Problem: 6.1 Inner product in terms of lengths

$$|\underline{u}+\underline{v}|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v})$$
$$= |\underline{u}|^2 + |\underline{v}|^2 + 2\,\underline{u}\cdot\underline{v}$$ (assuming real vectors)

Thus, $\underline{u}\cdot\underline{v} = \frac{1}{2}\left(|\underline{u}+\underline{v}|^2 - |\underline{u}|^2 - |\underline{v}|^2 \right)$

Problem 6.2 Closure property of $O(3)$

$R \in O(3)$ iff $R^T = R^{-1}$

Let $R_1, R_2 \in O(3)$

Then $R_1^T = R_1^{-1}$, $R_2^T = R_2^{-1}$

Consider $R_3 = R_1 R_2$

Then
$$\begin{aligned} R_3^T &= (R_1 R_2)^T \\ &= R_2^T R_1^T \\ &= R_2^{-1} R_1^{-1} \\ &= (R_1 R_2)^{-1} \\ &= R_3^{-1} \end{aligned}$$

so $R_3 \in O(3)$

Problem: 6.3 $O(3)$ with det $=-1$ does not form a group.

Let $R_1, R_2 \in O(3)$ with $\det R_1 = -1$, $\det R_2 = -1$

To be a group, $R_3 \equiv R_1 R_2$ must also be in $O(3)$ with $\det R_3 = -1$.

We showed in a previous problem that $R_3^T = R_3^{-1}$ so $R_3 \in O(3)$.

$$\text{But } \det(R_3) = \det(R_1 R_2)$$
$$= \det R_1 \cdot \det R_2$$
$$= -1 \cdot (-1)$$
$$= +1$$

so $\det(R_3) \neq -1$

Thus $O(3)$ with $\det = -1$ does not form a group.

Example: 6.3 Non-commutating rotations

active rotation by 90° ccw

$$Y = R_y(-90°) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$Z = R_z(-90°) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$ZY = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$YZ = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem (6.4) Show $\ldots D'''C''B'A = ABCD\ldots$

Transformations:

i) A

ii) $B' = ABA^{-1}$

iii) $C'' = (B'A)C(B'A)^{-1}$

iv) $D''' = (C''B'A)D(C''B'A)^{-1}$

etc.

Thus,

$$D'''C''B'A = C''B'AD(C''B'A)^{-1}C''B'A$$

$$= C''B'ADA^{-1}\underbrace{(B')^{-1}\underbrace{\underbrace{(C'')^{-1}C''}_{\mathbb{1}}B'}_{\mathbb{1}}A}_{\mathbb{1}}$$

$$= C''B'AD$$

$$= B'AC(B'A)^{-1}B'AD$$

$$= B'AC\underbrace{A^{-1}\underbrace{(B')^{-1}B'}_{\mathbb{1}}A}_{\mathbb{1}}D$$

$$= B'ACD$$

$$= AB\underbrace{A^{-1}A}_{\mathbb{1}}CD$$

$$= ABCD$$

Problem (6.5) Verify $R(\theta, \phi, \psi)$ and trace formula

$$R(\phi, \theta, \psi) = R_z(\psi)\, R_y(\theta)\, R_z(\phi)$$

$$= \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta c\phi & c\theta s\phi & -s\theta \\ -s\phi & c\phi & 0 \\ s\theta c\phi & s\theta s\phi & c\theta \end{bmatrix}$$

$$= \begin{bmatrix} c\theta c\phi c\psi - s\phi s\psi & c\theta s\phi c\psi + c\phi s\psi & -s\theta c\psi \\ -c\theta c\phi s\psi - s\phi c\psi & -c\theta s\phi s\psi + c\phi c\psi & +s\theta s\psi \\ s\theta c\phi & s\theta s\phi & c\theta \end{bmatrix}$$

$$1 + Tr[R(\phi, \theta, \psi)] = 1 + c\theta c\phi c\psi - s\phi s\psi - c\theta s\phi s\psi + c\phi c\psi + c\theta$$

$$= (1 + c\theta) + (1 + c\theta)\, c\phi\, c\psi - (1 + c\theta)\, s\phi\, s\psi$$

$$= (1 + c\theta)\,[\,1 + c\phi\, c\psi - s\phi\, s\psi\,]$$

$$= (1 + c\theta)\,[\,1 + c(\phi + \psi)\,]$$

Now: $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$ $\rightarrow$ $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$

$\rightarrow$ $\cos^2\left(\dfrac{x}{2}\right) = \dfrac{1 + \cos x}{2}$

Thus,

$$1 + Tr[R(\phi, \theta, \psi)] = 2\cos^2\left(\frac{\theta}{2}\right) \cdot 2\cos^2\left(\frac{\phi + \psi}{2}\right)$$

$$= 4\cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\phi + \psi}{2}\right)$$

Problem 66 Eigenvectors, eigenvalues of non-commuting rotations

$$ZY = A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \qquad YZ = B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A - \lambda \mathbb{1} = \begin{bmatrix} -\lambda & -1 & 0 \\ 0 & -\lambda & 1 \\ -1 & 0 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda \mathbb{1}) = -\lambda \cdot \lambda^2 + 1 \cdot 1$$
$$= -\lambda^3 + 1$$

$$\lambda - 1 \overline{\big) \lambda^3 - 1} \quad \lambda^2 + \lambda + 1$$
$$-\lambda^3 + \lambda^2$$
$$\lambda^2 - 1$$
$$-\lambda^2 + \lambda$$
$$\lambda - 1$$

$$0 = \lambda^3 - 1$$
$$= (\lambda - 1)(\lambda^2 + \lambda + 1)$$

$$\lambda = 1, \qquad \lambda = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2}$$

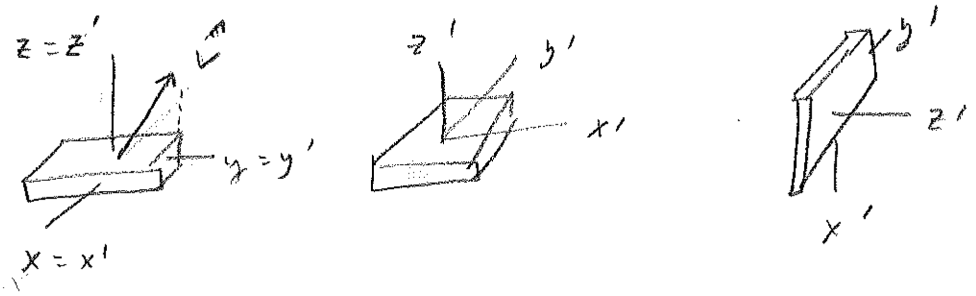
$\lambda = 1$:

$$(A - \mathbb{1})\underline{v} = \underline{0}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-v_1 - v_2 = 0 \quad \rightarrow \quad v_2 = -v_1$$
$$-v_2 + v_3 = 0 \quad \rightarrow \quad v_3 = v_2 = -v_1$$
$$-v_1 - v_3 = 0 \quad \rightarrow \quad v_3 = -v_1$$

$$V = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \rightarrow \text{normalize} \quad \hat{V} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ +1 \\ +1 \end{bmatrix}$$



$$\hat{V} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\cos\left(\frac{\Psi}{2}\right) = \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\phi + \psi}{2}\right)$$

$\phi = \pi/2$
$\theta = \pi/2$
$\psi = 0$

$$= \cos\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2}$$

$$= \frac{1}{2}$$

$$\rightarrow \frac{\Psi}{2} = 60^\circ \rightarrow \boxed{\Psi = 120^\circ}$$

$$B - \lambda \mathbb{1} = \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix}$$

$$\det(B - \lambda \mathbb{1}) = -\lambda \lambda^2 + 1 \cdot 1$$
$$= -\lambda^3 + 1$$

$$\rightarrow \lambda = 1, \quad \lambda = \frac{-1 \pm i\sqrt{3}}{2}$$

$(B - \lambda \mathbb{1})\,\underline{v} = \underline{0}$

$\underline{\lambda = 1}$:

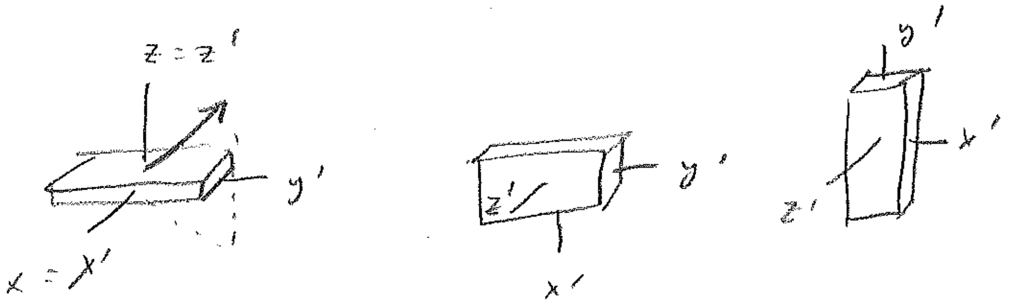
$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-v_1 + v_3 = 0 \quad \rightarrow \quad v_3 = v_1$

$v_1 - v_2 = 0 \quad \rightarrow \quad v_2 = v_1$

$v_2 - v_3 = 0 \quad \rightarrow \quad v_3 = v_2 = v_1$

so $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow$ normalize $\hat{v} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$



$$\cos\left(\frac{\Psi}{2}\right) = \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\phi + \psi}{2}\right) \qquad \phi = 0, \quad \theta = \pi/2, \quad \psi = \pi/2$$

$$= \cos(\pi/4)\cos(\pi/4)$$

$$= \frac{\sqrt{2}}{2}\,\frac{\sqrt{2}}{2}$$

$$= \frac{1}{2}$$

$\rightarrow \frac{\Psi}{2} = 60° \rightarrow \boxed{\Psi = 120°}$

Problem (6.7) Verify axis-angle $R_{\hat{n}}(\Psi)$ matrix formulae

$$\underline{A}' = \underline{A}\cos\Psi + \hat{n}\,(\underline{A}\cdot\hat{n})(1-\cos\Psi) + (\hat{n}\times\underline{A})\sin\Psi$$

Matrix rep:

$$\underline{A}' = \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix}, \qquad \underline{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\underline{A}\cos\Psi = \cos\Psi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} A_x\cos\Psi \\ A_y\cos\Psi \\ A_z\cos\Psi \end{pmatrix}$$

$$\hat{n}\,(\underline{A}\cdot\hat{n})(1-\cos\Psi) = \begin{pmatrix} n_x\,(A_x n_x + A_y n_y + A_z n_z)(1-\cos\Psi) \\ n_y\,(\quad)(\quad) \\ n_z\,(\quad)(\quad) \end{pmatrix}$$

$$(\hat{n}\times\underline{A})\sin\Psi = \begin{pmatrix} (n_y A_z - n_z A_y)\sin\Psi \\ (n_z A_x - n_x A_z)\sin\Psi \\ (n_x A_y - n_y A_x)\sin\Psi \end{pmatrix}$$

Thus,

$$A_x' = \cos\Psi A_x + n_x (A_x n_x + A_y n_y + A_z n_z)(1-\cos\Psi) + (n_y A_z - n_z A_y)\sin\Psi$$

$$= \left[\cos\Psi + n_x^2(1-\cos\Psi)\right] A_x + \left[n_x n_y(1-\cos\Psi) - n_z \sin\Psi\right] A_y + \left[n_x n_z(1-\cos\Psi) + n_y \sin\Psi\right] A_z$$

$$A_y' = \cos\Psi A_y + n_y (A_x n_x + A_y n_y + A_z n_z)(1-\cos\Psi) + (n_z A_x - n_x A_z)\sin\Psi$$

$$= \left[n_y n_x(1-\cos\Psi) + n_z \sin\Psi\right] A_x + \left[\cos\Psi + n_y^2(1-\cos\Psi)\right] A_y + \left[n_y n_z(1-\cos\Psi) - n_x \sin\Psi\right] A_z$$

$$A_z' = A_z \cos\Psi + n_z (A_x n_x + A_y n_y + A_z n_z)(1-\cos\Psi) + (n_x A_y - n_y A_x)\sin\Psi$$

$$= \left[n_z n_x(1-\cos\Psi) - n_y \sin\Psi\right] A_x + \left[n_z n_y(1-\cos\Psi) + n_x \sin\Psi\right] A_y + \left[\cos\Psi + n_z^2(1-\cos\Psi)\right] A_z$$

$$\begin{bmatrix} A'_x \\ A'_y \\ A'_z \end{bmatrix} = R^{\text{active}} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$R^{\text{active}} = \begin{bmatrix} \cos\Psi + n_x^2(1-\cos\Psi) & n_x n_y(1-\cos\Psi) - n_z \sin\Psi & n_x n_z(1-\cos\Psi) + n_y \sin\Psi \\ n_y n_x(1-\cos\Psi) + n_z \sin\Psi & \cos\Psi + n_y^2(1-\cos\Psi) & n_y n_z(1-\cos\Psi) - n_x \sin\Psi \\ n_z n_x(1-\cos\Psi) - n_y \sin\Psi & n_z n_y(1-\cos\Psi) + n_x \sin\Psi & \cos\Psi + n_z^2(1-\cos\Psi) \end{bmatrix}$$

$$R^{\text{passive}} = \left(R^{\text{active}}\right)^{-1}$$

Determine n_x, n_y, n_z:

__From:__ $R_n^{-1}(\psi)$ we see that

$$R_{y'z} - R_{z'y} = 2 \sin \Psi \, n_x$$

$$\rightarrow \boxed{n_x = \frac{1}{2 \sin \Psi} (R_{y'z} - R_{z'y})}$$

$$R_{z'x} - R_{x'z} = 2 \sin \psi \, n_y$$

$$\rightarrow \boxed{n_y = \frac{1}{2 \sin \Psi} (R_{z'x} - R_{x'z})}$$

$$R_{x'y} - R_{y'x} = 2 \sin \psi \, n_z$$

$$\rightarrow \boxed{n_z = \frac{1}{2 \sin \Psi} (R_{x'y} - R_{y'x})}$$

Substituting using Euler angle form ~~for~~ $R(\phi, \theta, \psi)$:

$$n_x = \frac{1}{2 \sin \Psi} (s\theta s\psi - s\theta s\phi)$$

← __NOTE:__ Difference between Ψ and ψ

$$= \frac{1}{2 \sin \Psi} \sin\theta (\sin\psi - \sin\phi)$$

$$n_y = \frac{1}{2 \sin \Psi} (s\theta c\phi + s\theta c\psi)$$

$$= \frac{1}{2 \sin \Psi} \sin\theta (\cos\phi + \cos\psi)$$

$$n_z = \frac{1}{2 \sin \Psi} (c\theta s\phi c\psi + c\phi s\psi + c\theta c\phi s\psi + s\phi c\psi)$$

$$= \frac{1}{2 \sin \Psi} (1 + \cos\theta)(\cos\phi \sin\psi + \sin\phi \cos\psi)$$

$$= \frac{1}{2 \sin \Psi} (1 + \cos\theta) \sin(\phi + \psi)$$

Exercise 6.7

Relating Ψ to (θ, ϕ, ψ):

$$\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \frac{1}{2\sin\Psi} \begin{bmatrix} \sin\theta(\sin\psi - \sin\phi) \\ \sin\theta(\cos\psi + \cos\phi) \\ (1+\cos\theta)\sin(\phi+\psi) \end{bmatrix}$$

