a^3$

so $\rho = \frac{M}{a^3}$

$$\boxed{I_{xx} = \frac{2}{3} M a^2}$$

similarly, $\boxed{I_{yy} = \frac{2}{3} M a^2 , \quad I_{zz} = \frac{2}{3} M a^2}$

$$I_{xy} = \rho \int_0^a\int_0^a\int_0^a dx\,dy\,dz \left[-xy \right]$$

$$= -\rho a \frac{x^2}{2}\Big|_0^a \frac{y^2}{2}\Big|_0^a$$

$$= -\frac{\rho a}{4} a^2 a^2 = -\rho \frac{a^5}{4} = -\frac{M a^2}{4}$$

2

Thus,

$$\boxed{I_{xy} = \frac{-Ma^2}{4}}$$

similarly, $\boxed{I_{zx} = I_{yz} = -\frac{Ma^2}{4}}$

$$I_{ij} = Ma^2 \begin{bmatrix} 2/3 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 2/3 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 2/3 \end{bmatrix}$$

---

b) Find principal axes and principal moments of inertia.

~~Find~~ → Eigenvectors, eigenvalues of $I_{ij} = Ma^2 \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

$$0 = \det(I - \lambda \mathbb{1})$$

$$= \det \begin{bmatrix} \frac{2}{3}Ma^2 - \lambda & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 - \lambda & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 - \lambda \end{bmatrix}$$

$\beta \equiv Ma^2$

$$\cancel{= \left(\frac{2}{3}Ma^2 - \lambda\right)\left(\left(\frac{2}{3}Ma^2 - \lambda\right)\left(\frac{2}{3}Ma^2 - \lambda\right) - \frac{1}{16}\right)}$$

$$\cancel{+\frac{1}{4}\left(-\frac{1}{4}\left(\frac{2}{3}Ma^2 - \lambda\right) - \frac{1}{16}\right)}$$

$$-\frac{1}{4}\left(\cancel{\frac{1}{16} + \frac{1}{4}\left(\frac{2}{3}Ma^2 - \lambda\right)}\right)$$

$-\frac{1}{4} \quad -\frac{2}{3}$

$-\frac{3}{12} \quad -\frac{8}{12}$

subtract 2nd row from 1st to simplify calculation on determinant

$$0 = \det \begin{vmatrix} \frac{11}{12}\beta - \lambda & -\frac{11}{12}\beta + \lambda & 0 \\ -\frac{1}{4}\beta & \frac{2}{3}\beta - \lambda & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta - \lambda \end{vmatrix}$$

$$= \left(\frac{11}{12}\beta - \lambda\right)\left[\left(\frac{2}{3}\beta - \lambda\right)^2 - \frac{1}{16}\beta^2\right] + \left(\frac{11}{12}\beta - \lambda\right)\left[-\frac{1}{4}\beta\left(\frac{2}{3}\beta - \lambda\right) - \frac{1}{16}\beta^2\right]$$

$$= \left(\frac{11}{12}\beta - \lambda\right)\left[\left(\frac{2}{3}\beta - \lambda\right)^2 - \frac{1}{16}\beta^2 - \frac{1}{4}\beta\left(\frac{2}{3}\beta - \lambda\right) - \frac{1}{16}\beta^2\right]$$

$$= \left(\frac{11}{12}\beta - \lambda\right)\left[\left(\frac{2}{3}\beta - \lambda\right)^2 - \frac{1}{4}\beta\left(\frac{2}{3}\beta - \lambda\right) - \frac{1}{8}\beta^2\right]$$

$$= \left(\frac{11}{12}\beta - \lambda\right)\left[\frac{4}{9}\beta^2 + \lambda^2 - \frac{4}{3}\beta\lambda - \frac{1}{6}\beta^2 + \frac{\beta}{4}\lambda - \frac{1}{8}\beta^2\right]$$

$$= \left(\frac{11}{12}\beta - \lambda\right)\left[\lambda^2 - \frac{13}{12}\beta\lambda + \frac{11}{72}\beta^2\right]$$

Thus, $\boxed{\lambda = \frac{11}{12}\beta}$, $\lambda_\pm = \dfrac{\frac{13}{12}\beta \pm \sqrt{\left(\frac{13}{12}\beta\right)^2 - 4\left(\frac{11}{72}\right)\beta^2}}{2}$

$$= \frac{\frac{13}{12}\beta \pm \sqrt{\frac{169}{144}\beta^2 - \frac{88}{144}\beta^2}}{2}$$

$$-\frac{4}{3} + \frac{1}{4} = -\frac{16}{12} + \frac{3}{12} = -\frac{13}{12}$$

$$\frac{4}{9} - \frac{1}{6} - \frac{1}{8} = \frac{32 - 12 - 9}{72} = \frac{11}{72}$$

$$= \frac{13}{24}\beta \pm \frac{1}{24}\sqrt{81\beta^2}$$

$$\begin{array}{r} 169 \\ \underline{88} \\ 81 \end{array}$$

$$= \frac{13}{24}\beta \pm \frac{9}{24}\beta$$

$$= \begin{cases} \frac{1}{6}\beta \\ \frac{11}{12}\beta \end{cases}$$

$$\frac{13-9}{24} = \frac{4}{24}, \quad \frac{22}{24}$$

Thus, $\boxed{\lambda = \underbrace{\frac{11}{12}\beta, \quad \frac{11}{12}\beta}_{\text{double root}}, \quad \frac{1}{6}\beta}$

Eigen Vectors:

$\lambda = \frac{1}{6}\beta$:

$$\begin{bmatrix} \frac{2}{3}\beta - \frac{1}{6}\beta & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta - \frac{1}{6}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta - \frac{1}{6}\beta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{2}{3} - \frac{1}{6} = \frac{4}{6} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$\begin{bmatrix} \frac{\beta}{2} & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{\beta}{2} & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{\beta}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\div \frac{4}{\beta}$:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(1) $2v_1 - v_2 - v_3 = 0$

(2) $-v_1 + 2v_2 - v_3 = 0$

(3) $-v_1 - v_2 + 2v_3 = 0 \rightarrow v_2 = -v_1 + 2v_3$

(1): $2v_1 - (-v_1 + 2v_3) - v_3 = 0$

$3v_1 - 3v_3 = 0 \rightarrow \boxed{v_3 = v_1}$

$v_2 = -v_1 + 2v_3 = -v_1 + 2v_1 = v_1$

so $\boxed{v_2 = v_1}$

Thus, $\boxed{\hat{n}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda_1 = \frac{1}{6} M a^2}$

---

other eigen vectors:

$\lambda = \frac{11}{12}\beta$:

$$\begin{bmatrix} \frac{2}{3}\beta - \frac{11}{12}\beta & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta - \frac{11}{12}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta - \frac{11}{12}\beta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{2}{3} - \frac{11}{12} = \frac{8}{12} - \frac{11}{12} = -\frac{3}{12} = -\frac{1}{4}$

$$-\frac{1}{4}\beta \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $v_1 + v_2 + v_3 = 0$ (only 1-independent equation)

$$v_3 = -(v_1 + v_2)$$

Choosing: $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ -(v_1+v_2) \end{bmatrix}$ guarantees that

$$\vec{v} \cdot \hat{n}_1 = \begin{bmatrix} v_1 & v_2 & -(v_1+v_2) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}\left(v_1 + v_2 - (v_1 + v_2)\right)$$

$$= 0.$$

(orthogonal)

Set $v_1 = 0$: $\vec{v} = \begin{bmatrix} 0 \\ v_2 \\ -v_2 \end{bmatrix}$ orthogonal to $\hat{n}_1$

choose $v_2 = \frac{1}{\sqrt{2}}$

Then $\boxed{\hat{n}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}$

To get $\hat{n}_3$: ~~tk~~ take $\hat{n}_1 \times \hat{n}_2$

$$\hat{n}_3 = \hat{n}_1 \times \hat{n}_2$$

$$= \frac{1}{\sqrt{3}}\left(\hat{x} + \hat{y} + \hat{z}\right) \times \frac{1}{\sqrt{2}}\left(\hat{y} - \hat{z}\right)$$

