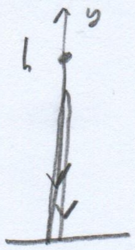
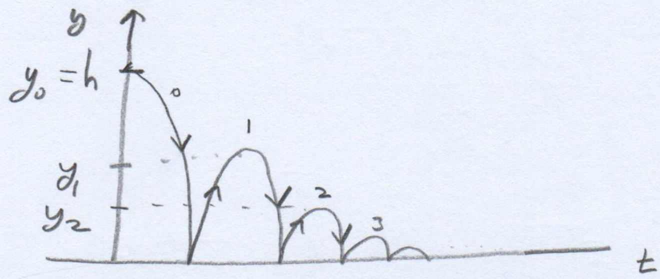
5.1

Exercise — Inelastic collision of a ball with floor





$$\eta^2 = \frac{K_{final}}{K_{initial}}$$

$$K_{initial} = \frac{1}{2} m v_0^2 = mgh$$

$$v_0^2 = 2gh$$

$$v_0 = \sqrt{2gh}$$

$$K_{final} = \eta^2 K_{initial}$$

$$= \eta^2 mgh$$

$$= \eta^2 mgy_0$$

$$= mgy_1$$

Thus,

$$y_1 = \eta^2 y_0 = \eta^2 h$$

$$y_2 = \eta^2 y_1 = \eta^4 h$$

$$y_3 = \eta^6 h$$

$$\ldots \quad y_N = \eta^N h$$

$$\text{Time} = T_0 + T_1 + T_2 + \ldots$$

$\underline{0}$: $h = \frac{1}{2} g T_0^2 \rightarrow T_0 = \sqrt{\frac{2h}{g}}$

$\underline{1}$: $\eta^2 h = \frac{1}{2} g \Delta t^2 \rightarrow T_1 = 2 \cdot \Delta t = 2\eta\sqrt{\frac{2h}{g}}$

Thus,

$$T = \sqrt{\frac{2h}{g}}\left(1 + 2\eta + 2\eta^2 + \cdots\right)$$

$$= \sqrt{\frac{2h}{g}}\left[\left(1+\eta+\eta^2+\eta^3+\cdots\right) + \left(\eta+\eta^2+\eta^3+\cdots\right)\right]$$

$$= \sqrt{\frac{2h}{g}}\left[\left(\frac{1}{1-\eta}\right) + \left(\frac{\eta}{1-\eta}\right)\right]$$

$$= \sqrt{\frac{2h}{g}}\left(\frac{1+\eta}{1-\eta}\right)$$

$$1+\eta+\eta^2+\cdots = \left(\frac{1}{1-\eta}\right)$$

$$\eta+\eta^2+\cdots = \left(\frac{1}{1-\eta}\right) - 1 = \frac{1-(1-\eta)}{1-\eta} = \frac{\eta}{1-\eta}$$

Limiting cases:

$\eta = 0$: $T = \sqrt{\frac{2h}{g}}$ (time to fall from $y=h$ to $y=0$)

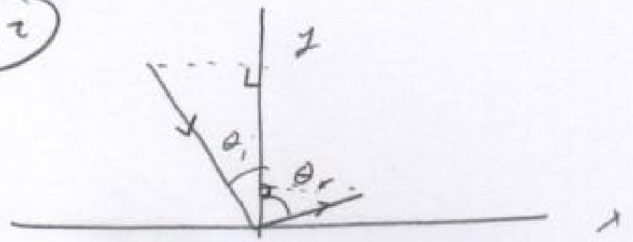
$\eta = 1$: $T = \infty$ (ball bounces forever)

$$2\left(\frac{1}{1-\eta}\right) - 1 = \frac{1}{1-\eta}\left(2-(1-\eta)\right) = \frac{1+\eta}{1-\eta}$$

~~Problem~~

Exercise 5.2

Angle of incidence vs. angle of reflection for inelastic collisions



$$p_{x,\text{final}} = p_{x,\text{init}}$$
$$p_{y,\text{final}} = \eta\, p_{y,\text{init}}$$

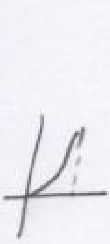
Thus, $v_{x,\text{final}} = v_{x,\text{init}} \equiv v_x$
$v_{y,\text{final}} = \eta\, v_{y,\text{init}}$

$$v_f^2 = v_{xf}^2 + v_{yf}^2 = v_x^2 + v_{yf}^2 = v_x^2 + \eta^2 v_{yi}^2$$
$$v_i^2 = v_{xi}^2 + v_{yi}^2 = v_x^2 + v_{yi}^2$$

$$\tan\theta_i = \frac{v_{xi}}{v_{yi}} = \frac{v_x}{v_{yi}}$$

$$\tan\theta_r = \frac{v_{xf}}{v_{yf}} = \frac{v_x}{v_{yf}} = \frac{v_x}{\eta v_{yi}} = \frac{1}{\eta}\tan\theta_i$$

Thus, $\boxed{\tan\theta_r = \frac{1}{\eta}\tan\theta_i}$

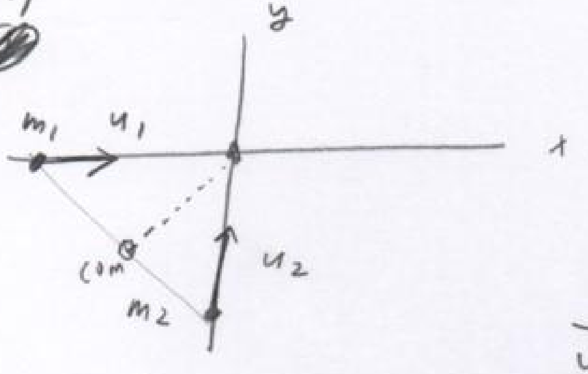


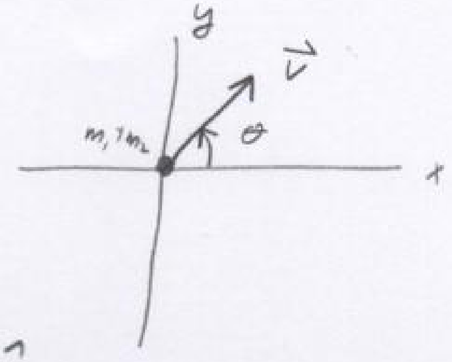
$\underline{\eta = 1:}$ $\tan\theta_r = \tan\theta_i \longrightarrow \theta_r = \theta_i$

$\underline{\eta = 0:}$ $\tan\theta_r = \infty \tan\theta_i \rightarrow \theta_r = \pi/2$

Exercise 5.3 Perfectly inelastic collision in different ref frames

(a)





(before)

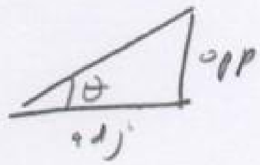
$$\vec{u}_1 = u_1\hat{x}$$
$$\vec{u}_2 = u_2\hat{y}$$

$\vec{p}_i = \vec{p}_f$ conservation of momentum

$$m_1 u_1 \hat{x} + m_2 u_2 \hat{y} = (m_1+m_2)\vec{v} \equiv M\vec{v}$$

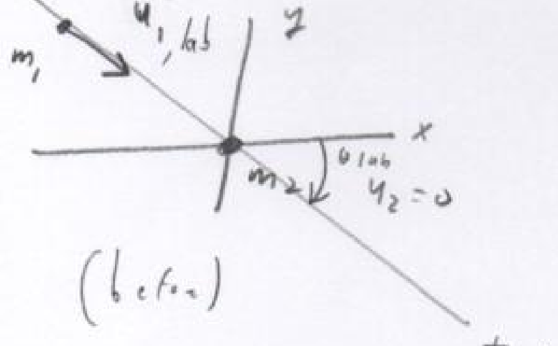
Thus, $\vec{v} = \left(\frac{m_1}{M}u_1\right)\hat{x} + \left(\frac{m_2}{M}u_2\right)\hat{y}$

<u>speed</u>: $v = \dfrac{\sqrt{m_1^2u_1^2 + m_2^2u_2^2}}{M}$



$$\tan\theta = \frac{\left(\frac{m_2u_2}{\cancel{M}}\right)}{\left(\frac{m_1u_1}{\cancel{M}}\right)} = \frac{m_2u_2}{m_1u_1}$$

(b) Lab frame (m_2 at rest ; subtract $u_2\hat{y}$ from all velocities

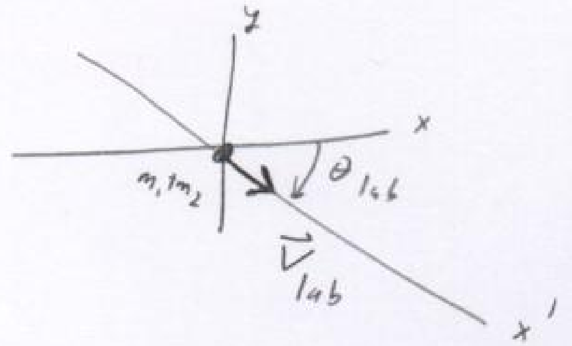


(before)

$$\vec{u}_{1,lab} = u_1\hat{x} - u_2\hat{y}, \quad \vec{u}_{2,lab} = \vec{0}$$
$$\vec{p}_{lab,i} = m_1\vec{u}_{1,lab}$$
$$\vec{p}_{lab,f} = (m_1+m_2)\vec{v}_{lab}$$

Thus, $\vec{v}_{lab} = \left(\frac{m_1}{M}\right)\vec{u}_{1,lab} = \left(\frac{m_1}{M}\right)\left[u_1\hat{x} - u_2\hat{y}\right]$

After:



speed: $V_{lab} = \frac{m_1}{M}\sqrt{u_1^2+u_2^2}$

direction: $\tan\theta_{lab} = -\frac{u_2}{u_1} \quad (<0)$

(c) Barycenter frame (COM at origin)

→ combined mass at rest at origin after collision

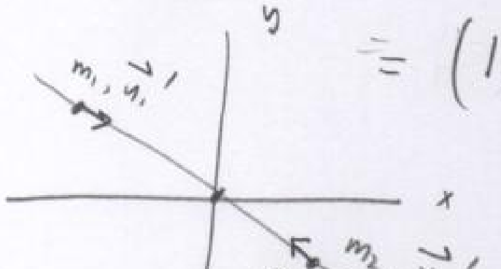
Thus, ~~add~~ subtract $\vec{V}_{lab}$ from $\vec{u}_{1,lab}$, $\vec{u}_{2,lab}$ to go to barycenter frame

$$\vec{u}_1{}' = \vec{u}_{1,lab} - \vec{V}_{lab}$$

$$= (u_1\hat{x} - u_2\hat{y}) - \left(\frac{m_1}{M}\right)[u_1\hat{x} - u_2\hat{y}]$$

$$= \left(1-\frac{m_1}{M}\right)(u_1\hat{x} - u_2\hat{y})$$

$$= \frac{m_2}{M}(u_1\hat{x} - u_2\hat{y})$$

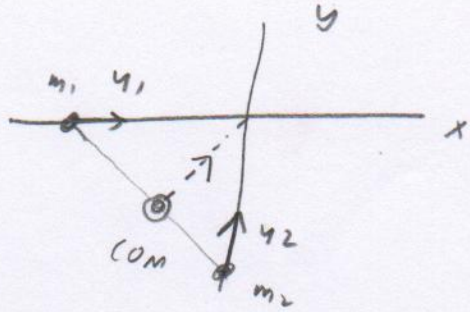


$$\vec{u}_2{}' = \underset{\overset{\shortparallel}{0}}{\vec{u}_{2,lab}} - \vec{V}_{lab} = -\frac{m_1}{M}(u_1\hat{x} - u_2\hat{y})$$

Original frame:

$$\vec{u_1} = u_1 \hat{x}$$

$$\vec{u_2} = u_2 \hat{y}$$



Calculate $\vec{u_1} - \vec{V}$, $\vec{u_2} - \vec{V}$:

$$\vec{u_1} - \vec{V} = u_1 \hat{x} - \left(\left(\frac{m_1 u_1}{M} \right) \hat{x} + \left(\frac{m_2 u_2}{M} \right) \hat{y} \right)$$

$$= \left(1 - \frac{m_1}{M}\right) u_1 \hat{x} - \frac{m_2 u_2}{M} \hat{y}$$

$$= \frac{m_2}{M} \left(u_1 \hat{x} - u_2 \hat{y} \right)$$

$$= \vec{u_1}'$$

$$\vec{u_2} - \vec{V} = u_2 \hat{y} - \left(\left(\frac{m_1 u_1}{M} \right) \hat{x} + \left(\frac{m_2 u_2}{M} \right) \hat{y} \right)$$

$$= -\frac{m_1 u_1}{M} \hat{x} + \underbrace{\left(1 - \frac{m_2}{M}\right)}_{\frac{m_1}{M}} u_2 \hat{y}$$

$$= -\frac{m_1}{M} \left(u_1 \hat{x} - u_2 \hat{y} \right)$$

$$= \vec{u_2}'$$

Thus, $\boxed{\vec{u_1} - \vec{V} = \vec{u_1}' , \quad \vec{u_2} - \vec{V} = \vec{u_2}'}$