$\uparrow$

unit vector

$$\begin{aligned}
1 &= n_x^2 + n_y^2 + n_z^2 \\
&= \frac{1}{4\sin^2\Psi}\Big[\sin^2\theta(\sin\psi - \sin\phi)^2 + \sin^2\theta(\cos\psi + \cos\phi)^2 \\
&\qquad + (1+\cos\theta)^2\sin^2(\phi+\psi)\Big] \\
&= \frac{1}{4\sin^2\Psi}\Big[\sin^2\theta(\sin^2\psi + \sin^2\phi - 2\sin\psi\sin\phi) \\
&\qquad + \sin^2\theta(\cos^2\psi + \cos^2\phi + 2\cos\psi\cos\phi) \\
&\qquad + (1+\cos\theta)^2\sin^2(\phi+\psi)\Big] \\
&= \frac{1}{4\sin^2\Psi}\Big[\sin^2\theta\sin^2\psi + \sin^2\theta\sin^2\phi - 2\sin^2\theta\sin\psi\sin\phi \\
&\qquad + \sin^2\theta\cos^2\psi + \sin^2\theta\cos^2\phi + 2\sin^2\theta\cos\psi\cos\phi \\
&\qquad + (1+\cos\theta)^2\sin^2(\phi+\psi)\Big] \\
&= \frac{1}{4\sin^2\Psi}\Big[2\sin^2\theta(1 - \sin\psi\sin\phi + \cos\psi\cos\phi) \\
&\qquad + (1+\cos\theta)^2\sin^2(\phi+\psi)\Big] \\
&= \frac{1}{4\sin^2\Psi}\Big[2\sin^2\theta(1+\cos(\psi+\phi)) + (1+\cos\theta)^2\sin^2(\phi+\psi)\Big]
\end{aligned}$$

Now,

$$\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x \\
&= 2\cos^2 x - 1 \\
&= 1 - 2\sin^2 x
\end{aligned}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin^2 x = \frac{1-\cos 2x}{2}$$

Thus, $\cos^2\left(\frac{x}{2}\right) = \frac{1+\cos x}{2}$, $\sin^2\left(\frac{x}{2}\right) = \frac{1-\cos x}{2}$

$$\rightarrow \quad 1+\cos x = 2\cos^2\left(\frac{x}{2}\right)$$
$$1-\cos x = 2\sin^2\left(\frac{x}{2}\right)$$

Thus,

$$1 = \frac{1}{4/\sin^2\Psi}\left[2\sin^2\theta \; 2\cos^2\left(\frac{\psi+\phi}{2}\right) + 4\cos^4\left(\frac{\theta}{2}\right)\sin^2(\phi+\psi)\right]$$

$$= \frac{1}{\sin^2\Psi}\left[\sin^2\theta \cos^2\left(\frac{\psi+\phi}{2}\right) + \cos^4\left(\frac{\theta}{2}\right)\sin^2(\phi+\psi)\right]$$

Now use: $\sin(2x) = 2\sin x\cos x$

$$\rightarrow \quad \sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$1 = \frac{1}{\sin^2\Psi}\left[4\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\psi+\phi}{2}\right) + \cos^4\left(\frac{\theta}{2}\right) 4\sin^2\left(\frac{\phi+\psi}{2}\right)\cos^2\left(\frac{\phi+\psi}{2}\right)\right]$$

$$= \frac{4}{\sin^2\Psi}\cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\psi+\phi}{2}\right)\left[\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\phi+\psi}{2}\right)\right]$$

$$\frac{\sin^2\Psi}{4} = \cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\psi+\phi}{2}\right)\left[1-\cos^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\phi+\psi}{2}\right)\right]$$

$$= \cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\psi+\phi}{2}\right)\left[1-\cos^2\left(\frac{\theta}{2}\right)\underbrace{\left(1-\sin^2\left(\frac{\phi+\psi}{2}\right)\right)}_{\cos^2\left(\frac{\phi+\psi}{2}\right)}\right]$$

$$= \cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\psi+\phi}{2}\right)\left[1-\cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\phi+\psi}{2}\right)\right]$$

$$= y\,[1-y]$$

where $y \equiv \cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\psi+\phi}{2}\right)$

$$LHS = \frac{1}{4} \sin^2 \Psi$$

$$= \frac{1}{\cancel{4}} \cancel{4} \sin^2\left(\frac{\Psi}{2}\right) \cos^2\left(\frac{\Psi}{2}\right)$$

$$= \cos^2\left(\frac{\Psi}{2}\right)\left(1 - \cos^2\left(\frac{\Psi}{2}\right)\right)$$

$$= x(1-x)$$

where $x \equiv \cos^2\left(\frac{\Psi}{2}\right)$

<u>Now:</u> $x(1-x) = y(1-y)$ iff $x = y$
$x = 1-y$

<u>check:</u> $(1-y)(1-(1-y)) = (1-y)y$

Thus, $\cos^2\left(\frac{\Psi}{2}\right) = \cos^2\left(\frac{\theta}{2}\right) \cos^2\left(\frac{\psi+\phi}{2}\right)$

$$\boxed{\cos\left(\frac{\Psi}{2}\right) = \pm\cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\psi+\phi}{2}\right)}$$

take + sign for equation in book

2^{nd} solution

$x = 1-y$ iff $y = 1-x$

$$\cos^2(\,)\cos^2(\,) = 1 - \cos^2\left(\frac{\Psi}{2}\right)$$
$$= \sin^2\left(\frac{\Psi}{2}\right)$$

Thus, $\boxed{\sin\left(\frac{\Psi}{2}\right) = \pm \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\psi+\phi}{2}\right)}$

Similar to original ~~equation~~ solution but with $\sin\left(\frac{\Psi}{2}\right)$ instead of $\cos\left(\frac{\Psi}{2}\right)$

Consider the case $\theta = 0, \phi = 0$ $\quad (\hat{n} = \hat{z})$

Then:

I, $\cos\left(\frac{\Psi}{2}\right) = \pm\cos\left(\frac{\psi}{2}\right)$

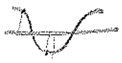
II, $\sin\left(\frac{\Psi}{2}\right) = \pm\cos\left(\frac{\psi}{2}\right)$

I $\Rightarrow$ $\frac{\Psi}{2} = \frac{\psi}{2} \pm n\pi \quad , \quad n = 1, 2, 3, \dots$

check: $\cos\left(\frac{\psi}{2} \pm n\pi\right) = \cos\left(\frac{\psi}{2}\right)\cos(n\pi) \mp \sin\left(\frac{\psi}{2}\right)\underbrace{\sin(n\pi)}_{0}$

$$= \cos\left(\frac{\psi}{2}\right)(-1)^n$$

$$= \pm\cos\left(\frac{\psi}{2}\right)$$



$\cos(\pi+\theta) = \cos\pi\cos\theta - \underbrace{\sin\pi}_{0}\sin\theta = -\cos\theta$

Thus, $\boxed{\Psi = \psi \pm 2n\pi}$ (so same angle) $n = 1, 2, \dots$

II $\Rightarrow$ $\frac{\Psi}{2} = \frac{\psi}{2} \pm \frac{n\pi}{2} \quad (n = 1, 3, 5, \dots)$

check: $\sin\left(\frac{\Psi}{2}\right) = \sin\left(\frac{\psi}{2} \pm \frac{n\pi}{2}\right)$

$$= \sin\left(\frac{\psi}{2}\right)\underbrace{\cos\left(\frac{n\pi}{2}\right)}_{0} \pm \cos\left(\frac{\psi}{2}\right)\underbrace{\sin\left(\frac{n\pi}{2}\right)}_{\pm 1}$$

$$= \pm\cos\left(\frac{\psi}{2}\right)$$

Thus, $\boxed{\Psi = \psi \pm n\pi}$ $\quad (n = 1, 3, 5, \dots)$

$\sin\left(\frac{\pi}{2}+\theta\right) = \sin\frac{\pi}{2}\cos\theta + \underbrace{\cos\frac{\pi}{2}}_{0}\sin\theta = \cos\theta$

$\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$

$\sin(\pi+\theta) = \underbrace{\sin\pi}_{0}\cos\theta + \cos\pi\sin\theta = -\sin\theta$

Thus, if we require that $\Psi = \psi \pmod{2\pi}$ for $\theta = 0, \phi = 0$ then the second solution involving $\sin(\Psi/2)$ is not allowed.