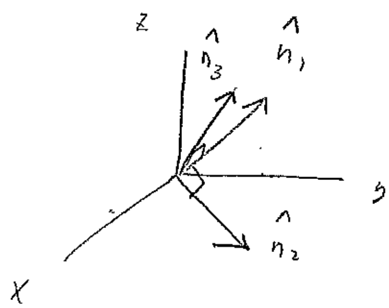
$$= \frac{1}{\sqrt{6}}\left(\hat{z} + \hat{y} - \hat{x} - \hat{x}\right) = \boxed{\frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \hat{n}_3}$$

So:

$$\hat{n}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{n}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

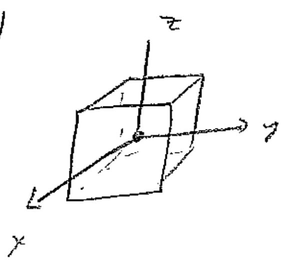
$$\hat{n}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$



check: $\hat{n}_1 \cdot \hat{n}_2 = 0$, $\hat{n}_1 \cdot \hat{n}_3 = 0$, $\hat{n}_2 \cdot \hat{n}_3 = 0$

---

c)



Origin at center of mass of cube

Calculate I_{ij} wrt these coords.

$$I_{ij} = \rho \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz \left[\left(x^2+y^2+z^2\right) \delta_{ij} - r_i r_j \right]$$

$$I_{xx} = \rho \iiint_{-a/2}^{a/2} dx\,dy\,dz \left(y^2+z^2\right)$$

$$= \rho \underset{\uparrow}{a} \left[a \frac{1}{3} y^3 \Big|_{-a/2}^{a/2} + a \frac{1}{3} z^3 \Big|_{-a/2}^{a/2} \right]$$

integrating over x

$$= \rho a^2 \frac{1}{3} \left[2 \frac{a^3}{8} + 2 \frac{a^3}{8} \right]$$

$$= \frac{1}{6} \rho a^5 = \boxed{\frac{1}{6} M a^2}$$

Similarly, $\boxed{I_{yy} = \frac{1}{6} Ma^2, \quad I_{zz} = \frac{1}{6} Ma^2}$

$$I_{xy} = \rho \iiint_{-a/2}^{a/2} dx\,dy\,dz\,(-xy)$$

$$= -\rho a \int_{-a/2}^{a/2} x\,dx \int_{-a/2}^{a/2} y\,dy$$

$$= -\rho a \left[\underbrace{\frac{x^2}{2} \Big|_{-a/2}^{a/2}}_{0} \cdot \underbrace{\frac{y^2}{2} \Big|_{-a/2}^{a/2}}_{0} \right]$$

$$= 0$$

So $\boxed{I_{xy} = I_{xz} = I_{yz} = 0}$

Thus,

$$I = \begin{bmatrix} \frac{1}{6} Ma^2 & 0 & 0 \\ 0 & \frac{1}{6} Ma^2 & 0 \\ 0 & 0 & \frac{1}{6} Ma^2 \end{bmatrix}$$

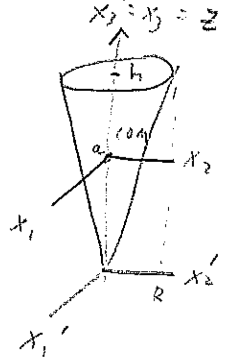
wrt this coord system

---

d) Principal axes are just $\hat{x}, \hat{y}, \hat{z}$ for this coord system
principal moments of inertia are all $= \frac{1}{6} Ma^2$

(7.4)

Problem Calculate principal moments of inertia of a circular cone of ~~base~~ height h and base radius R



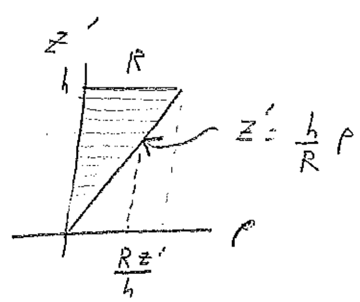
Hint: 1st calculate about vertex, then com using // axis theorem

$$I_3 = I_3'$$
$$= I_{33}'$$
mass density
$$= \int \mu \, dV \left(r'^2 - z^2\right)$$
$$= \int \mu \, dV \left(x'^2 + y'^2\right)$$

$a = \frac{3}{4} h$ (location of com)

$$\mu = \frac{M}{\text{volume}}$$

Need volume of cone:



$$V = \int dV$$
$$= \int \rho' d\rho' d\phi \, dz'$$
$$= \int_0^h dz' \int_0^{2\pi} d\phi \int_0^{\frac{Rz'}{h}} \rho' d\rho'$$
$$= \cancel{2}\pi \int_0^h dz' \left.\frac{\rho'^2}{\cancel{2}}\right|_0^{\frac{Rz'}{h}}$$
$$= \pi \int_0^h dz' \frac{R^2 z'^2}{h^2}$$
$$= \frac{R^2 \pi}{h^2} \left.\frac{z'^3}{3}\right|_0^h$$
$$= \boxed{\frac{1}{3}\pi R^2 h}$$

Thus, $\boxed{\mu = \frac{M}{\frac{1}{3}\pi R^2 h}}$ mass density

$$dV = \rho\, d\rho\, d\phi\, dz$$

$$I_3 = I_3' = \frac{M}{\frac{1}{3}\pi R^2 h} \int \rho'\, d\rho'\, d\phi\, dz'\, \rho'^2$$

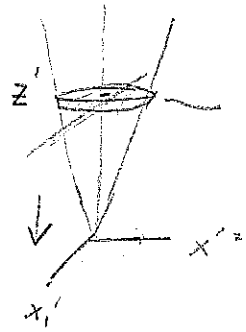
$$= \frac{M}{\frac{1}{3}\pi R^2 h} \cdot 2\pi \int_0^h dz' \int_0^{Rz'/h} \rho'^3\, d\rho'$$

$$= \frac{6M}{R^2 h} \int_0^h dz' \left. \frac{\rho'^4}{4} \right|_0^{Rz'/h}$$

$$= \frac{3}{2} \frac{M}{R^2 h} \int_0^h \frac{R^4 z'^4}{h^4}\, dz'$$

$$= \frac{3}{2} \frac{MR^2}{h^5} \left. \frac{z'^5}{5} \right|_0^h$$

$$= \boxed{\frac{3}{10} MR^2}$$



density

$$dM = \mu\, dz'\, \pi \rho(z')^2$$
$$= \mu\, dz'\, \pi \left(\frac{Rz'}{h}\right)^2$$
$$= \frac{\mu \pi R^2}{h^2} z'^2\, dz'$$

$$I_1' = \int_0^h \left[\frac{dM}{4} \left(\frac{Rz'}{h}\right)^2 + dM \cdot z'^2 \right]$$

parallel axis theorem

$$= \int_0^h \left(\frac{1}{4} \frac{R^2}{h^2} + 1 \right) z'^2 \frac{\mu \pi R^2}{h^2} z'^2\, dz'$$

$$= \left(\frac{R^2}{4h^2} + 1 \right) \frac{\mu \pi R^2}{h^2} \int_0^h z'^4\, dz' \leftarrow \frac{h^5}{5}$$

$$= \left(\frac{R^2}{4h^2} + 1 \right) \frac{M}{\frac{1}{3}\cancel{\pi R^2} h} \frac{\cancel{\pi R^2}}{h^2} \frac{h^5}{5}$$

$$= \frac{3}{5} M h^2 \left(\frac{R^2}{4h^2} + 1 \right) = \boxed{\frac{3}{5} M \left(\frac{R^2}{4} + h^2 \right)}$$

COM at a height $a = \frac{3}{4} h$ (see below)

Thus, $I_1 = I_1' - Ma^2$

$$= \frac{3}{5} M \left(\frac{R^2}{4} + h^2 \right) - M \frac{9}{16} h^2$$

$$= M \left[\frac{3}{20} R^2 + \underbrace{\frac{3}{5} h^2 - \frac{9}{16} h^2}_{\frac{48}{80} - \frac{45}{80} = \frac{3}{80}} \right]$$

$$\frac{\substack{16\\5}}{80}$$

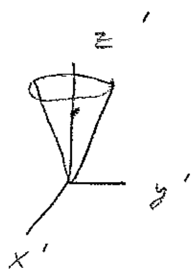
$$= M \left[\frac{3}{20} R^2 + \frac{3}{80} h^2 \right]$$

$$= \boxed{\frac{3}{20} M \left[R^2 + \frac{h^2}{4} \right]}$$

(same for I_2', I_2)

$$\boxed{\begin{aligned} \text{Thus,} \quad I_1 = I_2 &= \frac{3}{20} M \left[R^2 + \frac{h^2}{4} \right] \\ I_3 &= \frac{3}{10} M R^2 \end{aligned}}$$