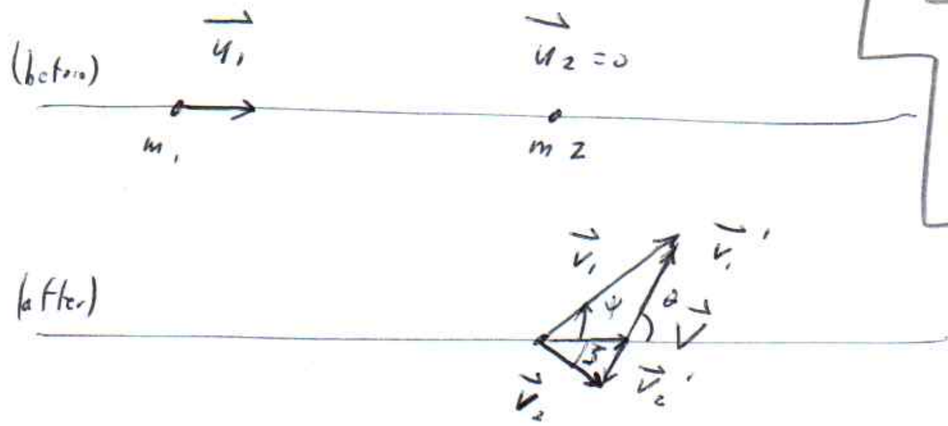
(So barycentre frame is moving with velocity $\vec{V}$ w/r/t original frame)

Relating lab and barycenter frames:

EXAMPLE 5.1

Exercise 5.4

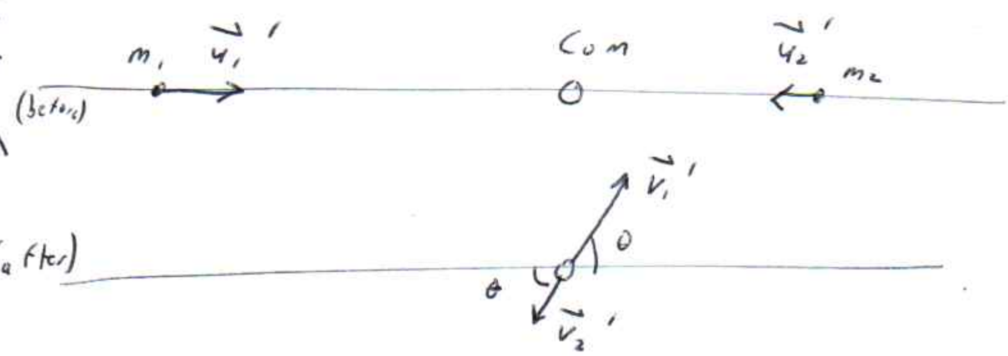
Lab



$$\vec{v}_1 = \vec{V} + \vec{v}_1{}'$$
$$\vec{v}_2 = \vec{V} + \vec{v}_2{}'$$

primes denote wrt barycenter frame

Barycenter



Barycenter Frame:

CoM $= 0$ $\rightarrow$ $m_1 \vec{u}_1{}' + m_2 \vec{u}_2{}' = \vec{0}$

$m_1 u_1' = m_2 u_2'$ where $u_1' = |\vec{u}_1{}'|$, $u_2' = |\vec{u}_2{}'|$

Also $m_1 v_1' = m_2 v_2'$

Elastic:
$$\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

$$\cancel{\tfrac{1}{2}} m_1 (u_1'^2 - v_1'^2) = \cancel{\tfrac{1}{2}} m_2 (v_2'^2 - u_2'^2)$$

$$m_1 (u_1' - v_1')(u_1' + v_1') = m_2 (v_2' - u_2')(v_2' + u_2')$$

$$(u_1' - v_1')(m_1 u_1' + m_1 v_1') = (v_2' - u_2')(m_2 v_2' + m_2 u_2')$$

$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \| \qquad \quad \|$$
$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad m_1 v_1' + m_1 u_1'$$

equal

so $u_1' - v_1' = v_2' - u_2'$

$$u_1' - v_1' = v_2' - u_2'$$

$$u_1' - v_1' = \frac{m_1}{m_2} v_1' - \frac{m_1}{m_2} u_1'$$

$$\left(1 + \frac{m_1}{m_2}\right) u_1' = \left(1 + \frac{m_1}{m_2}\right) v_1'$$

$$\rightarrow \boxed{u_1' = v_1'}$$

Similarly: $\boxed{u_2' = v_2'}$

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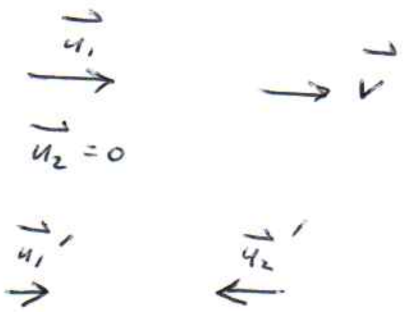
com velocity $\vec{V}$:

$$(m_1 + m_2)\vec{V} = m_1 \vec{u}_1 + m_2 \cancelto{0}{\vec{u}_2}$$

$$\vec{V} = \frac{m_1}{M}\vec{u}_1 \quad , \text{ where } M = m_1 + m_2$$

Barycentric quantities

$$\vec{u}_1' = \vec{u}_1 - \vec{V}$$
$$\vec{u}_2' = \vec{u}_2 - \vec{V} = -\vec{V}$$
$$\vec{v}_1' = \vec{v}_1 - \vec{V}$$
$$\vec{v}_2 = \vec{v}_2 - \vec{V}$$



In terms of magnitudes:

$$u_1' = u_1 - V = u_1 - \frac{m_1}{M}u_1 = u_1\left(1 - \frac{m_1}{M}\right) = u_1\frac{m_2}{M} = (v_1')$$

$$u_2' = +V = \frac{+m_1 u_1}{M} \quad (= v_2')$$

Components of $\vec{v}_1$, $\vec{v}_2$ in lab frame:

$$v_{1x} = v_1 \cos\psi = v_{1x}' + V = v_1'\cos\theta + \frac{m_1 u_1}{M}$$

$$= u_1\frac{m_2\cos\theta}{M} + \frac{m_1 u_1}{M}$$

$$= \frac{u_1}{M}(m_2\cos\theta + m_1)$$

③

$$v_{1y} = v_1 \sin\psi = v_{1y}' = v_1' \sin\theta = \frac{u_1 m_2 \sin\theta}{M}$$

Thus, $$V_1 = \sqrt{v_{1x}^2 + v_{1y}^2}$$

$$= \frac{u_1}{M}\sqrt{(m_2\cos\theta + m_1)^2 + \cancel{m_2}(m_2\sin\theta)^2}$$

$$= \frac{u_1}{M}\sqrt{m_1^2 + 2m_1 m_2\cos\theta + \underbrace{m_2^2\cos^2\theta + m_2^2\sin^2\theta}_{= m_2^2}}$$

$$= \frac{u_1}{M}\sqrt{m_1^2 + m_2^2 + 2m_1 m_2\cos\theta}$$

so $$\cos\psi = \cancel{\frac{v_{1x}}{v_1}} \quad \frac{v_{1x}}{v_1}$$

$$= \frac{\cancel{\frac{u_1}{M}}(m_2\cos\theta + m_1)}{\cancel{\frac{u_1}{M}}\sqrt{m_1^2 + m_2^2 + 2m_1 m_2\cos\theta}}$$

$$= \frac{m_2\cos\theta + m_1}{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2\cos\theta}}$$

Repeat for target particle:

$\cos\theta\cos\pi - \sin\theta \cancel{\sin\pi}$

$$v_{2x} = v_2\cos\zeta = v_{2x}' + V$$

$$= v_2'\cos(\theta + \pi) + \frac{m_1 u_1}{M}$$

$$= -v_2'\cos\theta + \frac{m_1 u_1}{M}$$

$$= -\frac{m_1 u_1}{M}\cos\theta + \frac{m_1 u_1}{M}$$

$$= \frac{m_1 u_1}{M}(1 - \cos\theta)$$

$$V_{2y} = V_2 \sin\zeta = V_{2y}' = V_2' \sin(\theta+\pi) = -u_1 \frac{m_1}{M}\sin\theta$$

$\sin(\theta+\pi)$
$= \sin\theta\cos\pi + \cos\theta\sin\pi$
$= -\sin\theta$

Thus,

$$V_2 = \sqrt{V_{2x}^2 + V_{2y}^2}$$
$$= \frac{u_1}{M} m_1 \sqrt{(1-\cos\theta)^2 + \sin^2\theta}$$
$$= \frac{u_1 m_1}{M}\sqrt{1+\cos^2\theta+\sin^2\theta - 2\cos\theta}$$
$$= \frac{u_1 m_1}{M}\sqrt{2(1-\cos\theta)}$$

$\rightarrow$

$$\cos\zeta = \frac{V_{2x}}{V_2} = \frac{\frac{m_1 \cancel{u_1}}{M}(1-\cos\theta)}{\frac{\cancel{u_1 m_1}}{M}\sqrt{2(1-\cos\theta)}} = \sqrt{\frac{1-\cos\theta}{2}}$$

$$p_1 = \frac{\frac{m_1 u_1}{M}}{u_1\frac{m_2}{M}} = \frac{m_1}{m_2}$$

$$p_2 = \frac{\frac{m_1 u_1}{M}}{\frac{m_1 u_1}{M}}$$

$\cos(2\pi-\zeta)$
$= \cos 2\pi\cos\zeta + \sin 2\pi \sin\zeta$
$= \cos\zeta$

Thus, for elastic scattering

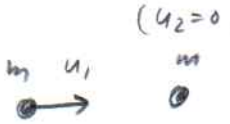
$$\cos\psi = \frac{m_2\cos\theta + m_1}{\sqrt{m_1^2+m_2^2+2m_1m_2\cos\theta}}$$

$$\cos\zeta = \sqrt{\frac{1-\cos\theta}{2}} = \sin\left(\frac{\theta}{2}\right)$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$
$\sin^2\theta = \frac{1-\cos 2\theta}{2}$
$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}$

$\cos\zeta = \sin\left(\frac{\theta}{2}\right)$
$\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \cos\frac{\pi}{2}\cos\frac{\theta}{2} + \sin\frac{\pi}{2}\sin\frac{\theta}{2}$ ✓

Exercise 5.5 Show that for equal-mass elastic scattering $|\psi|+|\zeta| = \pi/2$



(before)

(after)

Elastic scattering:

$$\cos\psi = \frac{m_2 \cos\theta + m_1}{\sqrt{m_1^2 + m_2^2 + 2 m_1 m_2 \cos\theta}}$$

θ: angle in barycenter frame

$$\cos\zeta = \sin\left(\frac{\theta}{2}\right)$$

$m_1 = m_2 \equiv m$ (Equal mass)

$$\cos\psi = \frac{\cos\theta + 1}{\sqrt{2+2\cos\theta}} = \sqrt{\frac{\cos\theta+1}{2}} = \cos\left(\frac{\theta}{2}\right)$$

Thus, $\boxed{\psi = \frac{\theta}{2}}$

$$\sin x = \cos(x - \pi/2) = \cos x \underbrace{\cos \pi/2}_{=0} + \sin x \underbrace{\sin \pi/2}_{=1}$$

So $\cos\zeta = \sin\left(\frac{\theta}{2}\right) = \cos\left(\frac{\theta}{2} - \frac{\pi}{2}\right)$

$\rightarrow \boxed{\zeta = \frac{\theta}{2} - \frac{\pi}{2}} < 0$

Hence, $\boxed{\psi - \zeta = \frac{\theta}{2} - \left(\frac{\theta}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2}}$

$$|\zeta| = \left|\frac{\theta}{2} - \frac{\pi}{2}\right| = \frac{\pi}{2} - \frac{\theta}{2}$$

$$|\psi| + |\zeta| = \cancel{\frac{\theta}{2}} + \frac{\pi}{2} - \cancel{\frac{\theta}{2}} = \frac{\pi}{2}$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

Newtonian Scattering : $\boxed{\text{Section 5.5}}$

$$\phi_{min} = \int_0^{u_{min}=1} \frac{du}{\sqrt{\frac{2\mu R^2}{l^2}\left(E - \frac{l^2}{2\mu r^2} + \frac{GM\mu}{r}\right)}}$$

$$= \int_0^1 \frac{du}{\sqrt{\frac{2\mu R^2}{l^2}E - u^2 + \frac{2GM\mu^2 R}{l^2}u}}$$

$$= \int_0^1 \frac{du}{\sqrt{\frac{2\mu R^2}{l^2}E + 2\frac{GM\mu^2 R}{l^2}u - u^2}}$$

$$= \int_0^1 \frac{du}{\sqrt{\underbrace{a + bu + cu^2}_{R}}}$$

$$a = \frac{2\mu R^2 E}{l^2}, \quad b = \frac{2GM\mu^2 R}{l^2}, \quad c = -1$$

$$b = \frac{2R}{\alpha}$$

where $\alpha \equiv \frac{l^2}{GM\mu^2}$ (latus rectum for ellipse)

$$\int_0^1 \frac{du}{\sqrt{R}} = -\frac{1}{\sqrt{-c}} \arcsin\left(\frac{2cu+b}{\sqrt{-\Delta}}\right)\Big|_0^1$$

$$= -\frac{1}{\sqrt{-c}}\left[\arcsin\left(\frac{2c+b}{\sqrt{-\Delta}}\right) - \arcsin\left(\frac{b}{\sqrt{-\Delta}}\right)\right]$$

$$-\Delta = b^2 - 4ac = \frac{4R^2}{\alpha^2} + 4\frac{2\mu R^2 E}{l^2}$$

$$= \frac{4R^2}{\alpha^2}\left(1 + \frac{2\mu E\alpha^2}{l^2}\right) = \frac{4R^2}{\alpha^2}\left(1 + \frac{2\mu E\alpha^2}{GM\mu^2\alpha}\right)$$

$$= \frac{4R^2}{\alpha^2}\left(1 + \frac{2E\alpha}{GM\mu}\right) = \boxed{\frac{4R^2 e^2}{\alpha^2}}$$

where $e^2 = 1 + \frac{2 E \alpha}{G M \mu} \geq 1$ (for scattering orbits)