Another way to see this when $\theta = \phi = 0$:

$$1 = n_x^2 + n_y^2 + n_z^2 \Big|_{\substack{\theta=0\\ \phi=0}} = \frac{1}{\cancel{4}\sin^2\Psi} \cancel{4}\sin^2(\psi)$$

so $\quad \sin\psi = \pm\sin\Psi$

$\longrightarrow \quad \psi = \Psi + n\pi \qquad (n = 1, 2, 3, \cdots)$

$\sin(\psi + n\pi)$
$= \sin\psi\cos(n\pi) + \cos\psi\,\cancel{\sin(n\pi)}_{0}$
$= (-1)^n \sin\psi$
$= \pm\sin\psi$

$n = 0, \pm 2, \pm 4, \pm 6, \cdots$ gives the $\Psi = \psi \pmod{2\pi}$ solution

$n = \pm 1, \pm 3, \pm 5 \cdots$ gives the $\sin(\Psi/2)$ solution since.

$$\cos\left(\frac{\psi}{2}\right) = \cos\left(\frac{\Psi}{2} + \frac{n\pi}{2}\right) \quad \left[n = 1, 3, 5, \cdots\right]$$

$$= \cos\left(\frac{\Psi}{2}\right)\underbrace{\cancel{\cos\left(\frac{n\pi}{2}\right)}}_{0} - \sin\left(\frac{\Psi}{2}\right)\sin\left(\frac{n\pi}{3}\right)$$

$$= \mp\sin\left(\frac{\Psi}{2}\right)$$

But $\Psi = \psi \pmod{2\pi}$ requires $n = 0, 2, 4, 6, \cdots$

Problem: (6.8) Angular momentum infinitesimal rotation matrices

$$\epsilon = \begin{bmatrix} 0 & -d\psi_z & d\psi_y \\ d\psi_z & 0 & -d\psi_x \\ -d\psi_y & d\psi_x & 0 \end{bmatrix}$$

$$= d\psi_x \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{L_x} + d\psi_y \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{L_y} + d\psi_z \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{L_z}$$

$$[L_x, L_y] = L_x L_y - L_y L_x$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = L_z$$

$$[L_y, L_z] = L_y L_z - L_z L_y$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = L_x$$

$$[L_z, L_x] = L_z L_x - L_x L_z$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = L_y$$

Thus, $[L_i, L_j] = \epsilon_{ijk} L_k$

Problem 6.9 verify expressions for $\underline{\omega}$

$$\epsilon = \begin{bmatrix} 0 & -d\psi_3 & d\psi_2 \\ d\psi_3 & 0 & -d\psi_1 \\ -d\psi_2 & d\psi_1 & 0 \end{bmatrix}$$

$$\underline{A}' = (1+\epsilon)\underline{A} \quad , \quad R^{\text{active}} = (1+\epsilon)$$

$$= \underline{A} + \epsilon \underline{A}$$

$$= \underline{A} + \begin{bmatrix} 0 & -d\psi_3 & d\psi_2 \\ d\psi_3 & 0 & -d\psi_1 \\ -d\psi_2 & d\psi_1 & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$= \underline{A} + \begin{bmatrix} (d\psi_2)A_z - (d\psi_3)A_y \\ (d\psi_3)A_x - (d\psi_1)A_z \\ (d\psi_1)A_y - (d\psi_2)A_x \end{bmatrix}$$

$$= \underline{A} + \underline{d\psi} \times \underline{A}$$

$$= \underline{A} + (\hat{n} \times \underline{A})\, d\psi$$

Compare with

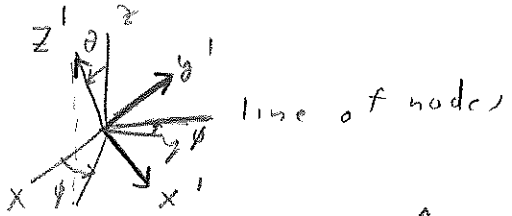
$$\underline{A}' = \cos\Psi\, \underline{A} + (1-\cos\Psi)(\underline{A}\cdot\hat{n})\hat{n} + \sin\Psi(\hat{n}\times\underline{A})$$

For $\Psi << 1$, $\cos\Psi \approx 1$ and $\sin\Psi \approx \Psi$

$$\rightarrow \underline{A}' \approx \underline{A} + \Psi(\hat{n}\times\underline{A}) \quad \text{which is}$$

consistent with $\underline{A} + (\hat{n}\times\underline{A})\,d\Psi$ with $\Psi \rightarrow d\Psi$

$$\underline{\omega}\, dt = \hat{n}\, d\Psi$$
$$= \hat{n}_\phi\, d\phi + \hat{n}_\theta\, d\theta + \hat{n}_\psi\, d\psi$$



<u>Now</u>:
$$\left\{\begin{array}{ll} \hat{n}_\phi = \hat{z} & \\ \hat{n}_\theta = -\sin\phi\, \hat{x} + \cos\phi\, \hat{y} & (= \hat{r}) \\ \hat{n}_\psi = \sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y} + \cos\theta\, \hat{z} & (= \hat{z}') \end{array}\right.$$

Thus

$$\underline{\omega} = \hat{z}\,\dot{\phi} + \left(-\sin\phi\, \hat{x} + \cos\phi\, \hat{y}\right)\dot{\theta} + \left(\sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y} + \cos\theta\, \hat{z}\right)\dot{\psi}$$

$$= \left(-\sin\phi\,\dot{\theta} + \sin\theta\cos\phi\,\dot{\psi}\right)\hat{x} + \left(\cos\phi\,\dot{\theta} + \sin\theta\sin\phi\,\dot{\psi}\right)\hat{y} + \left(\dot{\phi} + \cos\theta\,\dot{\psi}\right)\hat{z}$$

$$= \boxed{\begin{array}{c} \sin\theta\cos\phi\,\dot{\psi} - \sin\phi\,\dot{\theta} \\ \hline \sin\theta\sin\phi\,\dot{\psi} + \cos\phi\,\dot{\theta} \\ \hline \cos\theta\,\dot{\psi} + \dot{\phi} \end{array}} = \boxed{\begin{array}{c} \omega_x \\ \hline \omega_y \\ \hline \omega_z \end{array}}$$

In the body frame

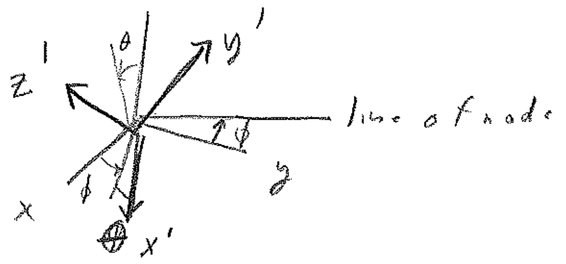
$$\begin{bmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{bmatrix} = \begin{bmatrix} c\theta\, c\phi\, c\psi - s\phi\, s\psi & c\theta\, s\phi\, c\psi + c\phi\, s\psi & -s\theta\, c\psi \\ -c\theta\, c\phi\, s\psi - s\phi\, c\psi & -c\theta\, s\phi\, s\psi + c\phi\, c\psi & s\theta\, s\psi \\ s\theta\, c\phi & s\theta\, s\phi & c\theta \end{bmatrix} \begin{bmatrix} s\theta\, c\phi\, \dot\psi - s\phi\, \dot\theta \\ s\theta\, s\phi\, \dot\psi + c\phi\, \dot\theta \\ c\theta\, \dot\psi + \dot\phi \end{bmatrix}$$