To show that COM at $a = \frac{3}{4} h$:



$x'_{com} = 0$

$y'_{com} = 0$

$$M z'_{com} = \int \mu \, dV(x', y', z') \, z'$$

$$= \mu \int_0^{2\pi} d\phi \int_0^h dz' z' \int_0^{Rz'/h} \rho' d\rho'$$

$$= \mu \cancel{2} \pi \int_0^h dz' \, z' \frac{\rho'^2}{\cancel{2}} \Big|_0^{Rz'/h}$$

$$= \mu \pi \frac{R^2}{h^2} \int_0^h z'^3 \, dz'$$

$$= \mu \pi \frac{R^2}{h^2} \left. \frac{z'^4}{4} \right|_0^h$$

$$= \mu \frac{\pi}{4} \cancel{h^2} R^2 h^2$$

$$= \frac{M}{\frac{1}{3}\cancel{\pi} R^2 h} \frac{\cancel{\pi}}{4} R^2 h^2$$

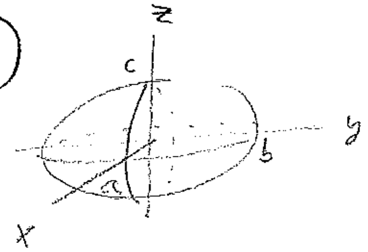
$$= \frac{3}{4} M h$$

Thus, $M z'_{com} = \frac{3}{4} M h$

or $\boxed{z'_{com} = \frac{3}{4}}$

Problem: Principal moments of inertia for uniform ellipsoid

(7.5)



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

Total mass: $M = \int dV'$

$$= \iiint dx\, dy\, dz$$

Let $u = \frac{x}{a}$, $v = \frac{y}{b}$, $w = \frac{z}{c}$

Then $u^2+y^2+z^2=1$

Jacobian $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$

$$= \det \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$= abc$$

Thus, $M = abc \iiint_{\text{unit sphere}} du\, dv\, dw$

$$= abc\ \frac{4}{3}\pi \cdot 1^3$$

$$= \frac{4}{3}\pi abc$$

(2)

Uniform density

$$\rho = \frac{M}{\frac{4}{3}\pi abc}$$

$$I(\hat{n}_1) = \frac{M}{\frac{4}{3}\pi abc}\int dV\left(r^2 - (\vec{r}\cdot\hat{n}_1)\right)$$

where $\vec{r}\cdot\hat{n}_1 = \vec{r}\cdot\hat{x}$
$= r\sin\theta\cos\phi$
$= x$

$r^2 = x^2+y^2+z^2$

$$I(\hat{n}_1) = \frac{M}{\frac{4}{3}\pi abc}\int dV\left(x^2+y^2+z^2-x^2\right)$$

$$= \frac{M}{\frac{4}{3}\pi abc}\int dV\left(y^2+z^2\right)$$

$$= \frac{M}{\frac{4}{3}\pi abc}\iiint dx\,dy\,dz\,\left(y^2+z^2\right)$$

$(x,y,z) \to (u,v,w)$

$dx\,dy\,dz = abc\,du\,dv\,dw$

$y^2+z^2 = b^2v^2 + c^2w^2$

Thus, $$I(\hat{n}_1) = \frac{M}{\frac{4}{3}\pi \cancel{abc}}\iiint \cancel{abc}\,du\,dv\,dw\,\left(b^2v^2+c^2w^2\right)$$

$$= \frac{M}{\frac{4}{3}\pi}\underset{\text{unit sphere}}{\iiint} r^2\sin\theta\,dr\,d\theta\,d\phi\,\left(b^2r^2\sin^2\theta\sin^2\phi + c^2r^2\cos^2\theta\right)$$

$$= \frac{M}{\frac{4}{3}\pi}\left[b^2 \int_0^1 r^4 dr \int_0^{2\pi} \sin^2\phi \, d\phi \int_0^{\pi} \underbrace{\sin^2\theta}_{1-\cos^2\theta} \sin\theta \, d\theta \right.$$

$$\left. + c^2 \int_0^1 r^4 dr \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \right]$$

$$= \frac{M}{\frac{4}{3}\pi}\left[b^2 \frac{r^5}{5}\Big|_0^1 \cdot \pi \cdot \int_{-1}^{1} (1-x^2)\, dx \right.$$

$$\left. + c^2 \frac{r^5}{5}\Big|_0^1 \cdot 2\pi \cdot \int_{-1}^{1} x^2 \, dx \right]$$

$$= \frac{M}{\frac{4}{3}\pi}\left[b^2 \frac{\pi}{5}\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} + c^2 \frac{2\pi}{5} \frac{x^3}{3}\Big|_{-1}^{1} \right]$$

$$= \frac{M}{\frac{4}{3}\cdot 5}\left[b^2 \, 2 \underbrace{\left(1-\frac{1}{3}\right)}_{\frac{2}{3}} + c^2 \, 2 \cdot \frac{2}{3} \right]$$

$$= \frac{M}{\frac{4}{3}\, 5}\left[\frac{4}{3} b^2 + \frac{4}{3} c^2 \right]$$

$$= \frac{M}{5}\left[b^2 + c^2 \right]$$

Thus, $\boxed{I(\hat{n}_1) = \frac{1}{5} M (b^2 + c^2)}$

$$\rightarrow \quad I(\hat{n}_2) = \frac{1}{5} M (a^2 + c^2)$$

$$I(\hat{n}_3) = \frac{1}{5} M (a^2 + b^2)$$

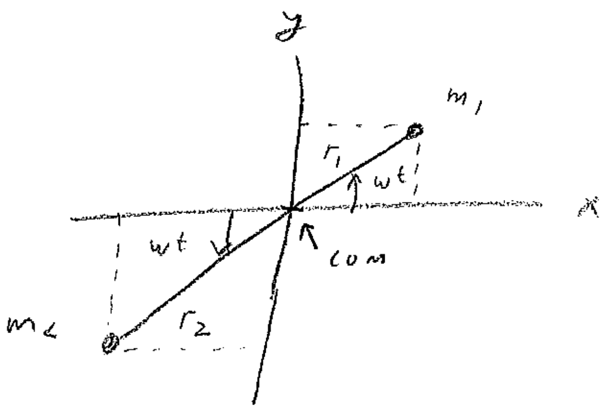
Problem: 7.6 Gravitational radiation from a binary system.

$$\frac{dE_{GW}}{dt} = \frac{1}{5}\frac{G}{c^5}\sum_{ij} \frac{d^3 Q_{ij}}{dt^3}\frac{d^3 Q_{ij}}{dt^3}$$

Einstein quadrupole formula

$$Q_{ij} = I_{ij} - \frac{1}{3}\,Tr(I)\,\delta_{ij}$$

$$I_{ij} = \sum_I m_I\left(\delta_{ij}\cdot r_I^2 - r_{Ii} r_{Ij}\right)$$



$$r_{1x} = r_1 \cos\omega t$$
$$r_{1y} = r_1 \sin\omega t$$
$$r_{1z} = 0$$
$$r_{2x} = r_2 \cos(\omega t + \pi) = -r_2 \cos(\omega t)$$

$$r_{2y} = r_2 \sin(\omega t + \pi) = -r_2 \sin(\omega t)$$
$$r_{2z} = 0$$

$$\cancel{I_{ij} = (m_1 r_1^2 + m_2 r_2^2)\delta_{ij} - m_1 r_{1i} r_{1j} - m_2 r_{2i} r_{2j}}$$

$$\cancel{I_{xx} = m_1 r_1^2 + m_2 r_2^2 - m_1 r_1^2 \cos^2\omega t - m_2 r_2^2 \cos^2\omega t}$$
$$\cancel{= m_1 r_1^2 (1 - \cos^2\omega t) + m_2 r_2^2 (1 - \cos^2\omega t)}$$

$$Tr(I) = \sum_i I_{ii} = \sum_I 3 m_I r_I^2 - \sum_I m_I r_I^2$$
$$= 2\sum_I m_I r_I^2$$

$$\cos(\omega t + \pi) = \cos(\omega t)\cos\pi - \sin(\omega t)\underbrace{\sin\pi}_{0}$$
$$= -\cos(\omega t)$$