Thus, $\sqrt{-\Delta} = \frac{2Re}{\alpha}$

$$\rightarrow \phi_{min} = -\left[\arcsin\left(\frac{-2 + \frac{2R}{\alpha}}{\left(\frac{2Re}{\alpha}\right)} \right) - \arcsin\left(\frac{\frac{2R}{\alpha}}{\frac{2Re}{\alpha}} \right) \right]$$

$$= -\left[\arcsin\left(\frac{1 - \frac{\alpha}{R}}{e} \right) - \arcsin\left(\frac{1}{e} \right) \right]$$

$$= -\left[\arcsin\left(\frac{1 - \frac{\alpha}{R}}{e} \right) - \arcsin\left(\frac{1}{e} \right) \right]$$

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where R = closest approach for r.

$$= \underbrace{b}_{\text{impact parameter}} \sqrt{\frac{e-1}{e+1}}$$

$$e^2 = 1 + \left(\frac{b v_\infty^2}{GM} \right)^2 \quad \text{(see below)}$$

$$\rightarrow \frac{b v_\infty^2}{GM} = \sqrt{e^2 - 1}$$

$$E = \frac{1}{2} \mu v_\infty^2 \leftarrow \text{Newtonian}$$

$$l = \mu b v_\infty$$

$$\rightarrow e^2 = 1 + \frac{\cancel{2} \frac{1}{\cancel{2}} \cancel{\mu} v_\infty^2 \alpha}{G M \cancel{\mu}}$$

$$= 1 + \frac{v_\infty^2 \alpha}{GM}$$

$$= 1 + \frac{v_\infty^2}{GM} \frac{l^2}{G M \mu^2}$$

$$= 1 + \frac{v_\infty^2}{(GM)^2} \frac{\cancel{\mu^2} b^2 v_\infty^2}{\cancel{\mu^2}} = 1 + \left(\frac{b v_\infty^2}{GM} \right)^2$$

$$\alpha = \frac{l^2}{G M \mu^2} = \frac{\cancel{\mu^2} b^2 v_\infty^2}{G M \cancel{\mu^2}}$$

$$\boxed{\alpha = \frac{b^2 v_\infty^2}{GM}} \; *$$

$$1 - \frac{\alpha}{R} = 1 - \frac{\alpha}{b\sqrt{\frac{e-1}{e+1}}} = 1 - \left(\frac{\alpha}{b}\right)\sqrt{\frac{e+1}{e-1}}$$

$$= 1 - \frac{bv_\infty^2}{GM}\sqrt{\frac{e+1}{e-1}}$$

$$= 1 - \sqrt{e^2-1}\sqrt{\frac{e+1}{e-1}}$$

$$= 1 - \sqrt{\cancel{(e-1)}(e+1)}\sqrt{\frac{e+1}{\cancel{e-1}}}$$

$$= 1 - (e+1)$$

$$= -e.$$

$$\rightarrow \boxed{\phi_{min} = -\left[\arcsin(-1) - \arcsin\left(\frac{1}{e}\right)\right] = -\left[-\frac{\pi}{2} - \arcsin\left(\frac{1}{e}\right)\right] = \frac{\pi}{2} + \arcsin\left(\frac{1}{e}\right)}$$

$$2\phi_{min} = \pi + 2\arcsin\left(\frac{1}{e}\right) \qquad \gamma = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$$

$$\theta = 2\phi_{min} - \pi$$

$$= 2\arcsin\left(\frac{1}{e}\right) \qquad \text{(Deflection angle)}$$

$$\frac{\theta}{2} = \arcsin\left(\frac{1}{e}\right) \qquad \left(\frac{2}{\theta} = \frac{bv_\infty^2}{GM} \rightarrow \theta = \frac{2GM}{bv_\infty^2}\right)$$

$$\frac{1}{e} = \sin\left(\frac{\theta}{2}\right)$$

Now: $\cot\left(\frac{\theta}{2}\right) = \sqrt{\csc^2\left(\frac{\theta}{2}\right) - 1} = \sqrt{\frac{1}{\sin^2\left(\frac{\theta}{2}\right)} - 1} = \sqrt{e^2-1} = \boxed{\frac{bv_\infty^2}{GM}}$

Exercise 5.6 Gravitational differential cross section

a) $$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$\cot\left(\frac{\theta}{2}\right) = \frac{b v_\infty^2}{GM}$$

$$\rightarrow \quad b = \frac{GM}{v_\infty^2}\cot\left(\frac{\theta}{2}\right) \qquad \cot = \frac{\cos}{\sin}$$

$$\frac{db}{d\theta} = \frac{GM}{v_\infty^2}\,\frac{-\sin^2\left(\frac{\theta}{2}\right)\frac{1}{2} - \cos^2\left(\frac{\theta}{2}\right)\frac{1}{2}}{\sin^2\left(\frac{\theta}{2}\right)}$$

$$= -\frac{GM}{2v_\infty^2}\,\frac{1}{\sin^2\left(\frac{\theta}{2}\right)}$$

Thus, $$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$= \frac{\frac{GM}{v_\infty^2}\cot\left(\frac{\theta}{2}\right)}{\sin\theta}\;\frac{GM}{2v_\infty^2}\,\frac{1}{\sin^2\left(\frac{\theta}{2}\right)}$$

$$= \frac{G^2M^2}{v_\infty^4}\,\frac{1}{2}\,\frac{\cancel{\cos\left(\frac{\theta}{2}\right)}}{\sin\left(\frac{\theta}{2}\right)}\,\frac{1}{2\sin\left(\frac{\theta}{2}\right)\cancel{\cos\left(\frac{\theta}{2}\right)}}\,\frac{1}{\sin^2\left(\frac{\theta}{2}\right)}$$

$$= \boxed{\frac{1}{4}\,\frac{G^2M^2}{\left[v_\infty \sin\left(\theta/2\right)\right]^4}}$$

NOTE: $\boxed{v_\infty^2 = \frac{2E}{m}}$ $\rightarrow$ $$= \frac{G^2M^2m^2\csc^4\left(\frac{\theta}{2}\right)}{16E^2} = \boxed{\left(\frac{GMm}{4E}\right)^2\csc^4\left(\frac{\theta}{2}\right)}$$

b)
$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$= \frac{1}{2}\left|\frac{db^2}{d(\cos\theta)}\right|$$

c) $\cancel{\tan\left(\frac{x}{2}\right) = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}}$

$$\frac{\sin x}{1+\cos x} = \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{1+\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}$$

$$= \frac{\cancel{2}\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cancel{2}\cos^2\left(\frac{x}{2}\right)}$$

$$= \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}$$

$$= \tan\left(\frac{x}{2}\right)$$

Thus,
$$\cot\left(\frac{\theta}{2}\right) = \frac{b v_\infty^2}{GM} = \frac{1}{\tan\left(\frac{\theta}{2}\right)}$$

$$\frac{b v_\infty^2}{GM} = \frac{1+\cos\theta}{\sin\theta}$$

$$\frac{b\, v_\infty^2}{G m} = \frac{1+\cos\theta}{\sin\theta}$$

$$= \frac{1+\cos\theta}{\sqrt{1-\cos^2\theta}}$$

$$= \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

$$\rightarrow \boxed{b^2 = \frac{G^2 m^2}{v_\infty^4}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)}$$

$$\frac{1+x}{1-x}$$

Thus,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left|\frac{db^2}{d(\cos\theta)}\right|$$

$$= \frac{1}{2}\frac{G^2 m^2}{v_\infty^4}\frac{(1-\cancel{\cos\theta}) - (-1)(1+\cancel{\cos\theta})}{(1-\cos\theta)^2}$$

$$= \frac{1}{2}\frac{G^2 m^2}{v_\infty^4}\frac{2}{(1-\cos\theta)^2}$$

$$= \frac{1}{\cancel{2}}\frac{G^2 m^2}{v_\infty^4}\frac{\cancel{2}}{\left(2\sin^2\left(\frac{\theta}{2}\right)\right)^2}$$

$$= \boxed{\frac{1}{4}\frac{G^2 m^2}{v_\infty^4}\frac{1}{\sin^4\left(\frac{\theta}{2}\right)}}$$

$$\cos\theta = \cos\left(2\cdot\frac{\theta}{2}\right)$$

$$= \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

$$2\sin^2\left(\frac{\theta}{2}\right) = 1 - \cos\theta$$

Exercise: 5.7

Newtonian Calculation for gravitational scattering:

$$\cot\left(\frac{\theta}{2}\right) = \frac{b v_\infty^2}{GM} \qquad (5.32)$$

Suppose $\theta << 1$ (small deflection)

Then $\cot\left(\frac{\theta}{2}\right) = \dfrac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \simeq \dfrac{1}{\left(\frac{\theta}{2}\right)} = \dfrac{2}{\theta}$

$$\rightarrow \quad \frac{2}{\theta} \simeq \frac{b v_\infty^2}{GM}$$

$$\theta \simeq \frac{2GM}{b v_\infty^2}$$

If we treat light as a Newtonian particle moving at $v_\infty = c$ then

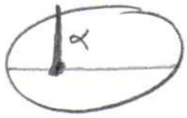
$$\boxed{\theta \simeq \frac{2GM}{bc^2}} \quad \text{(Newtonian)}$$

Compare to GR calculation: $\boxed{\theta \simeq \dfrac{4GM}{bc^2}}$ (GR)

so off by a factor of 2.

problem (5.8) Jupiter injection orbit

$$e^2 = 1 + \frac{2E\alpha}{GMm} \quad , \quad \alpha = a(1-e^2)$$



1) Circular orbit at r_{ES} around Sun:

$$e = 0, \quad \alpha = a_i = r_{ES} = 1\,AU$$

$$0 = 1 + \frac{2E_i a_i}{GMm}$$

$$\boxed{E_i = -\frac{GMm}{2a_i}} = \boxed{-\frac{GMm}{a_i} + \frac{1}{2}mv_i^2}$$

2) In Jupiter injection orbit: $a_f = 3.15\,AU$
$e = 0.683$

$$e^2 - 1 = \frac{2E_f a_f(1-e^2)}{GMm}$$

$$\boxed{-\frac{GMm}{2a_f} = E_f} = \boxed{-\frac{GMm}{a_i} + \frac{1}{2}mv_f^2}$$

$$\Delta E = E_f - E_i = -\frac{GMm}{2}\left(\frac{1}{a_f} - \frac{1}{a_i}\right)$$

$$\Delta E = \frac{GMm}{2}\left(\frac{a_f - a_i}{a_f a_i}\right) = \frac{GMm}{2}\frac{\Delta a}{a_f a_i}$$

But $\Delta E = \frac{1}{2}m\Delta v^2 \rightarrow \frac{1}{2}m\Delta v^2 = \frac{GM}{2}\left(\frac{\Delta a}{a_f a_i}\right)$

Solve for v_i, v_T :

$$-\frac{GMm}{2a_i} = -\frac{GMm}{a_i} + \frac{1}{2}mv_i^2$$

$$+\frac{GM\cancel{m}}{\cancel{2}a_i} = \frac{1}{\cancel{2}}\cancel{m}v_i^2$$

$$v_i = \sqrt{\frac{GM}{a_i}} = \boxed{3\times10^4\ m/s}$$

$$-\frac{GM\cancel{m}}{2a_f} = -\frac{GM\cancel{m}}{a_i} + \frac{1}{2}\cancel{m}v_f^2$$

$$GM\left(\frac{1}{a_i} - \frac{1}{2a_f}\right) = \frac{1}{2}v_f^2$$

$$\rightarrow \quad v_f = \sqrt{\cancel{2}GM\,\frac{2a_f - a_i}{\cancel{2}a_f a_i}}$$

$$= \sqrt{GM\left(\frac{2a_f - a_i}{a_f a_i}\right)}$$

$$= \sqrt{\frac{GM}{1AU}\left(\frac{6.3-1}{3.15}\right)}$$

$$= 3\times10^4\ m/s \cdot \sqrt{\frac{5.3}{3.15}}$$

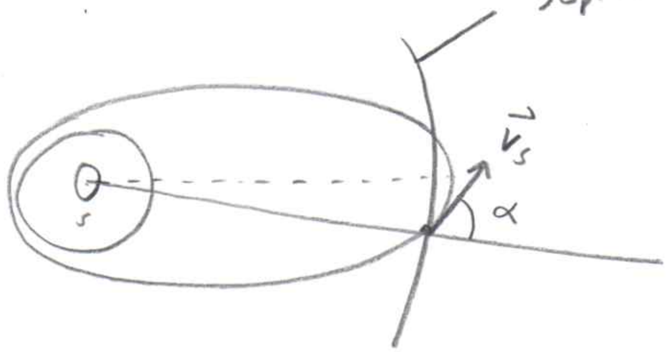
$$= \boxed{3\times10^4\ \frac{m}{s}\ (1.3)} \qquad (30\%\ \text{larger})$$

$$\Delta V = V_f - V_i = 3\times10^4\ \frac{m}{s}\ (1.3-1) = \boxed{9\ \frac{km}{s}}$$

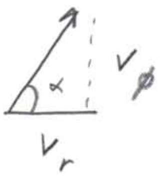
```
function deltaV = calBoost(a_i, a_f)
%
% calculate additional velocity needed to boost from
% elliptical orbit with semi-major axis a_i to an
% elliptical orbit with semi-major axis a_f, assuming
% both orbits have sun at one focal point
%
% input: a_i, a_f in units of AU
%
% example: calBoost(1, 3.15)
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% constants (MKS)
G = 6.67e-11;
M_sun = 2e30;
AU = 1.5e11;

% convert to MKS units
a_i = a_i*AU;
a_f = a_f*AU;

% calculate initial and final velocities and deltaV
v_i = sqrt(G*M_sun/a_i);
v_f = sqrt(2*G*M_sun*(1/a_i-1/(2*a_f)));
deltaV = v_f-v_i;

% display results
fprintf('v_i = %f km/s\n', v_i/1000);
fprintf('v_f = %f km/s\n', v_f/1000);
fprintf('deltaV = %f km/s\n', deltaV/1000);

return
```