$$= \begin{bmatrix} s\theta\, c\theta\, c^2\phi\, c\psi\, \dot\psi \cancel{- s\phi\, s\psi\, s\theta\, c\phi\, \dot\psi} \cancel{- c\theta\, c\phi\, s\phi\, c\psi\, \dot\theta} + s^2\phi\, s \\ + s\theta\, c\theta\, s^2\phi\, c\psi\, \dot\psi \cancel{+ c\phi\, s\psi\, s\theta\, s\phi\, \dot\psi} \cancel{+ c\theta\, s\phi\, c\phi\, c\psi\, \dot\theta} + c^2\phi\, s \\ - s\theta\, c\psi\, \dot\phi \\ - s\theta\, c\theta\, c\psi\, \dot\psi \\ \hline - s\theta\, c\theta\, c^2\phi\, s\psi\, \dot\psi \cancel{- s\phi\, c\psi\, s\theta\, c\phi\, \dot\psi} \cancel{+ c\theta\, c\phi\, s\phi\, s\psi\, \dot\theta} + s^2\phi\, c\psi \\ - s\theta\, c\theta\, s^2\phi\, s\psi\, \dot\psi \cancel{+ c\phi\, c\psi\, s\theta\, s\phi\, \dot\psi} \cancel{- c\theta\, s\phi\, c\phi\, s\psi\, \dot\theta} + c^2\phi\, c\psi \\ + s\theta\, s\psi\, \dot\phi \\ + s\theta\, c\theta\, s\psi\, \dot\psi \\ \hline s^2\theta\, c^2\phi\, \dot\psi \cancel{- s\theta\, s\phi\, c\phi\, \dot\theta} \\ + s^2\theta\, s^2\phi\, \dot\psi \cancel{+ s\theta\, s\phi\, c\phi\, \dot\theta} \\ + c^2\theta\, \dot\psi + c\theta\, \dot\phi \end{bmatrix}$$

$$= \begin{bmatrix} s\psi\,\dot{\theta} - s\theta\, c\psi\, \dot{\phi} \\ c\psi\, \dot{\theta} + s\theta\, s\psi\, \dot{\phi} \\ \dot{\psi} + c\theta\, \dot{\phi} \end{bmatrix}$$

$$= \begin{bmatrix} -s\theta\, c\psi\, \dot{\phi} + s\psi\, \dot{\theta} \\ s\theta\, s\psi\, \dot{\phi} + c\psi\, \dot{\theta} \\ \cos\theta\, \dot{\phi} + \dot{\psi} \end{bmatrix}$$

For Tait-Bryan angles



$$\left.\begin{aligned} \hat{n}_\phi &= \hat{z} \\ \hat{n}_\theta &= -\sin\phi\, \hat{x} + \cos\phi\, \hat{y} \end{aligned}\right\} \text{before}$$

$$\hat{n}_\psi = \hat{x}'$$

$$\hat{n}_\psi = \cos\theta\cos\phi\, \hat{x} + \cos\theta\sin\phi\, \hat{y} - \sin\theta\, \hat{z}$$

Problem: 6.10 2-d rotations and complex numbers

$$R(\theta_1) = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

$$R(\theta_1)R(\theta_2) = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \\ -\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}$$

$$= R(\theta_1+\theta_2)$$

Similarly $R(\theta_2)R(\theta_1) = R(\theta_1+\theta_2)$ (just interchange θ_1 and θ_2 in the above equation)

so multiplication of rotations in 2-d is commutative

$Z = x+iy \longrightarrow Z = e^{i\theta}$ (unit magnitude)

$= \cos\theta + i\sin\theta,$

$$Z_1 Z_2 = e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

mapping: $R(\theta) \in$ 2-d rotation matrix $\longrightarrow e^{i\theta} \in \mathbb{Z}$

Problem (6.11) multiplication of two quaternions

Let $q_1 = s + \vec{u} \equiv s + u_x i + u_y j + u_z k$

$q_2 = t + \vec{v} \equiv t + v_x i + v_y j + v_z k$

Then:

$$q_1 q_2 = (s + \vec{u})(t + \vec{v})$$
$$= st + s\vec{v} + t\vec{u} + \vec{u}\vec{v}$$

$$\vec{u}\vec{v} = (u_x i + u_y j + u_z k)(v_x i + v_y j + v_z k)$$

$$= u_x v_x \underbrace{ii}_{-1} + u_x v_y \underbrace{ij}_{k} + u_x v_z \underbrace{ik}_{-j}$$

$$+ u_y v_x \underbrace{ji}_{-k} + u_y v_y \underbrace{jj}_{-1} + u_y v_z \underbrace{jk}_{i}$$

$$+ u_z v_x \underbrace{ki}_{j} + u_z v_y \underbrace{kj}_{-i} + u_z v_z \underbrace{kk}_{-1}$$

$$= -u_x v_x - u_y v_y - u_z v_z$$
$$+ i(u_y v_z - u_z v_y) + j(u_z v_x - u_x v_z) + k(u_x v_y - u_y v_x)$$

$$= -\vec{u}\cdot\vec{v} + \vec{u}\times\vec{v}$$

NOTE. If $\vec{u} = ui$, $\vec{v} = vi$, then

$$q_1 q_2 = (s + ui)(t + vi)$$
$$= st + (sv + ut)i + uv\underbrace{ii}_{-1}$$
$$= (st - uv) + (sv + ut)i$$

which is multiplication of ordinary complex numbers.

Problem 6.12 Unit quaternions as rotations

$$q = \cos\left(\frac{\psi}{2}\right) + \sin\left(\frac{\psi}{2}\right)(n_x i + n_y j + n_z k)$$

$$= \cos\left(\frac{\psi}{2}\right) + \sin\left(\frac{\psi}{2}\right)\hat{n}$$

$$q\vec{v}\,q^{-1} = \left(\cos\left(\frac{\psi}{2}\right) + \sin\left(\frac{\psi}{2}\right)\hat{n}\right)\vec{v}\left(\cos\left(\frac{\psi}{2}\right) - \sin\left(\frac{\psi}{2}\right)\hat{n}\right)$$

$$= \left(\cos\left(\frac{\psi}{2}\right) + \sin\left(\frac{\psi}{2}\right)\hat{n}\right)\left[\cos\left(\frac{\psi}{2}\right)\vec{v} - \sin\left(\frac{\psi}{2}\right)\left(-\vec{v}\cdot\hat{n} + \vec{v}\times\hat{n}\right)\right]$$

$$= \cos^2\left(\frac{\psi}{2}\right)\vec{v} - \sin\left(\frac{\psi}{2}\right)\cos\left(\frac{\psi}{2}\right)\left(\cancel{-\vec{v}\cdot\hat{n}} + \vec{v}\times\hat{n}\right)$$
$$+ \sin\left(\frac{\psi}{2}\right)\cos\left(\frac{\psi}{2}\right)\left(\cancel{-\hat{n}\cdot\vec{v}} + \hat{n}\times\vec{v}\right)$$
$$+ \sin^2\left(\frac{\psi}{2}\right)\left(\vec{v}\cdot\hat{n}\right)\hat{n}$$
$$- \sin^2\left(\frac{\psi}{2}\right)\left[\underbrace{-\hat{n}\cdot(\vec{v}\times\hat{n})}_{=0} + \hat{n}\times(\vec{v}\times\hat{n})\right]$$

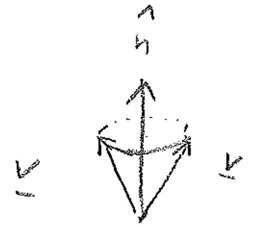
$$= \cos^2\left(\frac{\psi}{2}\right)\vec{v} + 2\sin\left(\frac{\psi}{2}\right)\cos\left(\frac{\psi}{2}\right)\hat{n}\times\vec{v} + \sin^2\left(\frac{\psi}{2}\right)\left(\vec{v}\cdot\hat{n}\right)\hat{n}$$
$$- \sin^2\left(\frac{\psi}{2}\right)\left[\vec{v}\,\underbrace{(\hat{n}\cdot\hat{n})}_{1} - \hat{n}\,(\hat{n}\cdot\vec{v})\right]$$

$$= \left[\cos^2\left(\frac{\psi}{2}\right) - \sin^2\left(\frac{\psi}{2}\right)\right]\vec{v} + 2\sin\left(\frac{\psi}{2}\right)\cos\left(\frac{\psi}{2}\right)\hat{n}\times\vec{v}$$
$$+ 2\sin^2\left(\frac{\psi}{2}\right)\left(\vec{v}\cdot\hat{n}\right)\hat{n}$$