$$\sin(\omega t + \pi) = \sin(\omega t)\cos\pi + \cos(\omega t)\underbrace{\sin\pi}_{0}$$
$$= -\sin\omega t$$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$

$$\cos^2 x$$

(2)

$$Q_{ij} = \sum_I m_I \left(\delta_{ij} r_I^2 - r_{Ii} r_{Ij} \right) - \frac{1}{3}\left(2 \sum_I m_I r_I^2 \delta_{ij}\right)$$

$$= \sum_I m_I \left(\frac{1}{3} \delta_{ij} r_I^2 - r_{Ii} r_{Ij} \right)$$

$$Q_{xx} = \sum_I m_I \left(\frac{1}{3} r_I^2 - r_{Ix} r_{Ix} \right)$$

$$= \frac{1}{3}\left(m_1 r_1^2 + m_2 r_2^2\right) - m_1 r_1^2 \cos^2 \omega t - m_2 r_2^2 \cos^2 \omega t$$

$$= m_1 r_1^2 \left(\frac{1}{3} - \cos^2 \omega t \right) + m_2 r_2^2 \left(\frac{1}{3} - \cos^2 \omega t \right)$$

$$Q_{yy} = \sum_I m_I \left(\frac{1}{3} r_I^2 - r_{Iy} r_{Iy} \right)$$

$$= \frac{1}{3}\left(m_1 r_1^2 + m_2 r_2^2\right) - m_1 r_1^2 \sin^2 \omega t - m_2 r_2^2 \sin^2 \omega t$$

$$= m_1 r_1^2 \left(\frac{1}{3} - \sin^2 \omega t \right) + m_2 r_2^2 \left(\frac{1}{3} - \sin^2 \omega t \right)$$

$$Q_{zz} = \sum_I m_I \frac{1}{3} r_I^2$$

$$= \frac{1}{3} \left(m_1 r_1^2 + m_2 r_2^2 \right)$$

$$Q_{xy} = \sum_I - m_I r_{Ix} r_{Iy}$$

$$= - m_1 r_1^2 \cos \omega t \sin \omega t - m_2 r_2^2 \cos \omega t \sin \omega t$$

$$= - \frac{1}{2} \left(m_1 r_1^2 + m_2 r_2^2 \right) \sin^1 2 \omega t$$

$$Q_{xz} = 0$$

$$Q_{yx} = Q_{xy}$$

$$Q_{yz} = 0$$

$$Q_{zx} = 0$$

$$Q_{zy} = 0$$

Thus,

$$Q_{ij} = \begin{bmatrix} X & X & 0 \\ X & X & 0 \\ 0 & 0 & X \end{bmatrix}$$

$$Q_{xx} = \frac{1}{3}\left(m_1 r_1^2 + m_2 r_2^2\right) - \left(m_1 r_1^2 + m_2 r_2^2\right)\cos^2 \omega t$$

$$Q_{yy} = \frac{1}{3}\left(m_1 r_1^2 + m_2 r_2^2\right) - \left(m_1 r_1^2 + m_2 r_2^2\right)\sin^2 \omega t$$

$$Q_{zz} = \frac{1}{3}\left(m_1 r_1^2 + m_2 r_2^2\right)$$

$$Q_{xy} = -\frac{1}{2}\left(m_1 r_1^2 + m_2 r_2^2\right)\sin 2\omega t$$

Now:

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\frac{1}{3} - \frac{1}{2}$$
$$\frac{2}{6} - \frac{3}{6} = -\frac{1}{6}$$

$$Q_{xx} = \frac{1}{3}\left(m_1 r_1^2 + m_2 r_2^2\right) - \frac{1}{2}\left(m_1 r_1^2 + m_2 r_2^2\right)\left(1 + \cos 2\omega t\right)$$

$$= -\frac{1}{6}\left(m_1 r_1^2 + m_2 r_2^2\right) - \frac{1}{2}\left(m_1 r_1^2 + m_2 r_2^2\right)\cos(2\omega t)$$

$$Q_{yy} = -\frac{1}{6}\left(m_1 r_1^2 + m_2 r_2^2\right) + \frac{1}{2}\left(m_1 r_1^2 + m_2 r_2^2\right)\cos(2\omega t)$$

$$Q_{zz} = \frac{1}{3}\left(m_1 r_1^2 + m_2 r_2^2\right)$$

$$Q_{xy} = -\frac{1}{2}\left(m_1 r_1^2 + m_2 r_2^2\right)\sin(2\omega t)$$

$$Q_{ij} = \begin{pmatrix} -\frac{1}{6}(\ ) - \frac{1}{2}(\ )\cos(2\omega t) & -\frac{1}{2}(\ )\sin(2\omega t) & 0 \\ -\frac{1}{2}(\ )\sin(2\omega t) & -\frac{1}{6}(\ ) + \frac{1}{2}(\ )\cos(2\omega t) & 0 \\ 0 & 0 & \frac{1}{3}(\ ) \end{pmatrix}$$

where $(\ ) = m_1 r_1^2 + m_2 r_2^2$

Note

COM: $m_1 r_1 = m_2 r_2$

$r = r_1 + r_2$

$$\begin{aligned} m_1 r_1^2 + m_2 r_2^2 &= m_1\left(\frac{m_2^2}{M^2} r^2\right) + m_2\left(\frac{m_1^2}{M^2} r^2\right) \\ &= \left(\frac{m_1 m_2}{M^2}\right)(m_2 + m_1)\, r^2 \\ &= \frac{m_1 m_2}{M^{\cancel{2}}} \cancel{M}\, r^2 \\ &= \frac{m_1 m_2}{M} r^2 \\ &= \boxed{\mu r^2} \end{aligned}$$

$r = r_1 + r_2$

$\mu = \frac{m_1 m_2}{M}$

$M = m_1 + m_2$

$m_1 r_1 = m_2 (r - r_1)$

$(m_1 + m_2) r_1 = m_2 r$

so $\boxed{r_1 = \frac{m_2}{M} r}$

similarly $\boxed{r_2 = \frac{m_1}{M} r}$

Thus,

$$Q_{ij} = \frac{1}{2}\mu r^2 \begin{pmatrix} \frac{1}{3} - \cos(2\omega t) & -\sin(2\omega t) & 0 \\ -\sin(2\omega t) & \frac{1}{3} + \cos(2\omega t) & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}$$

$$= -\frac{1}{2}\mu r^2 \begin{pmatrix} \frac{1}{3} + \cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & \frac{1}{3} - \cos(2\omega t) & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

$$\frac{d^3 Q_{ij}}{dt^3} = -\frac{1}{2}\mu r^2 \begin{pmatrix} 8\omega^3 \sin(2\omega t) & -8\omega^3 \cos(2\omega t) & 0 \\ -8\omega^3 \cos(2\omega t) & -8\omega^3 \sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= -4\mu r^2 \omega^3 \begin{pmatrix} \sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\frac{d}{dt}\cos = -\sin$

$\frac{d}{dt}(-\sin) = -\cos$

$\frac{d}{dt}(-\cos) = +\sin$

$\frac{d}{dt}\sin = \cos$

$\frac{d}{dt}\cos = -\sin$

$\frac{d}{dt}(-\sin) = -\cos$

Thus,

$$\frac{dE_{GW}}{dt} = \frac{1}{5}\frac{G}{c^5}\sum_{i,j}\frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3}$$

$$= \frac{1}{5}\frac{G}{c^5}16\mu^2r^4\omega^6\underbrace{\left[\sin^2(2\omega t)+\sin^2(2\omega t)+\cos^2(2\omega t)+\cos^2(2\omega t)\right]}_{2}$$

$$= \boxed{\frac{32}{5}\frac{G}{c^5}\mu^2r^4\omega^6} \quad \text{(power output in GWs)}$$

$$\frac{dE_{GW}}{dt} = \frac{-dE_{orb}}{dt} = -\frac{d}{dt}\left(\frac{-GM\mu}{2r}\right)$$

$$= -\frac{GM\mu}{2r^2}\left(\frac{dr}{dt}\right) \leftarrow \text{change in radius due to energy loss}$$

Thus, $\boxed{-\frac{GM\mu}{2r^2}\left(\frac{dr}{dt}\right) = \frac{32}{5}\frac{G}{c^5}\mu^2r^4\omega^6}$