Problem: Calculate $\tan\alpha$ for Jupiter gravitational slingshot manuever

5.9



$$\tan\alpha = \frac{v_\phi}{v_r} = \frac{r\,d\phi/dt}{dr/dt} = \frac{r\,d\phi}{dr}$$



Orbit equation: (ellipse)

$$r = \frac{a(1-e^2)}{1+e\cos\phi}$$

$$\rightarrow \quad 1+e\cos\phi = \frac{a}{r}(1-e^2)$$

$$\cos\phi = \frac{\frac{a}{r}(1-e^2) - 1}{e}$$

$$\frac{dr}{d\phi} = \frac{-a(1-e^2)}{(1+e\cos\phi)^2}(-e\sin\phi)$$

$$= \frac{+ae\sin\phi(1-e^2)}{(1+e\cos\phi)^2}$$

$$= \frac{+e\sin\phi}{a(1-e^2)}\frac{a^2(1-e^2)^2}{(1+e\cos\phi)^2}$$

$$= \frac{+e\sin\phi}{a(1-e^2)}r^2$$

$$\sin\phi = \sqrt{1-\cos^2\phi}$$

$$= \sqrt{1 - \frac{\left[\left(\frac{a}{r}\right)(1-e^2) - 1\right]^2}{e^2}}$$

$$= \frac{1}{e}\sqrt{e^2 - \left(\frac{a}{r}\right)^2(1-e^2)^2 - 1 + 2\left(\frac{a}{r}\right)(1-e^2)}$$

$$= \frac{1}{e}\sqrt{-(1-e^2) - \left(\frac{a}{r}\right)^2(1-e^2)^2 + 2\left(\frac{a}{r}\right)(1-e^2)}$$

$$= \frac{1}{e}\sqrt{1-e^2}\,\sqrt{-1 - \left(\frac{a}{r}\right)^2(1-e^2) + 2\left(\frac{a}{r}\right)}$$

Thus

$$\tan\alpha = r\,\frac{d\phi}{dr}$$

$$= r\,\frac{a(1-e^2)}{e\sin\phi\, r^2}$$

$$= +\frac{1}{r}\,a(1-e^2)\,\frac{1}{\sqrt{1-e^2}\,\sqrt{-1-\left(\frac{a}{r}\right)^2(1-e^2)+2\left(\frac{a}{r}\right)}}$$

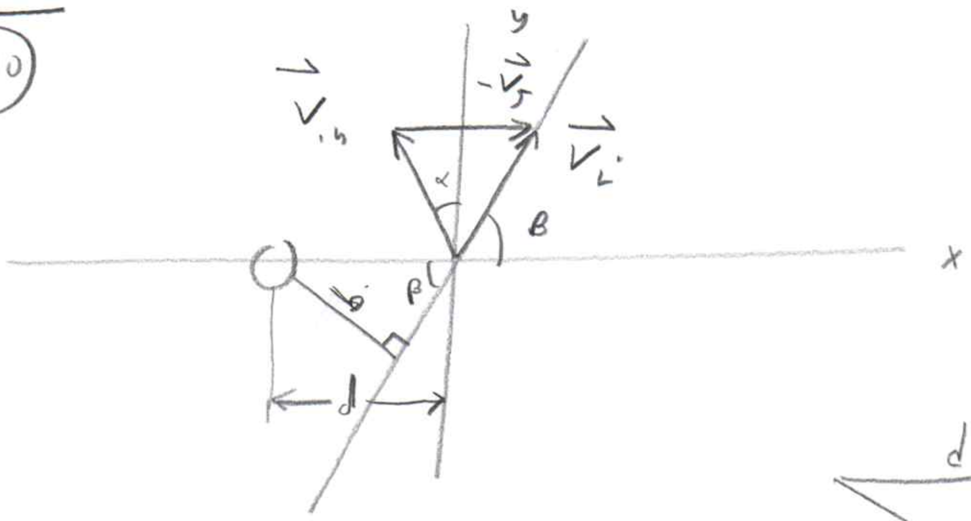
$$= +\frac{a}{r}\sqrt{1-e^2}\,\frac{1}{\sqrt{-1-\left(\frac{a}{r}\right)^2(1-e^2)+2\left(\frac{a}{r}\right)}}$$

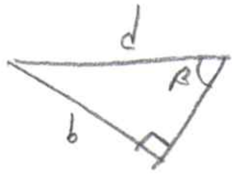
$$= +\cancel{\left(\frac{a}{r}\right)}\sqrt{1-e^2}\,\frac{1}{\cancel{\left(\frac{a}{r}\right)}\sqrt{-(1-e^2)-\left(\frac{r}{a}\right)^2+2\frac{r}{a}}}$$

$$= \frac{\sqrt{1-e^2}}{\sqrt{-\left(\frac{r}{a}\right)^2 + 2\frac{r}{a} - 1 + e^2}}$$

$$= \frac{\sqrt{1-e^2}}{\sqrt{e^2 - \left(1 - \left(\frac{r}{a}\right)\right)^2}}$$

Problem: (5.10) Velocities in comoving reference frame





$$\sin\beta = \frac{b}{d}$$

$$b = d\sin\beta$$

$$\vec{v}_i = \vec{v}_{in} - \vec{v}_J$$

$$\vec{v}_{in} = -v_{in}\sin\alpha\,\hat{x} + v_{in}\cos\alpha\,\hat{y}$$

$$-\vec{v}_J = v_J\,\hat{x}$$

$$\vec{v}_i = v_i\cos\beta\,\hat{x} + v_i\sin\beta\,\hat{y}$$

Thus, $$\tan\beta = \frac{\sin\beta}{\cos\beta}$$

$$= \frac{v_{i,y}}{v_{i,x}}$$

$$= \frac{v_{in,y} - v_{J,y}}{v_{in,x} - v_{J,x}}$$

$$= \frac{v_{in}\cos\alpha}{-v_{in}\sin\alpha + v_J}$$

Also, show $v_1^2 = v_{in}^2 + v_J^2 - 2 v_{in} v_J \sin\alpha$

<u>Proof:</u>

$$v_i^2 = (v_{in,x} - v_{J,x})^2 + (v_{in,y} - v_{J,y})^2$$

$$= (-v_{in}\sin\alpha + v_J)^2 + (v_{in}\cos\alpha)^2$$

$$= v_{in}^2 \sin^2\alpha + v_J^2 - 2 v_{in} v_J \sin\alpha + v_{in}^2 \cos^2\alpha$$

$$= v_{in}^2 + v_J^2 - 2 v_{in} v_J \sin\alpha$$

Problem: 5.11 Determine final velocity for gravitational slingshot

$$\vec{V}_{out} = (v_i \cos(\theta+\beta) - v_J)\hat{x} + v_i \sin(\theta+\beta)\hat{y}$$

where

$$v_i^2 = v_{in}^2 + v_J^2 - 2 v_{in} v_J \sin\alpha \quad (1)$$

$$\cot\left(\frac{\theta}{2}\right) = \frac{b v_i^2}{G M_J} \quad (2)$$

$$\tan\beta = \frac{v_{in}\cos\alpha}{v_J - v_{in}\sin\alpha} \quad (3)$$

$$\tan\alpha = \sqrt{\frac{1-e^2}{e^2 - (1 - r/a)^2}} \quad (4) \text{ where } r = |\vec{r}_{JS}|$$

$$b = d\sin\beta \quad (5)$$

~~Note~~ a, d, e : are initial conditions

$r_a = 5.3$ AU ; $r_p = 1$ AU $\rightarrow$ $a = 3.15$ AU, $e = 0.683$

☆ Actually take r_a, r_p, d as inputs

Need v_{in}: initial velocity of spacecraft before scattering.

$$E = -\frac{G M_\odot m_s}{2a}$$

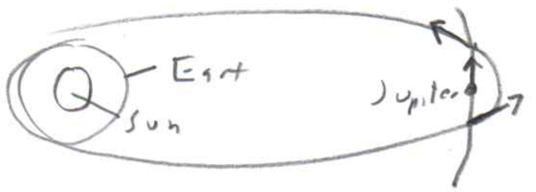
a — 3.15 AU for Jupiter injection orbit

$$-\frac{G M_\odot m_s}{2a} = -\frac{G M_\odot m_s}{r_{JS}} + \frac{1}{2} m_s v_{in}^2$$

r_{JS} — 5.2 AU

$$\boxed{v_{in} = \sqrt{2\left(-\frac{G M_\odot}{2a} + \frac{G M_\odot}{r_{JS}}\right)} = \sqrt{2 G M_\odot \left(\frac{1}{r_{JS}} - \frac{1}{2a}\right)}}$$

To determine final ~~ap~~ aphelion distance:



Scatters out at $r_{JS} = 5.2$ AU $\equiv a_i$
with v_{out}

$$E = -\frac{GM\cancel{m}}{2a_f} = -\frac{GM\cancel{m}}{a_i} + \frac{1}{2}\cancel{m}v_{out}^2$$

a_i ← 5.2 AU

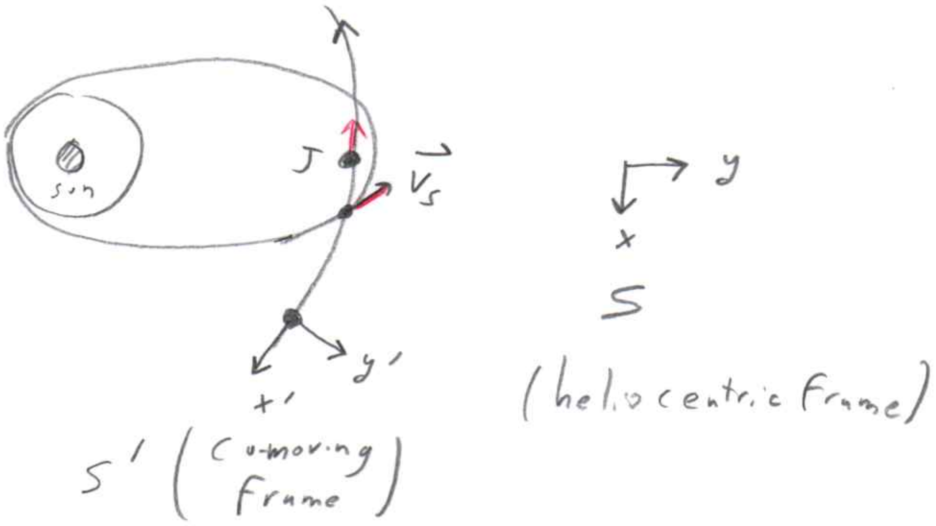
$$-\frac{GM}{2a_f} = -\frac{GM}{a_i} + \frac{1}{2}v_{out}^2$$

$$-\frac{GM}{2}\frac{1}{\left(-\frac{GM}{a_i} + \frac{1}{2}v_{out}^2\right)} = a_f$$

solar mass

$$\boxed{a_f = \frac{-GM/2}{\left(-GM/a_i + \frac{1}{2}v_{out}^2\right)}}$$

r_{JS}



$$\vec{V}_{out} = \vec{V}_{8} + \vec{V}_{J}$$

$$= v_i \cos(\theta+\beta)\,\hat{x} + v_i \sin(\theta+\beta)\,\hat{y}$$

$$- v_J\,\hat{x}$$

$$= \left[v_i \cos(\theta+\beta) - v_J\right]\hat{x} + v_i \sin(\theta+\beta)\,\hat{y}$$

```
function jupiterScatter(r_p, r_a, d)
%
% inputs:
%  r_perihelion, r_aphelion,  in units of AU
%  d: distance from Jupiter in units of Jupiter radius RJ
%
% e.g., jupiterScatter(1, 5.3, 10)
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% constants (MKS)
G = 6.67e-11;
M_sun = 2e30;
M_J = 1.9e27;
AU = 1.5e11;
R_J = 7e7; % mean radius jupiter
r_JS = 5.2*AU; % mean orbital radius of jupiter

% convert to MKS units
r_p = r_p*AU;
r_a = r_a*AU;
d = d*R_J;

% calculate a and e from r_p, r_a
a = 0.5*(r_p+r_a);
e = 0.5*(r_a-r_p)/a;

% calculate initial velocity of spacecraft (v_infinity)
v_in = sqrt(2*G*M_sun*(1/r_JS-1/(2*a)));

% calculate orbital velocity of jupiter
v_J = sqrt(G*M_sun/r_JS);

% calculate angles (radians)
alpha = atan(sqrt((1-e^2)/(e^2 - (1-r_JS/a)^2)));
beta = atan(v_in * cos(alpha)/(v_J - v_in*sin(alpha)));
b = d*sin(beta);
v_i = sqrt(v_in^2 + v_J^2 - 2*v_in*v_J*sin(alpha));
theta = 2*acot(b*v_i^2 /(G*M_J));
v_out = sqrt((v_i*cos(theta+beta) - v_J)^2 + (v_i*sin(theta+beta))^2);
phi_out = atan2(v_i*sin(theta+beta), v_i*cos(theta+beta) - v_J);

% calculate aphelion of scattered orbit
a_f = (-G*M_sun/2)/( -G*M_sun/r_JS + 0.5*v_out^2 );
if a_f<0
  a_f = inf;
  display(' ')
  display('spacecraft ejected out of solar system!')
end

% display results
fprintf('\n')
fprintf('v_in   = %f km/s\n', v_in/1000);
fprintf('v_out = %f km/s\n', v_out/1000);
fprintf('phi_out = %f degree\n', phi_out*180/pi);
fprintf('aphelion = %f AU\n', a_f/AU);
```

```
fprintf('v_J   = %f km/s\n', v_J/1000);
fprintf('v_i   = %f km/s\n', v_i/1000);
fprintf('alpha = %f degree\n', alpha*180/pi);
fprintf('beta  = %f degree\n', beta*180/pi);
fprintf('theta = %f degree\n', theta*180/pi);
fprintf('b = %f R_J\n', b/R_J);

return
```

```
>> jupiterScatter(1, 5.3, 1000)

v_in  = 7.728081 km/s
v_out = 9.006630 km/s
phi_out = 156.458064 degree
aphelion = 3.408297 AU
v_J   = 13.077677 km/s
v_i   = 6.015025 km/s
alpha = 74.277803 degree
beta  = 20.373932 degree
theta = 16.358148 degree
b = 348.145573 R_J

>> jupiterScatter(1, 5.3, 100)

v_in  = 7.728081 km/s
v_out = 17.602089 km/s
phi_out = 164.989016 degree
aphelion = 27.603841 AU
v_J   = 13.077677 km/s
v_i   = 6.015025 km/s
alpha = 74.277803 degree
beta  = 20.373932 degree
theta = 110.343328 degree
b = 34.814557 R_J

>> jupiterScatter(1, 5.3, 10)

spacecraft ejected out of solar system!

v_in  = 7.728081 km/s
v_out = 18.996130 km/s
phi_out = -176.096827 degree
aphelion = Inf AU
v_J   = 13.077677 km/s
v_i   = 6.015025 km/s
alpha = 74.277803 degree
beta  = 20.373932 degree
theta = 172.040099 degree
b = 3.481456 R_J
```