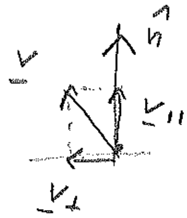
$$= \cos\psi\,\vec{v} + \sin\psi\,\hat{n}\times\vec{v} + (1-\cos\psi)(\vec{v}\cdot\hat{n})\hat{n}$$

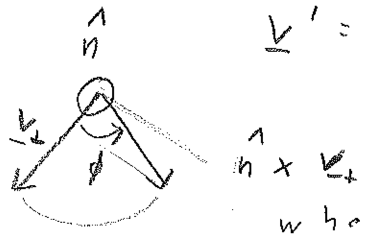
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta \rightarrow 2\sin^2\theta = 1 - \cos 2\theta$$

Rotation about $\hat{n}$ by ϕ: (active)



$$\underline{v}' = R_{\hat n}(\phi)\,\underline{v}$$





$$\underline{v}' = \underline{v}_{\parallel} + \left(\cos\phi\,\underline{v}_{\perp} + \sin\phi\,\hat{n}\times\underline{v}_{\perp}\right)$$

where

$$\underline{v}_{\parallel} = (\underline{v}\cdot\hat{n})\,\hat{n}$$

$$\begin{aligned}\underline{v}_{\perp} &= \underline{v} - \underline{v}_{\parallel}\\ &= \underline{v} - (\underline{v}\cdot\hat{n})\,\hat{n}\\ &= -\hat{n}\times(\hat{n}\times\underline{v})\end{aligned}$$

Using $\underline{A}\times(\underline{B}\times\underline{C}) = \underline{B}(\underline{A}\cdot\underline{C}) - \underline{C}(\underline{A}\cdot\underline{B})$

Alternate ways of writing:

$$\begin{aligned}\underline{v}' &= \underline{v}_{\parallel} + \cos\phi\,\underline{v}_{\perp} + \sin\phi\,\hat{n}\times\underline{v}_{\perp}\\ &= (\underline{v}\cdot\hat{n})\hat{n} + \cos\phi\left(\underline{v} - (\underline{v}\cdot\hat{n})\hat{n}\right) + \sin\phi\,\hat{n}\times\left(\underline{v} - \underbrace{\cancel{(\underline{v}\cdot\hat{n})\hat{n}}}_{0}\right)\\ &= (\underline{v}\cdot\hat{n})\hat{n} + \cos\phi\,\underline{v} - \cos\phi(\underline{v}\cdot\hat{n})\hat{n} + \sin\phi\,\hat{n}\times\underline{v}\\ &= (1-\cos\phi)(\underline{v}\cdot\hat{n})\hat{n} + \cos\phi\,\underline{v} + \sin\phi\,\hat{n}\times\underline{v}\\ &= \cos\phi\,\underline{v} + (1-\cos\phi)(\underline{v}\cdot\hat{n})\hat{n} + \sin\phi\,\hat{n}\times\underline{v}\end{aligned}$$

Given

$$q = \cos\left(\frac{\psi}{2}\right) + \sin\left(\frac{\psi}{2}\right)(n_x i + n_y j + n_z k)$$

$$= \cos\left(\frac{\psi}{2}\right) + \sin\left(\frac{\psi}{2}\right)\hat{n} \quad (= w + xi + yj + zk)$$

active

$$R_n(\psi) = \begin{bmatrix} c\psi + (1-c\psi)n_x^2 & (1-c\psi)n_x n_y - s\psi\, n_z & (1-c\psi)n_x n_z + s\psi\, n_y \\ (1-c\psi)n_y n_x + s\psi\, n_z & c\psi + (1-c\psi)n_y^2 & (1-c\psi)n_y n_z - s\psi\, n_x \\ (1-c\psi)n_z n_x - s\psi\, n_y & (1-c\psi)n_z n_y + s\psi\, n_x & c\psi + (1-c\psi)n_z^2 \end{bmatrix}$$

Recast in terms of (w, x, y, z):

$w = \cos\left(\frac{\psi}{2}\right)$

$x = \sin\left(\frac{\psi}{2}\right) n_x$

$y = \sin\left(\frac{\psi}{2}\right) n_y$

$z = \sin\left(\frac{\psi}{2}\right) n_z$

with
$w^2 + x^2 + y^2 + z^2 = 1$

$$\cos\psi = \cos^2\left(\frac{\psi}{2}\right) - \sin^2\left(\frac{\psi}{2}\right)$$
$$= 2\cos^2\left(\frac{\psi}{2}\right) - 1$$
$$= 2w^2 - 1$$

$$1 - \cos\psi = 1 - \cos^2\left(\frac{\psi}{2}\right) + \sin^2\left(\frac{\psi}{2}\right)$$
$$= 2\sin^2\left(\frac{\psi}{2}\right)$$

Thus,
$$c\psi + (1-c\psi)n_x^2 = 2w^2 - 1 + 2\sin^2\left(\frac{\psi}{2}\right)n_x^2$$
$$= 2w^2 - 1 + 2x^2$$
$$= 2(w^2 + x^2) - 1$$
$$= 2(w^2 + x^2) - w^2 - x^2 - y^2 - z^2$$
$$= \boxed{w^2 + x^2 - y^2 - z^2}$$

Similarly,
$$c\psi + (1-c\psi)n_y^2 = 2w^2 - 1 + 2\sin^2\left(\frac{\psi}{2}\right)n_y^2$$
$$= 2(w^2 + y^2) - 1$$
$$= \boxed{w^2 - x^2 + y^2 - z^2}$$

and
$$c\psi + (1-c\psi)n_z^2 = 2w^2 - 1 + 2\sin^2\left(\frac{\psi}{2}\right)n_z^2$$
$$= 2(w^2 + z^2) - 1$$
$$= \boxed{w^2 - x^2 - y^2 + z^2}$$

Diagonal elements

$$\sin\psi = 2\sin\left(\tfrac{\psi}{2}\right)\cos\left(\tfrac{\psi}{2}\right)$$

$$(1-c\psi)\, n_x n_y - s\psi\, n_z = 2\sin^2\left(\tfrac{\psi}{2}\right) n_x n_y - 2\sin\left(\tfrac{\psi}{2}\right)\cos\left(\tfrac{\psi}{2}\right) n_z$$

$$= 2xy - 2wz$$

$$= \boxed{2(xy - wz)}$$

$$(1-c\psi)\, n_x n_z + s\psi\, n_y = 2\sin^2\left(\tfrac{\psi}{2}\right) n_x n_z + 2\sin\left(\tfrac{\psi}{2}\right)\cos\left(\tfrac{\psi}{2}\right) n_y$$

$$= 2xz + 2wy$$

$$= \boxed{2(xz + wy)}$$

$$(1-c\psi)\, n_y n_x + s\psi\, n_z = \boxed{2(yx + wz)}$$

$$(1-c\psi)\, n_y n_z - s\psi\, n_x = \boxed{2(yz - wx)}$$

$$(1-c\psi)\, n_z n_x - s\psi\, n_y = \boxed{2(zx - wy)}$$

$$(1-c\psi)\, n_z n_y + s\psi\, n_x = \boxed{2(zy + wx)}$$

Thus R^{active} =

$w^2 + x^2 - y^2 - z^2$	$2(xy - wz)$	$2(xz + wy)$
$2(yx + wz)$	$w^2 - x^2 + y^2 - z^2$	$2(yz - wx)$
$2(zx - wy)$	$2(zy + wx)$	$w^2 - x^2 - y^2 + z^2$

NOTE: q and $-q$ map to same matrix R^{active} since above expression is quadratic in (w, x, y, z). $\equiv$ DOUBLE COVER

$$\left(R^{active}\right)^T = \begin{bmatrix} w^2+x^2-y^2-z^2 & 2(yx+wz) & 2(zx-wy) \\ 2(xy-wz) & w^2-x^2+y^2-z^2 & 2(zy+wx) \\ 2(xz+wy) & 2(yz-wx) & w^2-x^2-y^2+z^2 \end{bmatrix}$$