Now use Kepler's 3rd law

$$\omega^2r^3 = GM$$

circular orbits

$$2\omega\dot\omega r^3 + 3\omega^2r^2\dot r = 0$$

So, $2\frac{\dot\omega}{\omega} + 3\frac{\dot r}{r} = 0$

$$\boxed{2\frac{\dot\omega}{\omega} = \frac{-3\dot r}{r}}$$

ω^2

$E_{orb} = ~~\ldots~~ T + U$

$= -\frac{1}{2}U + U$

$= \frac{1}{2}U$

$= -\frac{1}{2}\frac{GM\mu}{r}$

↓

For circular orbit

$$E_{orb} = \frac{1}{2}\mu\dot r^2 + \frac{l^2}{2\mu r^2} - \frac{GM\mu}{r}$$

$$= \frac{\mu^2r^4\dot\phi^2}{2\mu r^2} - \frac{GM\mu}{r}$$

$$= \frac{\mu}{2}r^2\omega^2 - \frac{GM\mu}{r}$$

$\dot\phi = \omega$

$$= \frac{\mu}{2r}GM - \frac{GM\mu}{r} = \boxed{-\frac{1}{2}\frac{GM\mu}{r}}$$

~~$\frac{1}{2}\mu\dot r^2 + \frac{l^2}{2\mu r^2} - \frac{GM\mu}{r}$ circular $E_{orb} = \frac{\mu^2r^4\dot\phi^2}{2\mu r^2} - \frac{GM\mu}{r}$~~

$$= \frac{\mu}{2}r^2\omega^2 - \frac{GM\mu}{r}$$

$$- M\mu \dot{r} = \frac{64}{5}\frac{1}{c^5}\mu^2 r^6 \omega^6$$

$$r = \left(\frac{GM}{\omega^2}\right)^{\frac{1}{3}}$$

$$\dot{r} = -\frac{2}{3} r \frac{\dot{\omega}}{\omega}$$

$$M\mu \frac{2}{3}\left(\frac{GM}{\omega^2}\right)^{\frac{1}{3}} \frac{\dot{\omega}}{\omega} = \frac{64}{5}\frac{1}{c^5}\mu^2 \left(\frac{GM}{\omega^2}\right)^2 \omega^6$$

$$M\mu \left(\frac{GM}{\omega^2}\right)^{\frac{1}{3}} \frac{\dot{\omega}}{\omega} = \frac{96}{5}\frac{1}{c^5}\mu^2 \left(\frac{GM}{\omega^2}\right)^2 \omega^6$$

~~Take~~ Raise to 3rd power:

$$M^3\mu^3 \left(\frac{GM}{\omega^2}\right) \frac{\dot{\omega}^3}{\omega^3} = \left(\frac{96}{5}\right)^3 \frac{1}{c^{15}} \mu^6 \left(\frac{GM}{\omega^2}\right)^6 \omega^{18}$$

$$\rightarrow \dot{\omega}^3 = \left(\frac{96}{5}\right)^3 \frac{1}{c^{15}} \frac{\mu^3}{M^3} \left(\frac{GM}{\omega^2}\right)^5 \omega^{21}$$

$$= \left(\frac{96}{5}\right)^3 \frac{1}{c^{15}} \mu^3 G^5 M^2 \omega^{11}$$

$$= \left(\frac{96}{5}\right)^3 \frac{G^5}{c^{15}} \omega^{11} \mu^3 M^2$$

$$\boxed{\dot{\omega}^3 = \left(\frac{96}{5}\right)^3 \omega^{11} \left(\frac{G}{c^3}\right)^5 \mathcal{M}_c^5}$$

where $\boxed{\mathcal{M}_c^5 = \mu^3 M^2}$

$$\rightarrow \mathcal{M}_c = \left(\mu^3 M^2\right)^{\frac{1}{5}} = \left(\frac{(m_1 m_2)^3}{M^3} M^2\right)^{\frac{1}{5}}$$

$$\boxed{\mathcal{M}_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{\frac{1}{5}}}}$$

solve for $\mathcal{M}_c$:

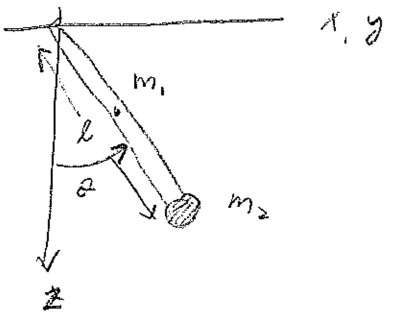
$$\boxed{\mathcal{M}_c = \left[\left(\frac{5}{96}\right)^3\left(\frac{c^3}{G}\right)^5 \frac{1}{\omega^{11}}\;\dot{\omega}^3\right]^{1/5} = \left(\frac{c^3}{G}\right)\left[\left(\frac{5}{96}\right)^3 \frac{\dot{\omega}^3}{\omega^{11}}\right]^{1/5}}$$

which can be determined from the observed data.

For GW150914, $\mathcal{M}_c \approx 30 M_\odot$

Problem Compound pendulum - small oscillations

7.7 (a)



Uniform rod: mass m_1, length ℓ

Mass point: m_2

$$U = m_1 g \frac{\ell}{2}(1-\cos\theta) + m_2 g \ell (1-\cos\theta)$$

$$= \left(\frac{m_1}{2} + m_2\right) g \ell (1-\cos\theta)$$

$$\approx \frac{1}{2} g \ell \left(\frac{m_1}{2} + m_2\right)\theta^2$$

$$T = \frac{1}{2} I \dot{\theta}^2$$

$$= \frac{1}{2}\left(\frac{1}{3} m_1 \ell^2 + m_2 \ell^2\right)\dot{\theta}^2$$

$$= \frac{1}{2}\left(\frac{1}{3} m_1 + m_2\right)\ell^2 \dot{\theta}^2$$

$$L = T - U$$

$$= \frac{1}{2}\left(\frac{1}{3} m_1 + m_2\right)\ell^2\dot{\theta}^2 - \frac{1}{2} g \ell\left(\frac{m_1}{2} + m_2\right)\theta^2$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$$

$$= \frac{d}{dt}\left[\left(\frac{1}{3} m_1 + m_2\right)\ell^2\dot{\theta}\right] + g\ell\left(\frac{m_1}{2} + m_2\right)\theta$$

$$= \left(\frac{1}{3} m_1 + m_2\right)\ell^2\ddot{\theta} + g\ell\left(\frac{m_1}{2} + m_2\right)\theta$$

$$\rightarrow \ddot{\theta} = \frac{-g\,\ell\left(\frac{m_1}{2} + m_2\right)}{\left(\frac{1}{3} m_1 + m_2\right)\ell^2}\theta = -\omega_{osc}^2\,\theta$$

$$\omega_{osc} = \sqrt{\frac{g}{\ell}\frac{\left(\frac{m_1}{2} + m_2\right)}{\left(\frac{m_1}{3} + m_2\right)}}$$

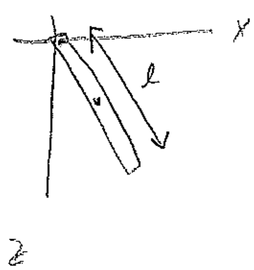
b

Limiting cases:

$m_2 >> m_1 \rightarrow \omega_{osc} \simeq \sqrt{\frac{g}{\ell}\frac{m_2}{m_2}} = \sqrt{\frac{g}{\ell}}$

$m_1 >> m_2 \rightarrow \omega_{osc} \simeq \sqrt{\frac{g}{\ell}\frac{\left(\frac{m_1}{2}\right)}{\left(\frac{m_1}{3}\right)}} = \sqrt{\frac{3}{2}\frac{g}{\ell}}$

Check: For $m_1 >> m_2$



$I = \frac{1}{12} m_1 \ell^2$ (pivotted at end)

Com at $\frac{\ell}{2}$

$$L = \frac{1}{2}\left(\frac{1}{3} m_1 \ell^2\right)\dot{\theta}^2 - m_1 g \frac{\ell}{2}\underbrace{(1-\cos\theta)}_{\simeq \frac{\theta^2}{2}}$$

$$= \frac{1}{6} m_1 \ell^2 \dot{\theta}^2 - m_1 g \frac{\ell}{4}\theta^2$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$$