Problem: Differential cross section for equal-mass particles in the lab frame

(5.12)

$$\left(\frac{d\sigma}{d\Omega}\right)' = \left(\frac{d\sigma}{d\Omega}\right)\left|\frac{d(\cos\theta)}{d(\cos\psi)}\right|$$

$$= \left(\frac{d\sigma}{d\Omega}\right)\left|\frac{d(\cos\theta)}{d(\cos(\frac{\theta}{2}))}\right| \qquad \psi = \frac{\theta}{2}$$

$$= \left(\frac{d\sigma}{d\Omega}\right)\left|\frac{\sin\theta \, \cancel{d\theta}}{\sin(\frac{\theta}{2})\,\frac{1}{2}\,\cancel{d\theta}}\right|$$

$$= \left(\frac{d\sigma}{d\Omega}\right)\frac{2\sin(2\psi)}{\sin\psi}$$

$$= \left(\frac{d\sigma}{d\Omega}\right)\frac{4\cancel{\sin\psi}\cos\psi}{\cancel{\sin\psi}}$$

$$= 4\cos\psi\left(\frac{d\sigma}{d\Omega}\right)$$

~~Problem~~: Calculate $\cot\left(\frac{\theta}{2}\right)$ for Rutherford scattering.

$\boxed{\text{OLD}}$

For gravitational scattering:

$$\cot\left(\frac{\theta}{2}\right) = \frac{b v_\infty^2}{GM}$$

Reexpress RHS in terms of E, kQq using

$$GM_m \leftrightarrow kQq$$
$$E = \frac{1}{2} m v_\infty^2$$

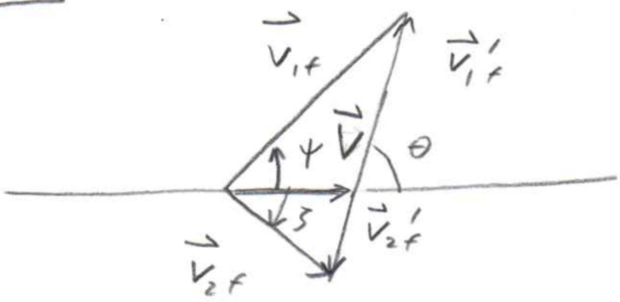
$$\rightarrow \quad v_\infty^2 = \frac{2E}{m}$$

Thus, $$\cot\left(\frac{\theta}{2}\right) = \frac{b}{Gm} \frac{2E}{m}$$

$$= \frac{2Eb}{kQq}$$

Problem: 5.13 Verify expression for $\cos\zeta$ ①

$\boxed{\text{ouad}}$



$$v_{2f}\cos\zeta = V - v_{2f}'\cos\theta$$

$$v_{2f}\sin\zeta = v_{2f}'\sin\theta$$

Square and add:

$$v_{2f}^2\cos^2\zeta + v_{2f}^2\sin^2\zeta$$

$$= (V - v_{2f}'\cos\theta)^2 + (v_{2f}')^2\sin^2\theta$$

$$= V^2 + \underbrace{(v_{2f}')^2\cos^2\theta - 2Vv_{2f}'\cos\theta + (v_{2f}')^2\sin^2\theta}_{=(v_{2f}')^2}$$

$$= V^2 - 2Vv_{2f}'\cos\theta + (v_{2f}')^2$$

Thus, $$\boxed{v_{2f}^2 = V^2 - 2Vv_{2f}'\cos\theta + (v_{2f}')^2}$$

substitute back into $\cos\zeta$ equation:

$$\cos\zeta = \frac{V - v_{2f}'\cos\theta}{\sqrt{V^2 - 2Vv_{2f}'\cos\theta + (v_{2f}')^2}}$$

Divide by v_{2f}' :

$$\cos\zeta = \frac{\frac{V}{v_{2f}'} - \cos\theta}{\sqrt{\left(\frac{V}{v_{2f}'}\right)^2 - 2\left(\frac{V}{v_{2f}'}\right)\cos\theta + 1}}$$

$$= \frac{\rho_2 - \cos\theta}{\sqrt{\rho_2^2 - 2\rho_2\cos\theta + 1}} \quad , \quad \rho_2 = \frac{V}{v_{2f}'}$$

Problem: Verify

$\boxed{\text{OLD}}$ $$\left(\frac{d\sigma}{d\Omega}\right)' = \frac{(1+2\rho\cos\theta+\rho^2)^{3/2}}{1+\rho\cos\theta}\left(\frac{d\sigma}{d\Omega}\right)$$

Proof: $$\left(\frac{d\sigma}{d\Omega}\right)' = \left(\frac{d\sigma}{d\Omega}\right)\left|\frac{d\cos\theta}{d\cos\psi}\right|$$

Now: $$\cos\psi = \frac{\cos\theta+\rho}{\sqrt{1+2\rho\cos\theta+\rho^2}}$$

$$d(\cos\psi) = \frac{d(\cos\theta)\sqrt{} - \frac{1}{\cancel{2}\sqrt{}}\cancel{2}\rho\, d(\cos\theta)(\cos\theta+\rho)}{1+2\rho\cos\theta+\rho^2}$$

$$= \frac{d(\cos\theta)}{[1+2\rho\cos\theta+\rho^2]^{3/2}}\left\{(1+2\rho\cos\theta+\cancel{\rho^2}) - \rho(\cos\theta+\cancel{\rho})\right\}$$

$$= \frac{d(\cos\theta)\,[1+\rho\cos\theta]}{[1+2\rho\cos\theta+\rho^2]^{3/2}}$$

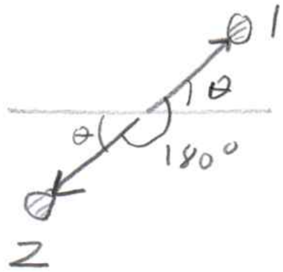
Thus, $$\boxed{\frac{d(\cos\theta)}{d(\cos\psi)} = \frac{[1+2\rho\cos\theta+\rho^2]^{3/2}}{1+\rho\cos\theta}}$$

Problem: Scattering of equal-mass hard spheres in the lab frame.

$\boxed{\text{OLD}}$

In COM frame:

$$\vec{v}_1{}' = -\vec{v}_2{}'$$



In lab frame:

$$\vec{v}_1 = \vec{V} + \vec{v}_1{}'$$

$$\vec{v}_2 = \vec{V} + \vec{v}_2{}' = \vec{V} - \vec{v}_1{}'$$

Thus,

$$\vec{v}_1 \cdot \vec{v}_2 = (\vec{V} + \vec{v}_1{}') \cdot (\vec{V} - \vec{v}_1{}')$$

$$= V^2 - v_1{}'^2$$

But: $V = \frac{1}{2} v_\infty$ ← (for equal mass)

$v_1{}' = \frac{1}{2} v_\infty$

so $\boxed{\vec{v}_1 \cdot \vec{v}_2 = 0}$

Problem: principle of relativity for elastic collisions

(5.1)

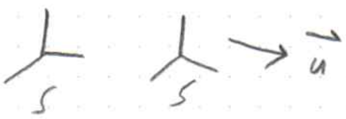
For an elastic collision, total KE is conserved

i.e., $$\sum_I \tfrac{1}{2} m_I v_{Ii}^2 = \sum_I \tfrac{1}{2} m_I v_{If}^2$$

(before) (after)

By the principle of relativity, this must hold in all inertial reference frames.

Let S, S' denote two inertial frames related by $\vec{u}$



so $\vec{v}' = \vec{v} - \vec{u}$

Then $$\vec{v}_{Ii}' = \vec{v}_{Ii} - \vec{u}$$
$$\vec{v}_{If}' = \vec{v}_{If} - \vec{u}$$

In S' (cons. of total KE):

$$\sum_I \tfrac{1}{2} m_I v_{Ii}'^2 = \sum_I \tfrac{1}{2} m_I v_{If}'^2$$

$$LHS = \sum_I \tfrac{1}{2} m_I (\vec{v}_{Ii} - \vec{u}) \cdot (\vec{v}_{Ii} - \vec{u})$$
$$= \sum_I \tfrac{1}{2} m_I (v_{Ii}^2 + u^2 - \vec{u} \cdot \vec{v}_{Ii})$$

$$RHS = \sum_I \tfrac{1}{2} m_I (v_{If}^2 + u^2 - \vec{u} \cdot \vec{v}_{If})$$

Thus,

$$\cancel{\sum_I \tfrac{1}{2} m_I v_{Ii}^2} + \tfrac{1}{2} \cancel{M} u^2 - \tfrac{1}{2} \vec{u} \cdot \sum_I m_I \vec{v}_{Ii}$$
$$= \cancel{\sum_I \tfrac{1}{2} m_I v_{If}^2} + \tfrac{1}{2} \cancel{M} u^2 - \tfrac{1}{2} \vec{u} \cdot \sum_I m_I \vec{v}_{If}$$

~~Thus~~ $\rightarrow \quad 0 = \vec{u} \cdot \left(\sum_I m_I \vec{v}_{Ii} - \sum_I m_I \vec{v}_{If} \right)$

since $\vec{u}$ is arbitrary

$$\sum_I m_I \vec{v}_{Ii} = \sum_I m_I \vec{v}_{If}$$ (cons. of total linear momentum)

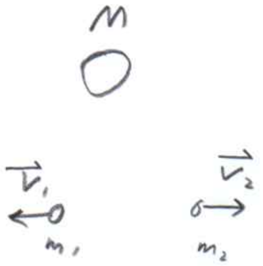
Prob (5.2) Disintegration of a particle of mass M into two constituent particles, mass m_1 and m_2.

~~Determine~~ Assume internal energy of original particle is E_i and internal energies of constituent particles is E_{1i}, E_{2i}

Determine velocities of constituent particles.

Before: E_i

After: E_{1i}, E_{2i}



Conservation ~~and~~ of mass and total energy in Newtonian physics. [In SR, total relativistic momentum and total energy are always conserved. Total mass only conserved for elastic collisions.

Now: $m_1 v_1 = m_2 v_2 \equiv p_0$ (cons of momentum)

$$E_i = E_{1i} + \tfrac{1}{2} m_1 v_1^2 + E_{2i} + \tfrac{1}{2} m_2 v_2^2 \quad \text{(cons of total energy)}$$

$$\Delta E_i \equiv E_i - E_{1i} - E_{2i} = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$

$$\boxed{\Delta E_i = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2}$$

$$\boxed{m_1 v_1 = m_2 v_2}$$

Thus,
$$\Delta E_i = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 \left(\frac{m_1 v_1}{m_2}\right)^2$$
$$= \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} \frac{m_1^2}{m_2} v_1^2$$
$$= \tfrac{1}{2} m_1 v_1^2 \left(1 + \frac{m_1}{m_2}\right)$$
$$= \tfrac{1}{2} m_1 v_1^2 \frac{M}{m_2}$$

Also:

$$\Delta E_i = \frac{1}{2} m_1 \left(\frac{m_2 v_2}{m_1}\right)^2 \frac{M}{m_2}$$

$$= \frac{1}{2} \frac{m_2^2}{m_1} v_2^2 \frac{M}{m_2}$$

$$= \frac{1}{2} m_2 v_2^2 \left(\frac{M}{m_1}\right)$$

Reduced mass: $\mu = \frac{m_1 m_2}{M}$

$$\Delta E_i = \frac{1}{2} m_1^2 v_1^2 \left(\frac{M}{m_1 m_2}\right)$$

$$= \frac{1}{2} \frac{p_0^2}{\mu} \quad \text{(same for 2nd particle)}$$

$$\Delta E_i = \frac{1}{2} \frac{m_1^2 v_1^2}{\mu}$$

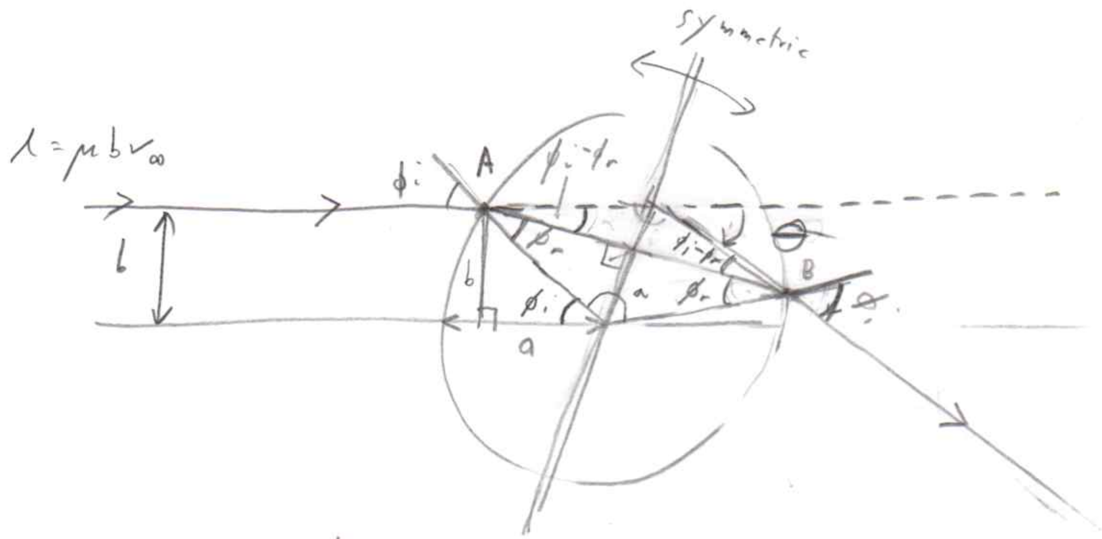
$$\frac{2\mu \Delta E_i}{m_1^2} = v_1^2$$