<u>NOTE:</u> inverse matrix has $-\psi$ in place of ψ

But $\cos\left(-\frac{\psi}{2}\right) = \cos\left(\frac{\psi}{2}\right)$

$\sin\left(-\frac{\psi}{2}\right) = -\sin\left(\frac{\psi}{2}\right)$

Thus, $\left.\begin{array}{l} w \rightarrow w \\ (x,y,z) \rightarrow (-x,-y,-z) \end{array}\right\}$ for inverse

$$\left(R^{active}\right)^{-1} = \begin{bmatrix} w^2+x^2-y^2-z^2 & 2(xy+wz) & 2(xz-wy) \\ 2(yz-wz) & w^2-x^2+y^2-z^2 & 2(yz+wx) \\ 2(zx+wy) & 2(zy-wx) & w^2-x^2-y^2+z^2 \end{bmatrix}$$

$$= \left(R^{active}\right)^T$$

Problem: (6.2) Calculate $R(\phi, \theta, \psi)$ (passive rotation) in zxz representation.
[used by Goldstein]

In zyz representation

$$R_z(\psi)\, R_y(\theta)\, R_z(\phi)$$

$$= \begin{bmatrix} c\theta\, c\phi\, c\psi - s\phi\, s\psi & c\theta\, s\phi\, c\psi + c\phi\, s\psi & -s\theta\, c\psi \\ -c\theta\, c\phi\, s\psi - s\phi\, c\psi & -c\theta\, s\phi\, s\psi + c\phi\, c\psi & s\theta\, s\psi \\ s\theta\, c\phi & s\theta\, s\phi & c\theta \end{bmatrix}$$

In zxz representation:

$$R_z(\psi)\, R_x(\theta)\, R_z(\phi)$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\phi & s\phi & 0 \\ -c\theta\, s\phi & c\theta\, c\phi & s\theta \\ s\theta\, s\phi & -s\theta\, c\phi & c\theta \end{bmatrix}$$

$$= \begin{bmatrix} c\phi\, c\psi - c\theta\, s\phi\, s\psi & s\phi\, c\psi + c\theta\, c\phi\, s\psi & s\theta\, s\psi \\ -c\phi\, s\psi - c\theta\, s\phi\, c\psi & -s\phi\, s\psi + c\theta\, c\phi\, c\psi & s\theta\, c\psi \\ s\theta\, s\phi & -s\theta\, c\phi & c\theta \end{bmatrix}$$

Prob (6.3)

$$R(\phi,\theta,\psi) = \begin{bmatrix} c\theta\, c\phi\, c\psi - s\phi\, s\psi & c\theta\, s\phi\, c\psi + c\phi\, s\psi & -s\theta\, c\psi \\ -c\theta\, c\phi\, s\psi - s\phi\, c\psi & -c\theta\, s\phi\, s\psi + c\phi\, c\psi & s\theta\, s\psi \\ s\theta\, c\phi & s\theta\, s\phi & c\theta \end{bmatrix}$$

a) $$1 + Tr[R(\phi,\theta,\psi)] = 1 + c\theta\, c\phi\, c\psi - s\phi\, s\psi - c\theta\, s\phi\, s\psi + c\phi\, c\psi + c\theta$$

$$= 1 + c\theta\,(c\phi\, c\psi - s\phi\, s\psi) + (c\phi\, c\psi - s\phi\, s\psi) + c\theta$$

$$= (1 + c\theta)(1 + c\phi\, c\psi - s\phi\, s\psi)$$

$$= (1 + c\theta)(1 + c(\phi+\psi))$$

$$= 4\cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\phi+\psi}{2}\right)$$

$c^2 + s^2 = 1$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$

b) Always possible to find an S such that

$$S R_{\hat{n}}(\Psi) S^{-1} = \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c) $Tr\,(S R_{\hat{n}}(\Psi) S^{-1}) = Tr$

LHS $= Tr\,(\underbrace{S^{-1} S}_{1} R_{\hat{n}}(\Psi)) = Tr\,(R_{\hat{n}}(\Psi))$

RHS $= 2\cos\Psi + 1$ $\cancel{\rightarrow \cos\Psi}$

$$\rightarrow 2\cos\Psi + 1 = Tr\,(R_{\hat{n}}(\Psi)) \rightarrow \cos\Psi = \frac{1}{2}\left(Tr\,(R_{\hat{n}}(\Psi)) - 1\right)$$

d) $2\cos\Psi = Tr(R_{\hat{n}}(\Psi)) - 1$

$= Tr(R(\phi,\theta,\psi)) - 1$

$= 4\cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\phi+\psi}{2}\right) - 1 - 1$

$= 4\cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\phi+\psi}{2}\right) - 2$

$\frac{\cos\Psi + 1}{2} = \cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\phi+\psi}{2}\right)$

$\parallel$

$\cos^2\left(\frac{\Psi}{2}\right)$

Thus, $\boxed{\cos\left(\frac{\Psi}{2}\right) = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi+\psi}{2}\right)}$

~~Problem~~: Eigenvectors, eigenvalues of $R_z(\psi)$

Prob 6.4

$$R = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ where $\hat{n}$ along $\hat{z}$

Eigenvectors, evals:

$$\begin{aligned}
0 &= \det(R - \lambda \mathbb{1}) \\
&= (1-\lambda)\left[(\cos\psi - \lambda)^2 + \sin^2\psi\right] \\
&= (1-\lambda)\left[\cos^2\psi + \lambda^2 - 2\cos\psi\,\lambda + \sin^2\psi\right] \\
&= (1-\lambda)\left[1 + \lambda^2 - 2\cos\psi\,\lambda\right]
\end{aligned}$$

$\rightarrow \lambda = 1$, $\quad \lambda = \dfrac{2\cos\psi \pm \sqrt{4\cos^2\psi - 4\cdot1\cdot1}}{2}$

$$\begin{aligned}
&= \cos\psi \pm \sqrt{\cos^2\psi - 1} \\
&= \cos\psi \pm i\sqrt{1-\cos^2\psi} \\
&= \cos\psi \pm i\sin\psi \\
&= e^{\pm i\psi}
\end{aligned}$$

thus, $\lambda = 1,\ e^{+i\psi},\ e^{-i\psi}$

$\lambda = 1$:

$$\begin{bmatrix} \cos\psi - 1 & \sin\psi & 0 \\ -\sin\psi & \cos\psi - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_1 (\cos\psi - 1) + V_2 \sin\psi = 0$$

$$\boxed{V_3 = \text{anything}}$$

$$-V_1 \sin\psi + V_2 (\cos\psi - 1) = 0$$

$$V_1 \sin\psi (\cos\psi - 1) + V_2 \sin^2\psi = 0$$

$$-V_1 \sin\psi (\cos\psi - 1) + V_2 (\cos\psi - 1)^2 = 0$$

$$0 + V_2 [\sin^2\psi + \cos^2\psi + 1 - 2\cos\psi] = 0$$

$$2 V_2 [\underbrace{1 - \cos\psi}_{\neq 0 \text{ in general}}] = 0$$

$$\rightarrow \boxed{V_2 = 0}$$

$$\rightarrow \boxed{V_1 = 0}$$

Thus, $\underline{V} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\underline{\lambda = e^{i\psi}}$:

$$\begin{bmatrix} \cos\psi - e^{i\psi} & \sin\psi & 0 \\ -\sin\psi & \cos\psi - e^{i\psi} & 0 \\ 0 & 0 & 1 - e^{i\psi} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(3)

$$\cos\psi - e^{i\psi} = \cos\psi - [\cos\psi + i\sin\psi]$$
$$= -i\sin\psi$$

$$1 - e^{i\psi} = (1-\cos\psi) - i\sin\psi$$

thus,

$$\left.\begin{aligned} -i\sin\psi\, v_1 + \sin\psi\, v_2 &= 0 \\ -\sin\psi\, v_1 - i\sin\psi\, v_2 &= 0 \end{aligned}\right\} \text{equivalent}$$

$$[(1-\cos\psi) - i\sin\psi]\, v_3 = 0$$

$$\to \boxed{v_3 = 0}$$

and $\sin\psi\,[-i v_1 + v_2] = 0$

$$\to \boxed{v_2 = i v_1}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \underline{v}_+$$