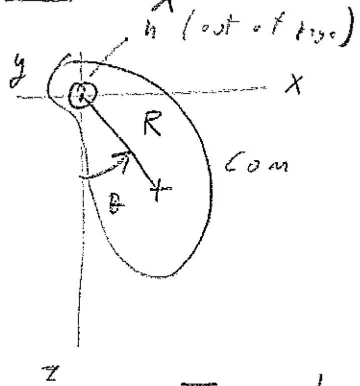
$$= \frac{d}{dt}\left(\frac{1}{3} m_1 \ell^2 \dot{\theta}\right) + m_1 g \frac{\ell}{2}\theta$$

$$= \frac{1}{3} \cancel{m_1} \ell^2 \ddot{\theta} + \cancel{m_1} g \frac{\ell}{2}\theta$$

$$\ddot{\theta} = -\left(\frac{3g}{2\ell}\right)\theta$$

$$\omega_{osc} = \sqrt{\frac{3g}{2\ell}}$$

(b) Prob 7.7 Compound pendulum - small oscillations



Total mass M

Center of mass located a distance R from axis

$$U = Mg R(1-\cos\theta)$$
$$\approx \frac{1}{2} Mg R\theta^2$$

$$T = \frac{1}{2} I(\hat{n})\,\omega^2 , \qquad \omega = \dot{\theta}$$
$$= \frac{1}{2} M k^2 \omega^2 \qquad k^2 = \text{radius of gyration}$$

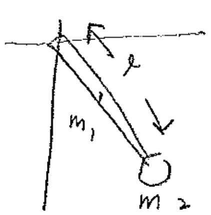
$$L = T - U$$
$$= \frac{1}{2} M k^2 \dot{\theta}^2 - \frac{1}{2} Mg R\theta^2$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$$
$$= \frac{d}{dt}\left(M k^2 \dot{\theta}\right) + Mg R\,\theta$$
$$= M k^2 \ddot{\theta} + Mg R\theta$$

$$\rightarrow \ddot{\theta} = -\left(\frac{gR}{k^2}\right)\theta = -\omega_{osc}^2\,\theta$$

$$\boxed{\omega_{osc} = \sqrt{\frac{gR}{k^2}}} = \sqrt{\frac{MgR}{Mk^2}} = \sqrt{\frac{MgR}{I}}$$

Suppose: ~~Prob~~ compound pendulum = uniform rod (mass m_1, length ℓ) + pendulum bob (mass m_2)



$$R(m_1+m_2) = m_1 \frac{\ell}{2} + m_2 \ell$$
$$= \ell\left(\frac{m_1}{2} + m_2\right)$$

$$\rightarrow R = \frac{\ell\left(\frac{m_1}{2} + m_2\right)}{M}$$

$$I = \underbrace{\frac{1}{3} m_1 l^2}_{\substack{\text{thin rod} \\ \text{(axis at} \\ \text{end)}}} + m_2 l^2 = \left(\frac{1}{3} m_1 + m_2\right) l^2$$

$$I = M\kappa^2 = (m_1 + m_2)\kappa^2 = \left(\frac{1}{3} m_1 + m_2\right) l^2$$

$$\rightarrow \quad \kappa^2 = \left(\frac{\frac{1}{3} m_1 + m_2}{m_1 + m_2}\right) l^2$$

<u>Also:</u> $$R = \frac{l\left(\frac{m_1}{2} + m_2\right)}{(m_1 + m_2)}$$

Thus, $$\kappa^2 = \left(\frac{\frac{1}{3} m_1 + m_2}{\cancel{m_1 + m_2}}\right) \frac{R^2 (m_1 + m_2)^{\cancel{2}}}{\left(\frac{m_1}{2} + m_2\right)^2}$$

$$= \frac{\left(\frac{m_1}{3} + m_2\right)(m_1 + m_2)}{\left(\frac{m_1}{2} + m_2\right)^2} R^2$$

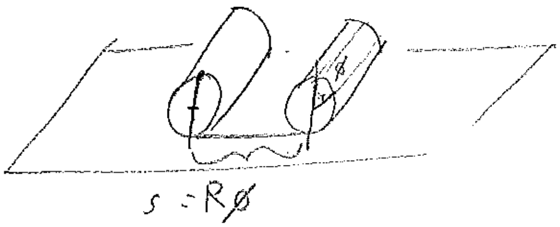
$$\omega_{osc} = \sqrt{\frac{gR}{\kappa^2}} = \sqrt{\frac{g \cancel{R} \left(\frac{m_1}{2} + m_2\right)^2}{\left(\frac{1}{3} m_1 + m_2\right)(m_1 + m_2) R^{\cancel{2}}}}$$

$$= \sqrt{\frac{g \left(\frac{m_1}{2} + m_2\right)^{\cancel{2}}}{\left(\frac{1}{3} m_1 + m_2\right)\cancel{(m_1 + m_2)} \frac{l \cancel{\left(\frac{m_1}{2} + m_2\right)}}{\cancel{(m_1 + m_2)}}}}$$

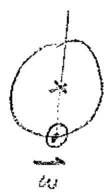
$$= \sqrt{\frac{g}{l} \frac{\left(\frac{m_1}{2} + m_2\right)}{\left(\frac{1}{3} m_1 + m_2\right)}}$$

(7.8)

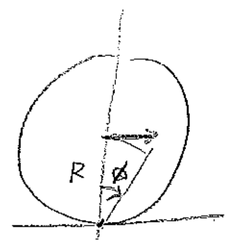
Problem: Calculate kinetic energy of cylinder rolling on a horizontal surface



Point of contact is instantaneously at rest



$$I(\hat{\omega}) = I_3 + MR^2$$
$$= \frac{1}{2}MR^2 + MR^2$$
$$= \frac{3}{2}MR^2$$



Thus,
$$T = \frac{1}{2} I(\hat{\omega})\,\omega^2$$
$$= \frac{1}{2}\,\frac{3}{2} M R^2 \dot{\phi}^2$$
$$= \frac{3}{4} M R^2 \dot{\phi}^2$$

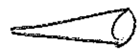
$$V = R\omega$$
$$\boxed{\omega = \frac{v}{R}}$$

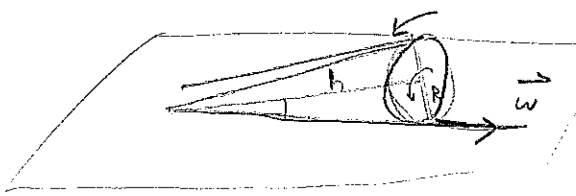
rolling without slipping constraint for any velocity

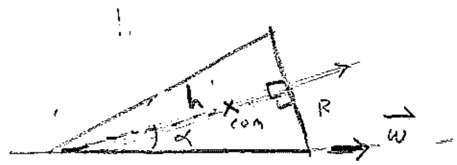
NOTE: (preferred way)

$$T = T_{com} + T_{rot,\,com}$$
$$= \frac{1}{2}MV^2 + \frac{1}{2}I_3\,\omega^2$$
$$= \frac{1}{2}MR^2\dot{\phi}^2 + \frac{1}{2}\cdot\frac{1}{2}MR^2\dot{\phi}^2$$
$$= MR^2\dot{\phi}^2\left[\frac{1}{2} + \frac{1}{4}\right]$$
$$= \frac{3}{4}MR^2\dot{\phi}^2 \quad \text{(same as it should be)}$$

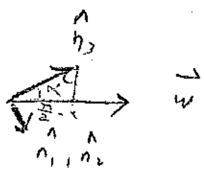
7.9

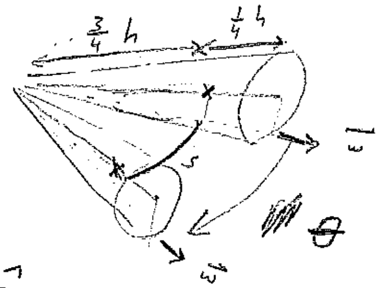
Problem Calculate KE of  cone rolling without slipping on a horizontal surface





$$\tan\alpha = \frac{R}{h}$$





$$\vec{\omega} = (\vec{\omega}\cdot\hat{n}_3)\hat{n}_3 + (\vec{\omega}\cdot\hat{n}_1)\hat{n}_1$$

$$= \omega\cos\alpha\,\hat{n}_3 + \omega\underbrace{\cos\left(\tfrac{\pi}{2}-\alpha\right)}_{\underbrace{\cos\frac{\pi}{2}}_{0}\cos\alpha + \underbrace{\sin\frac{\pi}{2}}_{=1}\sin\alpha}\hat{n}_1$$

$$= \omega\left(\sin\alpha\,\hat{n}_1 + \cos\alpha\,\hat{n}_3\right)$$

$$\hat{\omega} = \frac{\vec{\omega}}{\omega} = \sin\alpha\,\hat{n}_1 + \cos\alpha\,\hat{n}_3$$