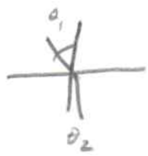
$$\boxed{v_1 = \frac{\sqrt{2\mu \Delta E_i}}{m_1} \qquad v_2 = \frac{\sqrt{2\mu \Delta E_i}}{m_2}}$$

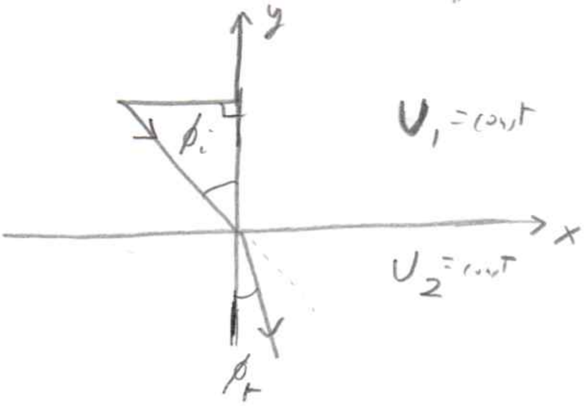
5.5 and 5.3

Problem Scattering off of spherical potential well

$$U(r) = \begin{cases} -U_0 & 0 \le r \le a \\ 0 & r > a \end{cases}$$







$$E_1 = E_2 = E$$

$$E = \frac{1}{2}\mu(\dot{x}_1^2 + \dot{y}_1^2) + U_1$$

$$= \frac{1}{2}\mu(\dot{x}_2^2 + \dot{y}_2^2) + U_2$$

$$\frac{1}{2}\mu v_1^2 + U_1 = \frac{1}{2}\mu v_2^2 + U_2$$

$$\boxed{v_2^2 - v_1^2 = \frac{2}{\mu}(U_1 - U_2)}$$ cons. of energy

$$\cancel{(\dot{x}_2^2 + \dot{y}_2^2) = (\dot{x}_1^2 + \dot{y}_1^2) - \frac{2}{\mu}(U_1 - U_2)}$$

$$\cancel{\dot{y}_2^2 = \dot{y}_1^2 - \frac{2}{\mu}(U_1 - U_2)}$$

$\mu \dot{x}_2 = \mu \dot{x}_1$

(momentum unchanged in x-direction)

$$\cancel{\tan\phi_1 = \frac{\dot{x}_1}{\dot{y}_1}}$$

$$\cancel{\tan\phi_r = \frac{\dot{x}_2}{\dot{y}_2} = \frac{\dot{x}_1}{\dot{y}_2}}$$

~~Thus~~ $\cancel{\dot{y}_1 \tan\phi_1 = \dot{y}_2 \tan\phi_2}$

Now:

$$\sin\phi_i = \frac{\dot{x}_1}{v_1}, \quad \sin\phi_r = \frac{\dot{x}_2}{v_2}$$

But $\dot{x}_1 = \dot{x}_2$

so $$\boxed{v_1 \sin\phi_i = v_2 \sin\phi_r}$$ cons. of momentum in x-direction

Thus,
$$v_1 \sin\phi_i = v_2 \sin\phi_r$$
$$v_2^2 - v_1^2 = \frac{2}{\mu}(U_1 - U_2)$$

$$\Rightarrow \quad \sin\phi_i = \frac{v_2}{v_1}\sin\phi_r$$
$$= \frac{\sqrt{v_1^2 + \frac{2}{\mu}(U_1 - U_2)}}{v_1}\sin\phi_r$$
$$= \sqrt{1 + \frac{2}{\mu v_1^2}(U_1 - U_2)}\,\sin\phi_r$$

Thus, $\boxed{n = \sqrt{1 + \frac{2(U_1 - U_2)}{\mu v_1^2}}}$, $\boxed{\sin\phi_i = n\sin\phi_r}$

~~At A:~~ $\cancel{U_1 = 0,\ U_2 = -U_0}$

$\cancel{n = \sqrt{1 + \frac{2U_0}{\mu v_\infty^2}}}$

For $U_1 = 0$, $U_2 = -U_0$ (at A):
$$\boxed{n = \sqrt{1 + \frac{2(0 - (-U_0))}{\mu v_\infty^2}} = \sqrt{1 + \frac{2U_0}{\mu v_\infty^2}}} > 1$$

Geometry:
$$2\phi_r + \alpha = \pi$$
$$2(\phi_i - \phi_r) + \beta = \pi$$
$$\theta + \beta = \pi$$

Thus, $\boxed{\theta = 2(\phi_i - \phi_r)} \rightarrow \boxed{\phi_r = \phi_i - \frac{\theta}{2}}$

Also, $\boxed{\sin\phi_i = \frac{b}{a}}$

so $\phi_r = \phi_i - \frac{\theta}{2}$

$$\frac{1}{n} = \frac{\sin\phi_r}{\sin\phi_i} = \frac{\sin\left(\phi_i - \frac{\theta}{2}\right)}{b/a}$$

$$= \frac{a}{b}\left(\sin\phi_i \cos\left(\frac{\theta}{2}\right) + \cos\phi_i \sin\left(\frac{\theta}{2}\right)\right)$$

$$= \frac{a}{b}\left[\frac{b}{a}\cos\left(\frac{\theta}{2}\right) + \sqrt{1-\left(\frac{b}{a}\right)^2}\sin\left(\frac{\theta}{2}\right)\right]$$

$$= \cos\left(\frac{\theta}{2}\right) + \frac{a}{b}\sqrt{1-\left(\frac{b}{a}\right)^2}\sin\left(\frac{\theta}{2}\right)$$

$$= \cos\left(\frac{\theta}{2}\right) + \sqrt{\left(\frac{a}{b}\right)^2 - 1}\,\sin\left(\frac{\theta}{2}\right)$$

$$\frac{1}{n} = \cos\left(\frac{\theta}{2}\right) + \sqrt{\left(\frac{a}{b}\right)^2 - 1}\,\sin\left(\frac{\theta}{2}\right)$$

$$\left(\frac{\frac{1}{n} - \cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\right)^2 = \left(\frac{a}{b}\right)^2 - 1$$

$$\left(\frac{1 - n\cos(\theta/2)}{n\sin(\theta/2)}\right)^2 + 1 = \left(\frac{a}{b}\right)^2$$

$$\frac{(1 - n\cos(\theta/2))^2 + n^2\sin^2\left(\frac{\theta}{2}\right)}{n^2\sin^2(\theta/2)} = \left(\frac{a}{b}\right)^2$$

<u>Numerator:</u> $1 + n^2\cos^2(\theta/2) - 2n\cos(\theta/2) + n^2\sin^2(\theta/2)$

$$= 1 + n^2 - 2n\cos(\theta/2)$$

Thus,

$$\boxed{\left(\frac{a}{b}\right)^2 = \frac{1 - 2n\cos(\theta/2) + n^2}{n^2 \sin^2(\theta/2)}}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Differentiate:

b^{-2}

$$-2\frac{a^2}{b^3}\left(\frac{db}{d\theta}\right) = \frac{\frac{\cancel{2}n}{\cancel{2}}\sin\left(\frac{\theta}{2}\right) n^2 \sin^2\left(\frac{\theta}{2}\right) - \frac{\cancel{2}n^2}{\cancel{2}}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\left(1 - 2n\cos\left(\frac{\theta}{2}\right) + n^2\right)}{n^4 \sin^4(\theta/2)}$$

$$-2\left(\frac{a^2}{b^2}\right)\frac{1}{b}\frac{db}{d\theta} = \frac{\cancel{n^3}^{\,n} \sin^{\cancel{3}\,2}\left(\frac{\theta}{2}\right) - \cancel{n^2}\cancel{\sin\left(\frac{\theta}{2}\right)}\cos\left(\frac{\theta}{2}\right)\left(1 - 2n\cos() + n^2\right)}{n^2 \cancel{n^4} \sin^{\cancel{4}\,3}(\theta/2)}$$

$$\to \frac{1}{b}\left(\frac{db}{d\theta}\right) = -\frac{1}{2}\left(\frac{b}{a}\right)^2 \frac{n\sin^2\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)\left(1 - 2n\cos() + n^2\right)}{n^2 \sin^3\left(\frac{\theta}{2}\right)}$$

$$= -\frac{1}{2}\,\frac{\cancel{n^2 \sin^2\left(\frac{\theta}{2}\right)}}{\left(1 - 2n\cos() + n^2\right)}\,\frac{\sin^2\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)\left(1 - 2n\cos() + n^2\right)}{\cancel{n^2}\sin^{\cancel{3}}\left(\frac{\theta}{2}\right)}$$

$$= -\frac{1}{2}\,\frac{n\sin^2\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)\left(1 - 2n\cos() + n^2\right)}{\sin\left(\frac{\theta}{2}\right)\left(1 - 2n\cos() + n^2\right)}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$= \frac{1}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\; b^2\; \frac{1}{b}\left|\frac{db}{d\theta}\right|$$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$$= \frac{1}{2\,\cancel{\cos^2()}\cos()} \frac{a^2 n^2 \cancel{\cos^2()}}{(1-2n\cos()+n^2)} \frac{1}{2} \left| \frac{n\sin^2() - \cos()(1-2n\cos()+n^2)}{\cancel{\cos()}(1-2n\cos()+n^2)} \right|$$

$$= \frac{a^2 n^2}{4\cos()} \frac{|n\sin^2() - \cos()(1-2n\cos()+n^2)|}{(1-2n\cos()+n^2)^2}$$

Numerator: $|n\sin^2() - \cos()(1-2n\cos()+n^2)|$

$$= |n\sin^2() - \cos() + 2n\cos^2() - n^2\cos()|$$

$$= |n(1-\cos^2()) - \cos() + 2n\cos^2() - n^2\cos()|$$

$$= |n - n\cos^2() - \cos() + 2n\cos^2() - n^2\cos()|$$

$$= |n - \cos() + n\cos^2() - n^2\cos()|$$

$$= |(n-\cos()) - n\cos\theta(n-\cos\theta)|$$

$$= |(n-\cos())(1-n\cos())|$$

$$\cancel{= n\cos(\tfrac{\theta}{2})}$$

$$= \left|\left(n\cos\left(\tfrac{\theta}{2}\right)-1\right)\left(n-\cos\left(\tfrac{\theta}{2}\right)\right)\right|$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{a^2 n^2}{4\cos(\frac{\theta}{2})} \frac{\left|\left(n\cos(\frac{\theta}{2})-1\right)\left(n-\cos(\frac{\theta}{2})\right)\right|}{(1-2n\cos()+n^2)^2}}$$

What is maximum θ?

$$\left(\frac{a}{b}\right)^2 = \frac{1 - 2n\cos(\theta/2) + n^2}{n^2 \sin^2(\theta/2)}$$

$$b = b(\theta) \quad \Longleftrightarrow \quad \theta = \theta(b)$$

want to set $0 = \frac{d\theta}{db}$

Differentiate wrt b:

$$-\frac{2a^2}{b^3} = \frac{\frac{1}{\cancel{2}}\cancel{2} n \sin\left(\frac{\theta}{2}\right)\frac{d\theta}{db}\, n^2 \sin^2\left(\frac{\theta}{2}\right) - \cancel{2}n^2 \cancel{\tfrac{1}{2}} \sin()\cos()\frac{d\theta}{db} \left(1 - 2n\cos() + n^2\right)}{n^4 \sin^4()}$$

$$= \frac{\left[n^3 \sin^3() - n^2 \sin()\cos()\left(1 - 2n\cos() + n^2\right)\right]\frac{d\theta}{db}}{n^4 \sin^4(\theta/2)}$$

...

get same expression for $\frac{db}{d\theta}$ as before

[recall $\frac{d\theta}{db} = \frac{1}{\left(\frac{db}{d\theta}\right)}$]

obviously: $b \leq a$

$b = 0 \rightarrow \theta = 0$

$b = a \rightarrow 1 = \frac{1 - 2n\cos(\theta/2) + n^2}{n^2 \sin^2(\theta/2)}$

$$n^2 \sin^2(\theta/2) = 1 - 2n\cos(\theta/2) + n^2$$

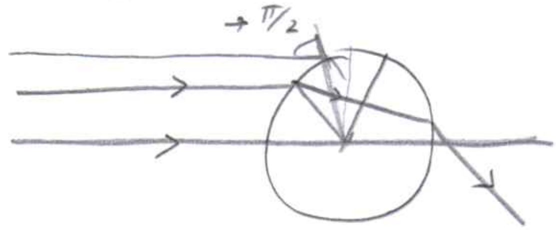
$$\sigma = 1 - 2n\cos(\theta/2) + n^2\left(1 - \sin^2\left(\frac{\theta}{2}\right)\right)$$

$$= 1 - 2n\cos(\theta/2) + n^2\cos^2\left(\frac{\theta}{2}\right)$$

$$= \left(1 - n\cos\left(\frac{\theta}{2}\right)\right)^2$$

Thus, $\cos\left(\frac{\theta}{2}\right) = \frac{1}{n}$

$$\boxed{\cos\left(\frac{\theta_{max}}{2}\right) = \frac{1}{n}}$$ ← ~~max deflection angle~~