$\underline{\lambda = e^{-i\psi}}$:

$$\begin{bmatrix} \cos\psi - e^{-i\psi} & \sin\psi & 0 \\ -\sin\psi & \cos\psi - e^{-i\psi} & 0 \\ 0 & 0 & 1-e^{-i\psi} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\to \boxed{v_3 = 0}$$

$$i\sin\psi\, v_1 + \sin\psi\, v_2 = 0$$
$$\sin\psi\,[i v_1 + v_2] = 0$$
$$\to \boxed{v_2 = -i v_1}$$

$$\underline{v}_- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

$e^{i\psi} = 1 \quad \text{if } \psi = 0 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ identity transformation

$e^{i\psi} = -1 \quad \text{if } \psi = \pi \qquad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ rotation by π around $\hat{z}$

For the latter case, $\lambda = -1$ is a double root!

$$\begin{bmatrix} -1+1 & 0 & 0 \\ 0 & -1+1 & 0 \\ 0 & 0 & 1+1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow \quad V_3 = 0$

$V_1 = V_2 =$ anything

so $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Problem 6.5 Pauli spin matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

a) $$\left[-\frac{i}{2}\sigma_x , -\frac{i}{2}\sigma_y \right] = -\frac{1}{4}[\sigma_x, \sigma_y]$$

$$= -\frac{1}{4}\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$= -\frac{1}{4}\left\{ \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \right\}$$

$$= -\frac{1}{4}\begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}$$

$$= -\frac{i}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= -\frac{i}{2}\sigma_z$$

$$\left[-\frac{i}{2}\sigma_y , -\frac{i}{2}\sigma_z \right] = -\frac{1}{4}[\sigma_y, \sigma_z]$$

$$= -\frac{1}{4}\left\{ \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right\}$$

$$= -\frac{1}{4}\left\{ \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \right\}$$

$$= -\frac{1}{4}\begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix}$$

$$= -\frac{i}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= -\frac{i}{2}\sigma_x$$

$$\left[-\frac{i}{2}\sigma_z\, ,\, -\frac{i}{2}\sigma_x\right] = -\frac{1}{4}\left[\sigma_z, \sigma_x\right]$$

$$= -\frac{1}{4}\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\}$$

$$= -\frac{1}{4}\left\{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} - \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right\}$$

$$= -\frac{1}{4}\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$$= -\frac{i}{2}\begin{pmatrix} & -i\\ i & 0\end{pmatrix}$$

$$= -\frac{i}{2}\sigma_y$$

b) Element of $SU(2)$:

$$U = \begin{pmatrix}a & -b^*\\ b & a^*\end{pmatrix} \qquad \text{with} \quad |a|^2 + |b|^2 = 1$$

$$\det\begin{pmatrix}a & -b^*\\ b & a^*\end{pmatrix} = |a|^2 + |b|^2 = 1$$

$$U^\dagger = \begin{pmatrix}a^* & b^*\\ -b & a\end{pmatrix}$$

$$UU^\dagger = \begin{pmatrix}a & -b^*\\ b & a^*\end{pmatrix}\begin{pmatrix}a^* & b^*\\ -b & a\end{pmatrix}$$

$$= \begin{pmatrix}|a|^2 + |b|^2 & 0\\ 0 & |a|^2 + |b|^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbb{1}$$

$$U^\dagger U = \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix} \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$$

$$= \begin{pmatrix} |a|^2+|b|^2 & 0 \\ 0 & |a|^2+|b|^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \mathbb{1}$$

so U is unitary and has $\det U = 1$.
Thus $U \in SU(2)$

Write $a = x+iy$, $b = u+iv$

Then $$U = \begin{pmatrix} x+iy & -(u-iv) \\ u+iv & x-iy \end{pmatrix}$$

$$= x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + yi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + u\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + vi\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= x\mathbb{1} + yi\sigma_z + u(-i)\underbrace{\begin{pmatrix} 0 & -i \\ i & 1 \end{pmatrix}}_{\sigma_y} + iv\sigma_x$$

$$= x\mathbb{1} - y(-i\sigma_z) + u(-i\sigma_y) - v(-i\sigma_x)$$

$$= x\mathbb{1} - v(-i\sigma_x) + u(-i\sigma_y) - y(-i\sigma_z)$$

So $\{\mathbb{1}, -i\sigma_x, -i\sigma_y, -i\sigma_z\}$ span $SU(2)$.

<u>NOTE:</u> $-i\sigma_x = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$, $-i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $-i\sigma_z = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

are all elts of $SU(2)$ since they are of the form $\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$

(4)

c) $-i\sigma_x, \ -i\sigma_y, \ -i\sigma_z \longleftrightarrow i, j, k$

(Pauli matrices) (quaternions)

$$(-i\sigma_x)(-i\sigma_y) = -\sigma_x\sigma_y$$

$$= -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$= -i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= -i\sigma_z \longleftrightarrow \text{like } ij = k$$

$$(-i\sigma_y)(-i\sigma_z) = -\sigma_y\sigma_z$$

$$= -\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$= -i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= -i\sigma_x \longleftrightarrow \text{like } jk = i$$

$$(-i\sigma_z)(-i\sigma_x) = -\sigma_z\sigma_x$$

$$= -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= -i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -i\sigma_y \longleftrightarrow \text{like } ki = j$$

$$(-i\sigma_y)(-i\sigma_x) = -\sigma_y\sigma_x$$

$$= -\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$= +i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= -(-i\sigma_z) \longleftrightarrow \text{like } \; ji = -k$$

$$(-i\sigma_z)(-i\sigma_y) = -\sigma_z\sigma_y$$

$$= -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$= -(-i)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= -(-i\sigma_x) \longleftrightarrow \text{like } \; kj = -i$$

$$(-i\sigma_x)(-i\sigma_z) = -\sigma_x\sigma_z$$

$$= -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

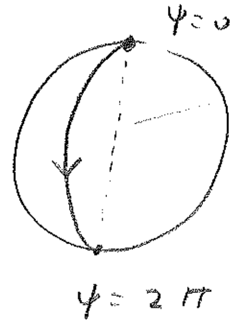
$$= -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= -(-i)\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= -(-i\sigma_y) \longleftrightarrow \text{like } \; ik = -j$$

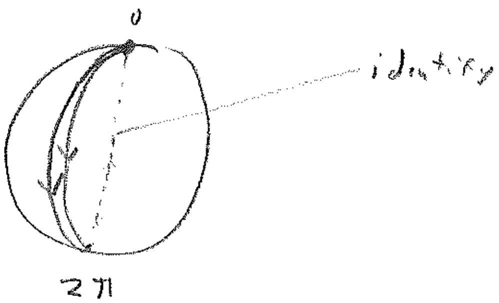
Problem 6.6 Rotation by 4π is continuously deformable to identity

a)



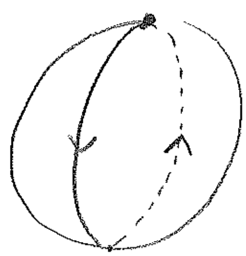
. identify 2π, 0 rotations
so closed curve

b)



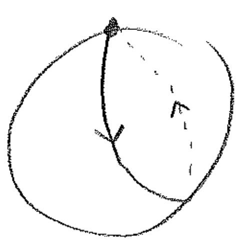
$0 - 4\pi$

$\equiv$



identifying the rotations on the 2^{nd} pass with antipodal points on S^3 for 1^{st} pass

$\downarrow$ deform



$\downarrow$ deform