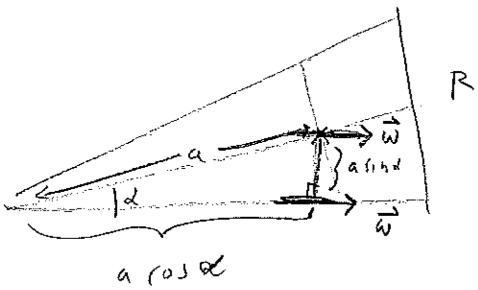
$$I(\hat{\omega}) = \sum_{i,j} I_{ij}\,\hat{\omega}^i\hat{\omega}^j$$

$$= I\left(\sin\alpha\,\hat{n}_1 + \cos\alpha\,\hat{n}_3\right)$$

$$= \sin^2\alpha\, I_1 + \cos^2\alpha\, I_3$$

$$= \left(I_1\sin^2\alpha + I_3\cos^2\alpha\right)$$

Recall: $I_1 = I_2 = \frac{3}{20}M\left(R^2 + \frac{1}{4}h^2\right)$, $I_3 = \frac{3}{10}MR^2$ — for axes with origin at com



$$\tan\alpha = \frac{R}{h} \rightarrow R = h\tan\alpha$$

$$s = a\cos\alpha\ \theta \rightarrow \boxed{V_{com} = \frac{ds}{dt} = a\cos\alpha\ \dot{\theta}}$$

└ distance moved by com

Angular velocity: $\omega = \dot{\phi}$ = pure rotation about instantaneous axis $\hat{\omega}$

$$\hat{\omega} = \sin\alpha\ \hat{n}_1 + \cos\alpha\ \hat{n}_3$$

$$= \frac{V_{com}}{a\sin\alpha}$$

$$= \frac{\cancel{a}\cos\alpha\ \dot{\theta}}{\cancel{a}\sin\alpha} = \boxed{\frac{\dot{\theta}}{\tan\alpha}} = \omega$$

Thus,

$$T = T_{com} + T_{rot,com}$$

since I_1, I_3 for axes passing through com

$$= \frac{1}{2}M V_{com}^2 + \frac{1}{2} I(\hat{\omega})\ \omega^2$$

$$= \frac{1}{2}M a^2\cos^2\alpha\ \dot{\theta}^2 + \frac{1}{2}\left(I_1\sin^2\alpha + I_3\cos^2\alpha\right)\frac{\dot{\theta}^2}{\tan^2\alpha}$$

$$= \frac{1}{2}M a^2\cos^2\alpha\ \dot{\theta}^2 + \frac{1}{2}I_1\cos^2\alpha\ \dot{\theta}^2 + \frac{1}{2}I_3\frac{\cos^4\alpha}{\sin^2\alpha}\dot{\theta}^2$$

$$= \frac{1}{2}\left(Ma^2\cos^2\alpha + I_1\cos^2\alpha + I_3\frac{\cos^4\alpha}{\sin^2\alpha}\right)\dot{\theta}^2$$

$$= \frac{1}{2}\left[M\left(\frac{9}{16}\right)h^2\cos^2\alpha + \frac{3}{20}M\left(R^2+\frac{h^2}{4}\right)\cos^2\alpha + \frac{3}{10}MR^2\frac{\cos^4\alpha}{\sin^2\alpha}\right]\dot{\theta}^2$$

$$\cancel{= \frac{1}{2}M\dot{\theta}^2\left[\cos^2\alpha\left(\frac{9}{16}+\frac{3}{80}\right)h^2 + \frac{3}{20}R^2\cos^2\alpha + \frac{3}{10}R^2\frac{\cos^4\alpha}{\sin^2\alpha}\right]}$$

$$\cancel{\frac{48}{80}\quad \frac{6}{10}\quad \frac{3}{5}\quad h^2\sin^2\alpha\quad h^2\cos^2\alpha}$$

$$\cancel{= \frac{1}{2}Mh^2\dot{\theta}^2\left[\frac{3}{5}\cos^2\alpha + \frac{3}{20}\sin^2\alpha\right]}$$

$$T = \frac{1}{2} M \dot{\theta}^2 \left[\frac{9}{16} h^2 \cos^2\alpha + \frac{3}{20}\left(R^2 + \frac{h^2}{4}\right)\cos^2\alpha + \frac{3}{10} R^2 \frac{\cos^4\alpha}{\sin^2\alpha} \right]$$

<u>Now:</u> $R = h \tan\alpha = h \frac{\sin\alpha}{\cos\alpha}$

$$\rightarrow \quad T = \frac{1}{2} M \dot{\theta}^2 \left[\frac{9}{16} h^2 \cos^2\alpha + \frac{3}{20}\left(h^2 \frac{\sin^2\alpha}{\cos^2\alpha} + \frac{h^2}{4}\right)\cos^2\alpha + \frac{3}{10} h^2 \cos^2\alpha \right]$$

$$= \frac{1}{2} M h^2 \dot{\theta}^2 \left[\frac{9}{16}\cos^2\alpha + \frac{3}{20}\sin^2\alpha + \frac{3}{80}\cos^2\alpha + \frac{3}{10}\cos^2\alpha \right]$$

$$= \frac{1}{2} M h^2 \dot{\theta}^2 \left[\underbrace{\left(\frac{9}{16} + \frac{3}{80} + \frac{3}{10}\right)}_{\frac{45+3+24}{80} = \frac{72}{80} = \frac{9}{10}} \cos^2\alpha + \frac{3}{20}\sin^2\alpha \right]$$

$$= \frac{1}{2} M h^2 \dot{\theta}^2 \left[\frac{9}{10}\cos^2\alpha + \frac{3}{20}\sin^2\alpha \right]$$

$$= \frac{3}{40} M h^2 \dot{\theta}^2 \left[\sin^2\alpha + \underbrace{3 \cdot 2}_{6} \cos^2\alpha \right]$$

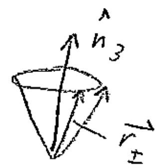
$$= \frac{3}{40} M h^2 \dot{\theta}^2 \left[\underbrace{\sin^2\alpha + \cos^2\alpha}_{1} + 5\cos^2\alpha \right]$$

$$= \boxed{\frac{3}{40} M h^2 \dot{\theta}^2 \left[1 + 5\cos^2\alpha \right]}$$

problem (7.10) Euler equation from EL equation for ψ

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\psi}\right) - \frac{\partial T}{\partial \psi} - G_\psi = 0$$

$$G_\psi \equiv \sum_I \frac{\partial \vec r_I}{\partial \psi} \cdot \vec F_I = \sum_I (\hat n_3 \times \vec r_I) \cdot \vec F_I$$



Euler angle ψ : rotation around $\hat n_3$

Infinitesimal rotation

$$\vec A' = \vec A + (\hat n \times \vec A)\, d\Phi$$

$$\rightarrow \frac{\partial \vec A}{\partial \Phi} = \hat n \times \vec A$$

$$\boxed{\text{so } \frac{\partial \vec r_I}{\partial \psi} = \hat n_3 \times \vec r_I}$$

$$\begin{aligned} G_\psi &= \sum_I (\hat n_3 \times \vec r_I) \cdot \vec F_I \\ &= \sum_I (\vec r_I \times \vec F_I) \cdot \hat n_3 \\ &= \left(\sum_I \vec \tau_I\right) \cdot \hat n_3 \\ &= \tau_3 \end{aligned}$$

$T = \frac{1}{2} \sum_i I_i \omega_i^2$ where

$$\begin{aligned} \omega_1 &= -\sin\theta \cos\psi \,\dot\phi + \sin\psi \,\dot\theta \\ \omega_2 &= \sin\theta \sin\psi \,\dot\phi + \cos\psi \,\dot\theta \\ \omega_3 &= \cos\theta \,\dot\phi + \dot\psi \end{aligned}$$

$$\frac{\partial T}{\partial \dot\psi} = I_3\,\omega_3 \underbrace{\frac{\partial \omega_3}{\partial \dot\psi}}_{=1} = I_3\,\omega_3$$

$$\rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\psi}\right) = I_3\,\dot\omega_3$$

$$\frac{\partial T}{\partial \psi} = I_1\,\omega_1 \frac{\partial \omega_1}{\partial \psi} + I_2\,\omega_2 \frac{\partial \omega_2}{\partial \psi} + I_3\,\omega_3 \cancelto{0}{\frac{\partial \omega_3}{\partial \psi}}$$