$\sin\phi_i = \frac{b}{a}$

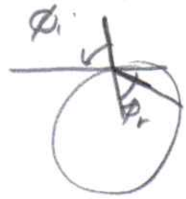
$b = a \rightarrow \phi_i = \pi/2 \quad \rightarrow \quad \sin\phi_i = n\sin\phi_r$

$1 = n\sin\phi_r$

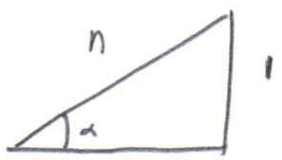
$\frac{1}{n} = \sin\phi_r$

$\theta = 2(\phi_i - \phi_r) = 2\left(\pi/2 - \arcsin\left(\frac{1}{n}\right)\right)$

$= \pi - 2\arcsin\left(\frac{1}{n}\right)$



$\frac{\theta}{2} = \frac{\pi}{2} - \arcsin\left(\frac{1}{n}\right) \rightarrow \cos\left(\frac{\theta}{2}\right) = \cos\left(\frac{\pi}{2} - \arcsin\left(\frac{1}{n}\right)\right)$



$= \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0}\cos\left(\arcsin\left(\frac{1}{n}\right)\right) + \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}\sin\left(\arcsin\left(\frac{1}{n}\right)\right)$

$= \frac{1}{n}$

$n = \sqrt{}$

$\sin\alpha =$

so $\boxed{\cos\left(\frac{\theta}{2}\right) = \frac{1}{n}}$ when $b = a$.

Total cross section:

$$\sigma_T = \pi a^2 \quad (\text{obvious since no scattering for } b > a)$$

check by integrating:

$$\sigma_T = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega \quad , \quad d\Omega = 2\pi \sin\theta \, d\theta$$

$$= 2\pi \int_0^{\theta_{max}} d\theta \sin\theta \left(\frac{d\sigma}{d\Omega}\right)$$

$$= 2\pi \int_0^{2\arccos\left(\frac{1}{n}\right)} d\theta \sin\theta$$

$$= \frac{\cancel{4}\pi a^2 n^2}{\cancel{4}} \int_0^{2\arccos\left(\frac{1}{n}\right)} d\theta \frac{\cancel{2}\sin\left(\frac{\theta}{2}\right)\cancel{\cos\left(\frac{\theta}{2}\right)}}{\cancel{\cos\left(\frac{\theta}{2}\right)}} \frac{\left(n\cos\left(\frac{\theta}{2}\right)-1\right)\left(n-\cos\left(\frac{\theta}{2}\right)\right)}{\left(1-2n\cos\left(\frac{\theta}{2}\right)+n^2\right)^2}$$

$$= \pi a^2 n^2 \int_0^{2\arccos\left(\frac{1}{n}\right)} \underbrace{\sin\left(\frac{\theta}{2}\right) d\theta}_{-2\,d\left(\cos\left(\frac{\theta}{2}\right)\right)} \frac{\left(n\cos\left(\frac{\theta}{2}\right)-1\right)\left(n-\cos\left(\frac{\theta}{2}\right)\right)}{\left(1-2n\cos\left(\frac{\theta}{2}\right)+n^2\right)^2}$$

$$= 2\pi a^2 n^2 \int_{\frac{1}{n}}^{1} dx \frac{(nx-1)(n-x)}{(1-2nx+n^2)^2}$$

let $x = \cos\left(\frac{\theta}{2}\right)$

$\theta = 0 \rightarrow x = 1$

$\theta = 2\arccos\left(\frac{1}{n}\right) \rightarrow x = \frac{1}{n}$

$x = \cos\left(\frac{\theta}{2}\right)$

$dx = -\frac{1}{2}\sin\left(\frac{\theta}{2}\right) d\theta$

$y = nx$

$$\sigma_T = 2\pi a^2 n^2 \int_{\frac{1}{n}}^{1} dx \; \frac{(nx-1)(n-x)}{[(1+n^2)-2nx]^2}$$

$$n-x = n - \frac{(1+n^2)-u}{2n} = \boxed{\frac{(n^2-1)+u}{2n}}$$

Let: $u = (1+n^2) - 2nx \qquad x = \frac{(1+n^2)-u}{2n} \rightarrow nx-1 = \frac{(1+n^2)-u}{2} - 1 = \boxed{\frac{(n^2-1)-u}{2}}$

$$du = -2n\,dx \rightarrow dx = \frac{du}{-2n}$$

$$x = \tfrac{1}{n} \rightarrow u = (1+n^2) - 2n\left(\tfrac{1}{n}\right) = 1+n^2-2 = n^2-1$$

$$x = 1 \rightarrow u = (1+n^2) - 2n = (n-1)^2$$

$$\sigma_T = 2\pi a^2 n^2 \int_{(n-1)^2}^{n^2-1} \frac{du}{2n} \frac{\left(\frac{(n^2-1)-u}{2}\right)}{u^2} \frac{(n^2-1)+u}{2n}$$

$$= 2\pi a^2 \cancel{n^2} \frac{1}{8\cancel{n^2}} \int_{(n-1)^2}^{(n^2-1)} \frac{du}{u^2} \left((n^2-1)-u\right)\left((n^2-1)+u\right)$$

$$= \frac{\pi}{4} a^2 \int_{(n-1)^2}^{n^2-1} du \; \frac{(n^2-1)^2 - u^2}{u^2}$$

$$\frac{a^2-u^2}{u}$$

$$= \frac{\pi}{4} a^2 \int_{(n-1)^2}^{n^2-1} du \left[\frac{(n^2-1)^2}{u^2} - 1\right]$$

$$= \frac{\pi}{4} a^2 \left\{ (n^2-1)^2 \left(\frac{-1}{u}\right)\Bigg|_{(n-1)^2}^{n^2-1} - (n^2-1) + (n-1)^2 \right\}$$

$$= \frac{\pi}{4} a^2 \left\{ -(n^2-1)^2 \left(\frac{1}{n^2-1} - \frac{1}{(n-1)^2}\right) - (n^2-1) + (n-1)^2 \right\}$$

$$= \frac{\pi}{4} a^2 \left\{ -(n^2-1) + (n+1)^2 - (n^2-1) + (n-1)^2 \right\}$$

$$= \frac{\pi}{4} a^2 \left\{ \cancel{-n^2} + 1 + \cancel{n^2} + \cancel{2n} + 1 \cancel{-n^2} + 1 + \cancel{n^2} \cancel{-2n} + 1 \right\} = \boxed{\pi a^2}$$

(5.4)

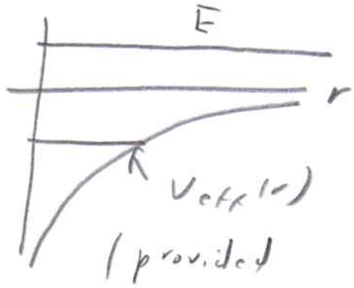
Problem (a) Total cross-section for particle falling into center $U(r) = -\frac{A}{r^2}$ $(A>0)$

$$U_{eff}(r) = \frac{l^2}{2\mu r^2} + U(r)$$

$$= \frac{l^2}{2\mu r^2} - \frac{A}{r^2}$$

$$= -\frac{1}{r^2}\left(A - \frac{l^2}{2\mu}\right)$$

$$< 0 \quad \text{if} \quad A - \frac{l^2}{2\mu} > 0$$



(provided $A - \frac{l^2}{2\mu} > 0$)

particle falls into center for any E

Now: $l = \mu b v_\infty$

so $$A - \frac{\mu^2 b^2 v_\infty^2}{2\mu} > 0$$

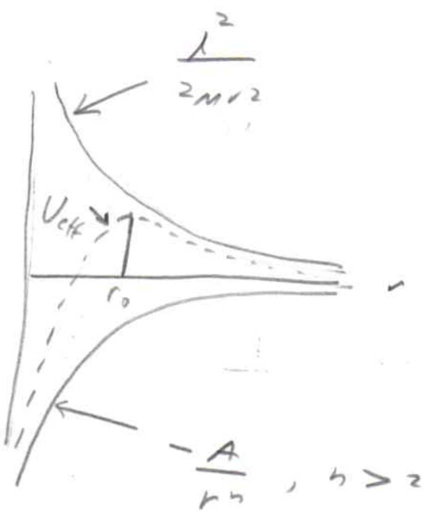
or $$\frac{\mu b^2 v_\infty^2}{2} < A$$

$$b < \sqrt{\frac{2A}{\mu v_\infty^2}} \equiv b_{max}$$

$$\sigma_T = \pi b^2$$

$$= \frac{\pi\, 2A}{\mu v_\infty^2} = \boxed{\frac{2\pi A}{\mu v_\infty^2}}$$

b) $U(r) = -\dfrac{A}{r^n}$, $A > 0$, $n > 2$



$$U_{eff}(r) = U(r) + \frac{\lambda^2}{2\mu r^2}$$

$$= -\frac{A}{r^n} + \frac{\lambda^2}{2\mu r^2}$$

Particle falls to infinity for $E > U_{eff}\big|_{max}$

$$\frac{dU_{eff}}{dr} = +n\frac{A}{r^{n+1}} - \frac{\cancel{2}\lambda^2}{\cancel{2}\mu r^3}$$

$$= \frac{nA}{r^{n+1}} - \frac{\lambda^2}{\mu r^3}$$

$$\left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = 0 \iff \left.\frac{nA}{r^{n+1}}\right|_{r=r_0} = \left.\frac{\lambda^2}{\mu r^3}\right|_{r=r_0}$$

$$\left.\frac{nA}{r^{n-2}}\right|_{r=r_0} = \frac{\lambda^2}{\mu}$$

so $\boxed{r_0 = \left(\dfrac{n\mu A}{\lambda^2}\right)^{\frac{1}{n-2}}}$

$$\cancel{U_{eff}\big|_{max} = U_{eff}(r_0)}$$

$$= \cancel{-\frac{A}{\left(\frac{n\mu A}{\lambda^2}\right)^{\frac{1}{n-2}}} + \frac{\lambda^2}{2\mu\left(\frac{n\mu A}{\lambda^2}\right)^{\frac{1}{n}}}}$$

$$= \cancel{\frac{-A\,2\mu\left(\frac{n\mu A}{\lambda^2}\right)^{\frac{1}{n}} + \lambda^2\left(\frac{n\mu A}{\lambda^2}\right)^{-\frac{1}{2}}}{2\mu\left(\frac{n\mu A}{\lambda^2}\right)^{\frac{1}{n}-\frac{1}{2}}}}$$

$$\cancel{= -2\mu A\left(\frac{n\mu A}{l^2}\right)^{\frac{1}{n}} + l^2\left(\frac{n\mu A}{l^2}\right)^{-\frac{1}{2}}} \over \cancel{2\mu\left(\frac{n\mu A}{l^2}\right)^{\frac{2-n}{2n}}}$$

$$U_{eff}\Big|_{max} = U_{eff}(r_0)$$

$$= -\frac{A}{r_0^{\,n}} + \frac{l^2}{2\mu r_0^{\,2}}$$

$$= \frac{-2\mu A r_0^{\,2} + l^2 r_0^{\,n}}{2\mu r_0^{\,n+2}}$$

$$= \frac{r_0^{\,2}\left[-2\mu A + l^2 r_0^{\,n-2}\right]}{2\mu r_0^{\,n+2}}$$

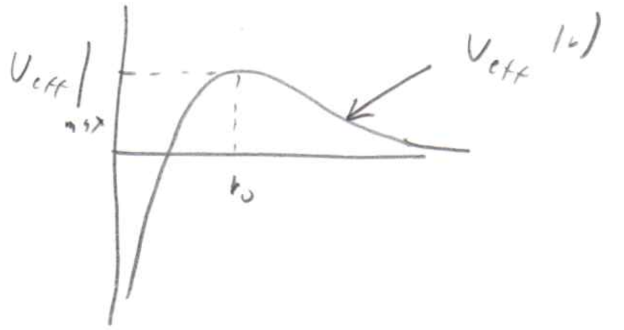
$$= \frac{1}{2\mu r_0^{\,n}}\left[-2\mu A + \cancel{l^2}\,\frac{n\mu A}{\cancel{l^2}}\right]$$

$$= \frac{(n-2)\cancel{\mu}A}{2\cancel{\mu}\, r_0^{\,n}}$$

$$= \frac{(n-2)A}{2}\,\frac{1}{\left(\frac{n\mu A}{l^2}\right)^{\frac{n}{n-2}}}$$

$$= \frac{(n-2)A}{2}\,\frac{1}{\left(\frac{n\mu A}{\mu^2 b^2 v_\infty^{\,2}}\right)^{\frac{n}{n-2}}}$$

$$= \frac{(n-2)A}{2}\left(\frac{\mu b^2 v_\infty^{\,2}}{An}\right)^{\frac{n}{n-2}}$$