<u>Now</u>:
$$\frac{\partial \omega_1}{\partial \psi} = \sin\theta \sin\psi\,\dot\phi + \cos\psi\,\dot\theta = \omega_2$$

$$\frac{\partial \omega_2}{\partial \psi} = \sin\theta \cos\psi\,\dot\phi - \sin\psi\,\dot\theta = -\omega_1$$

Thus,
$$\frac{\partial T}{\partial \psi} = I_1\,\omega_1\,\omega_2 - I_2\,\omega_2\,\omega_1 = \omega_1\,\omega_2\,(I_1 - I_2)$$

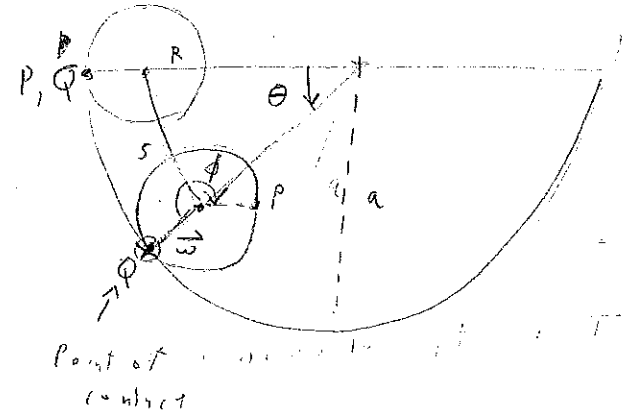
$$\rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\psi}\right) - \frac{\partial T}{\partial \psi} - G_\psi = 0$$

$$I_3\,\dot\omega_3 - \omega_1\,\omega_2\,(I_1 - I_2) - \tau_3 = 0$$

Thus,
$$\boxed{\tau_3 = I_3\,\dot\omega_3 - \omega_1\,\omega_2\,(I_1 - I_2)}$$

Additional

Prob: Calculate KE of ~~[illegible]~~ unit cylinder of radius R rolling inside a cylindrical surface of radius a.



$s = (a-R)\theta$

Distance moved by com

$R\phi = s = (a-R)\theta$

Thus, $\phi = \dfrac{(a-R)\theta}{R}$

$V_{com} = \dfrac{ds}{dt} = (a-R)\dot{\theta}$

Instantaneous axis of rotation:

$$I(\hat{\omega}) = I_3 + MR^2 \quad \text{(parallel-axis theorem)}$$
$$= \frac{1}{2}MR^2 + MR^2$$
$$= \frac{3}{2}MR^2$$

$$T = \frac{1}{2} I(\hat{\omega})\,\omega^2$$
$$= \frac{1}{2}\left(\frac{3}{2}MR^2\right)\omega^2$$
$$= \frac{3}{4} M \cancel{R^2} \frac{(a-R)^2\dot{\theta}^2}{\cancel{R^2}}$$
$$= \boxed{\frac{3}{4} M (a-R)^2 \dot{\theta}^2}$$

rolling without slipping

$$\omega = \dot{\phi} = \frac{V_{com}}{R}$$
$$= \left(\frac{a-R}{R}\right)\dot{\theta}$$

Alternative derivation (preferred way)

about com

$$T = T_{com} + T_{rot,\,com}$$
$$= \frac{1}{2}M(a-R)^2\dot{\theta}^2 + \frac{1}{2} I_3\, \omega^2$$
$$= \frac{1}{2}M(a-R)^2\dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{2}M\cancel{R^2}\right)\frac{(a-R)^2}{\cancel{R^2}}\dot{\theta}^2$$
$$= \frac{1}{2}M(a-R)^2\dot{\theta}^2 + \frac{1}{4}M(a-R)^2\dot{\theta}^2$$
$$= \boxed{\frac{3}{4} M (a-R)^2 \dot{\theta}^2}$$

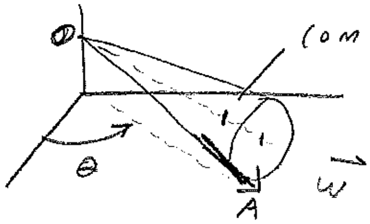
(note: KE = 0 if $a = R$)

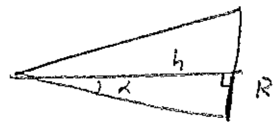
$$\omega = \dot{\phi} = \frac{(a-R)\dot{\theta}}{R}$$

Additional

Prob: Find KE of cone rolling on horizontal surface with vertex fixed at height = base radius of cone

com $a = \frac{3}{4} h$ from vertex



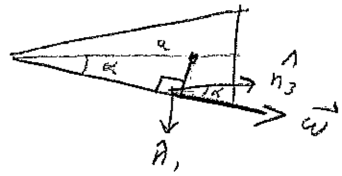


$\tan\alpha = \frac{R}{h}$

$R = h\tan\alpha$

~~$\vec{\omega}$ along OA~~

$V_{com} = a\dot{\theta}$



$\vec{\omega}$ along OA

$\omega = \dfrac{V_{com}}{\underline{a\sin\alpha}}$ ——— $\perp$ distance from a to axis

$$\vec{\omega} = (\vec{\omega}\cdot\hat{n}_3)\,\hat{n}_3 + (\vec{\omega}\cdot\hat{n}_1)\,\hat{n}_1$$
$$= \omega\cos\alpha\,\hat{n}_3 + \omega\sin\alpha\,\hat{n}_1$$
$$= \omega_3\,\hat{n}_3 + \omega_1\,\hat{n}_1$$

$$\omega_1 = \omega\sin\alpha = \frac{V_{com}}{a\cancel{\sin\alpha}}\cancel{\sin\alpha} = \frac{a\dot{\theta}}{a} = \dot{\theta}$$

$$\omega_3 = \omega\cos\alpha = \frac{V_{com}}{a\sin\alpha}\cos\alpha = \frac{\cancel{a}\dot{\theta}}{\cancel{a}}\cot\alpha = \dot{\theta}\cot\alpha$$

Thus,

$$T = T_{com} + T_{rot,\,com}$$
$$= \frac{1}{2}MV_{com}^2 + \frac{1}{2}\sum_{i,j} \underbrace{I_{ij}}_{\text{about com}}\,\omega_i\omega_j$$
$$= \frac{1}{2}Ma^2\dot{\theta}^2 + \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_3\,\omega_3^2$$
$$= \frac{1}{2}Ma^2\dot{\theta}^2 + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3\cot^2\alpha\,\dot{\theta}^2$$

$$= \frac{1}{2} M \frac{9}{16} h^2 \dot\theta^2 + \frac{1}{2}\frac{3}{20} M \left(R^2 + \frac{1}{4}h^2\right)\dot\theta^2 + \frac{1}{2}\frac{3}{10} M R^2 \cot^2\alpha\, \dot\theta^2$$

$$= \frac{1}{2} M \dot\theta^2 \left(\frac{9}{16} h^2 + \frac{3}{20} h^2 \tan^2\alpha + \frac{3}{80} h^2 + \frac{3}{10} h^2 \cancel{\tan^2\alpha \cot^2\alpha} \right)$$

$$= \frac{1}{2} M \dot\theta^2 h^2 \left[\frac{9}{16} + \frac{3}{20}\tan^2\alpha + \frac{3}{80} + \frac{3}{10} \right]$$

$$\frac{45 + 3 + 24}{80} = \frac{72}{80} = \frac{9}{10}$$

$$= \frac{1}{2} M \dot\theta^2 h^2 \left[\frac{9}{10} + \frac{3}{20}\tan^2\alpha \right]$$

$$= \frac{3}{40} M \dot\theta^2 h^2 \left[6 + \tan^2\alpha \right]$$

$$= \frac{3}{40} M \dot\theta^2 h^2 \left[5 + (1 + \tan^2\alpha) \right]$$

$$1 + \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha} = \sec^2\alpha$$

$$= \boxed{\frac{3}{40} M \dot\theta^2 h^2 \left[5 + \sec^2\alpha \right]}$$