Need $E > V_{eff}\big|_{max}$ for particle to fall to center

Total cross section $\sigma_T = \pi b_{max}^2$

Where b_{max} given by setting $E = V_{eff}\big|_{max}$

$$\cancel{\tfrac{1}{2}}\mu v_\infty^2 = \frac{(n-2)A}{\cancel{2}}\left(\frac{\mu b_{max}^2 v_\infty^2}{An}\right)^{\frac{n}{n-2}}$$

$$\frac{\mu v_\infty^2}{A(n-2)} = \left(\frac{\mu b_{max}^2 v_\infty^2}{An}\right)^{\frac{n}{n-2}}$$

$$\left(\frac{\mu v_\infty^2}{A(n-2)}\right)^{\frac{n-2}{n}} = \frac{\mu b_{max}^2 v_\infty^2}{An}$$

$$\rightarrow \quad b_{max}^2 = \frac{An}{\mu v_\infty^2}\left(\frac{\mu v_\infty^2}{A(n-2)}\right)^{1-\frac{2}{n}}$$

$$= n\left(\frac{\mu v_\infty^2}{A}\right)^{-\frac{2}{n}}\frac{1}{(n-2)^{\frac{n-2}{n}}}$$

$$= n(n-2)^{\frac{2-n}{n}}\left(\frac{A}{\mu v_\infty^2}\right)^{\frac{2}{n}}$$

so $$\boxed{\sigma_T = \pi b_{max}^2 = \pi n(n-2)^{\frac{2-n}{n}}\left(\frac{A}{\mu v_\infty^2}\right)^{\frac{2}{n}}}$$

Particle motion in GR: (scattering calculation) ①

Problems (5.6) (5.7)

$$ds^2 = -\left(1-\frac{2GM}{rc^2}\right)c^2dt^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$\theta = \pi/2$

Now

$$-\epsilon c^2 = -c^2\left(1-\frac{2GM}{rc^2}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

where $\begin{cases}\epsilon = 1 & \text{for massive particle} \\ \lambda = \tau & \text{(proper time)}\end{cases}$

or $\epsilon = 0$, λ = affine parameter for photon

~~Define~~ **Conserved quantities**

$$E = \left(1-\frac{2GM}{rc^2}\right)\left(\frac{dt}{d\lambda}\right)c^2$$

$$L = r^2\frac{d\phi}{d\lambda}$$

E, L $\begin{cases}\text{energy, momentum of photon} \\ \dfrac{\text{energy}}{\mu}, \dfrac{\text{momentum}}{\mu} \text{ for massive particle of mass } \mu\end{cases}$

So

$$-\epsilon c^2 = -c^2\left(1-\frac{2GM}{rc^2}\right)\frac{\left(E^2/c^4\right)}{\left(1-\frac{2GM}{rc^2}\right)^2} + \left(1-\frac{2GM}{rc^2}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{L^2}{r^4}\right)$$

$$-\epsilon c^2 = -\left(1-\frac{2GM}{rc^2}\right)^{-1}\frac{E^2}{c^2} + \left(1-\frac{2GM}{rc^2}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}$$

$$-\epsilon c^2\left(1-\frac{2GM}{rc^2}\right) = -\frac{E^2}{c^2} + \left(\frac{dr}{d\lambda}\right)^2 + \left(1-\frac{2GM}{rc^2}\right)\frac{L^2}{r^2}$$

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{2}\left(1-\frac{2GM}{rc^2}\right)\frac{L^2}{r^2} + \frac{1}{2}\epsilon c^2\left(1-\frac{2GM}{rc^2}\right) = \frac{1}{2}\frac{E^2}{c^2}$$

$$\boxed{\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_{eff}(r) = \frac{1}{2}\frac{E^2}{c^2}}$$

where $V_{eff}(r) = \frac{1}{2}\left(1-\frac{2GM}{rc^2}\right)\left(\frac{L^2}{r^2}+\epsilon c^2\right)$

$$= \left(1-\frac{2GM}{rc^2}\right)\left(\frac{1}{2}\right)\left(\frac{L^2}{r^2}+\epsilon c^2\right)$$

$$= \frac{1}{2}\left\{\frac{L^2}{r^2} - \frac{2GML^2}{c^2r^3} + \epsilon c^2 - \frac{2GM\epsilon}{r}\right\}$$

$$= \frac{1}{2}\epsilon c^2 \underbrace{- \epsilon\frac{GM}{r} + \frac{L^2}{2r^2}}_{\text{Newtonian gravity for massive particle}} \underbrace{- \frac{GML^2}{c^2r^3}}_{\text{G-R correction}}$$

$$\frac{dr}{d\lambda} = \sqrt{2\left(\frac{1}{2}\frac{E^2}{c^2} - V_{eff}(r)\right)}$$

<u>Turning point:</u> R when $\left.\frac{dr}{d\lambda}\right|_{r=R} = 0$

$$\frac{1}{2}\frac{E^2}{c^2} - V_{eff}(R) = 0$$

$$\frac{1}{2}\frac{E^2}{c^2} = V_{eff}(R)$$

$$= \frac{1}{2}\epsilon c^2 - \epsilon\frac{GM}{R} + \frac{L^2}{2R^2} - \frac{GML^2}{c^2R^3}$$

$$\frac{d\phi}{dr} = \frac{d\phi}{d\lambda}\,\frac{d\lambda}{dr}$$

$$= \frac{L}{r^2}\,\frac{1}{\sqrt{2\left(\frac{1}{2}\frac{E^2}{c^2} - V_{eff}(r)\right)}}$$

(take − sign since $\phi \downarrow$ as r goes from ∞ to $r_{min} \equiv R$)

$$= \frac{L}{r^2}\,\frac{1}{\sqrt{2\left(\cancel{\frac{1}{2}\epsilon c^2} - \epsilon\frac{GM}{R} + \frac{L^2}{2R^2} - \frac{GML^2}{c^2R^3} - \cancel{\frac{1}{2}\epsilon c^2} + \epsilon\frac{GM}{r} - \frac{L^2}{2r^2} + \frac{GML^2}{c^2r^3}\right)}}$$

$$= \frac{L}{r^2}\,\frac{1}{\sqrt{2\left(-\epsilon GM\left(\frac{1}{R}-\frac{1}{r}\right) + \frac{L^2}{2}\left(\frac{1}{R^2}-\frac{1}{r^2}\right) - \frac{GML^2}{c^2}\left(\frac{1}{R^3}-\frac{1}{r^3}\right)\right)}}$$

$$d\phi = \frac{L}{r^2}\,\frac{dr}{\sqrt{}}$$

$$u = \frac{R}{r}$$

$$du = -\frac{R}{r^2}\,dr$$

$$= \frac{L}{R}\,\frac{(-du)}{\sqrt{}}$$

$$\frac{1}{r} = \frac{u}{R}$$

$$d\phi = \frac{-du}{\sqrt{\frac{2R^2}{L^2}\left(-\epsilon GM\left(\frac{1}{R}-\frac{1}{r}\right)+\frac{L^2}{2}\left(\frac{1}{R^2}-\frac{1}{r^2}\right)-\frac{GML^2}{c^2}\left(\frac{1}{R^3}-\frac{1}{r^3}\right)\right)}}$$

$$= \frac{-du}{\sqrt{\frac{2R^2}{L^2}\left(-\epsilon\frac{GM}{R}(1-u)+\frac{L^2}{2R^2}(1-u^2)-\frac{GML^2}{c^2R^3}(1-u^3)\right)}}$$

$$= \frac{-du}{\sqrt{(1-u^2)-\epsilon\frac{2GMR(1-u)}{L^2}-\frac{2GM}{Rc^2}(1-u^3)}}$$

$$= \frac{-du}{\sqrt{1-u^2}\sqrt{1-\epsilon\frac{2GMR}{L^2}\frac{(1-u)}{(1-u^2)}-\frac{2GM}{Rc^2}\frac{(1-u^3)}{(1-u^2)}}}$$

$$= \frac{-du/\sqrt{1-u^2}}{\sqrt{1+A\left(\frac{1-u}{1-u^2}\right)+B\left(\frac{1-u^3}{1-u^2}\right)}}$$

so

$$\phi_m = \pi \int_0^1 \frac{du/\sqrt{1-u^2}}{\sqrt{1+A\left(\frac{1-u}{1-u^2}\right)+B\left(\frac{1-u^3}{1-u^2}\right)}}$$

$$A = -\epsilon\frac{2GMR}{L^2}$$

$$B = -\frac{2GM}{Rc^2}$$

Approximation:

$$\frac{1}{\sqrt{1+A\left(\frac{1-u}{1-u^2}\right)+B\left(\frac{1-u^3}{1-u^2}\right)}} \approx 1-\frac{1}{2}A\left(\frac{1-u}{1-u^2}\right)-\frac{1}{2}B\left(\frac{1-u^3}{1-u^2}\right)$$

$$\phi_m = \pi - \left[\underbrace{\int_0^1 \frac{du}{\sqrt{1-u^2}}}_{I_1} - \frac{1}{2}A\underbrace{\int_0^1 \frac{du}{\sqrt{1-u^2}}\left(\frac{1-u}{1-u^2}\right)}_{I_2} - \frac{1}{2}B\underbrace{\int_0^1 \frac{du}{\sqrt{1-u^2}}\left(\frac{1-u^3}{1-u^2}\right)}_{I_3}\right]$$

$$I_1 = \int_0^1 \frac{du}{\sqrt{1-u^2}} \qquad \begin{matrix} u=\cos\theta \\ du = -\sin\theta\, d\theta \end{matrix} \qquad u=0,1 \rightarrow \theta = \frac{\pi}{2},0$$

$$= -\int_{-\pi/2}^{0} \frac{\cancel{\sin\theta}\, d\theta}{\cancel{\sin\theta}} = +\int_0^{\pi/2} d\theta = \boxed{\pi/2}$$

$$I_2 = \int_0^1 \frac{du}{\sqrt{1-u^2}}\underbrace{\left(\frac{1-u}{1-u^2}\right)}_{\frac{1}{1+u}} = \int_0^{\pi/2} \underbrace{d\theta\left(\frac{1}{1+\cos\theta}\right)}_{d\left(\tan\left(\frac{\theta}{2}\right)\right)} = \tan\left(\frac{\theta}{2}\right)\Big|_0^{\pi/2} = \boxed{1}$$

$$I_3 = \int_0^1 \frac{du}{\sqrt{1-u^2}}\left(\frac{1-u^3}{1-u^2}\right) = \int_0^1 \frac{du}{\sqrt{1-u^2}} \frac{\cancel{(1-u)}(1+u+u^2)}{\cancel{(1-u)}(1+u)}$$

$$= \int_0^1 \frac{du}{\sqrt{1-u^2}}\left[\frac{1}{1+u} + u\right] \qquad \frac{1+u+u^2}{1+u} = \frac{1+u(1+u)}{1+u} = \frac{1}{1+u}+u$$

$$= \int_0^{\pi/2} d\theta \left[\frac{1}{1+\cos\theta} + \cos\theta\right]$$

$$= 1 + \int_0^{\pi/2} \cos\theta\, d\theta$$

$$= 1 + \sin\theta\Big|_0^{\pi/2} = \boxed{2}$$

Thus,

$$\phi_m = \pi - \left[\frac{\pi}{2} - \frac{1}{2} A \cdot 1 - \frac{1}{\cancel{2}} B \cdot \cancel{2} \right]$$

$$= \frac{\pi}{2} + \frac{A}{2} + B$$

$$\theta = \pi - 2\phi_m$$

$$= \cancel{\pi} - 2\left(\cancel{\frac{\pi}{2}} + \frac{A}{2} + B \right)$$

$$= -A - 2B$$

$$= \epsilon \frac{2GMR}{L^2} + \frac{4GM}{Rc^2}$$

For small angle deflection $R \cong b$ (closest approach = impact parameter)

Then:

$$\theta = \epsilon \frac{2GMb}{L^2} + \frac{4GM}{bc^2}$$

$$= \frac{2GM}{bc^2} \left(\frac{\epsilon b^2 c^2}{L^2} + 2 \right)$$

Light: $\epsilon = 0 \rightarrow \boxed{\theta = \frac{4GM}{bc^2}}$

Massive particle $\epsilon = 1 \rightarrow \theta = \frac{2GM}{bc^2} \left(\frac{b^2c^2}{L^2} + 2 \right)$

$$L = \frac{\ell}{m} = \frac{b}{m} \sqrt{\frac{E^2 - E_0^2}{c^2}} \quad (= bp)$$

$$= \frac{b}{m} \sqrt{\frac{\gamma^2 m^2 c^4 - m^2 c^4}{c^2}}$$

$$= bc\sqrt{\gamma^2 - 1} = \boxed{\frac{bc\beta}{\sqrt{1-\beta^2}}} = \frac{b v_\infty}{\sqrt{1-\beta^2}}$$

$$\gamma^* = \frac{1}{\sqrt{1-\beta^2}} \rightarrow \gamma^2 = \frac{1}{1-\beta^2} \rightarrow \gamma^2 - 1 = \frac{1}{1-\beta^2} - 1 = \frac{\beta^2}{1-\beta^2}$$

Thus,

$$\theta = \frac{2GM}{bc^2}\left(\frac{\cancel{b^2c^2}}{\cancel{b^2c^2}\beta^2}(1-\beta^2) + 2\right)$$

$$= \frac{2GM}{bc^2}\left(\frac{1}{\beta^2} - 1 + 2\right)$$

$$= \frac{2GM}{bc^2}\left(\frac{1}{\beta^2} + 1\right)$$

$$= \boxed{\frac{2GM}{bc^2\beta^2}(1+\beta^2)}$$

Gradshteyn & Ryzhik:

p. 96 (2.264 #8)

$$\int \frac{x^3\,dx}{\sqrt{R^3}} = \frac{c\Delta x^2 + b(10ac - 3b^2)x + a(8ac - 3b^2)}{c^2 \Delta \sqrt{R}} - \frac{3b}{2c^2}\int \frac{dx}{\sqrt{R}}$$

p. 94 (2.261 #)

$$\int \frac{dx}{\sqrt{R}} = \frac{-1}{\sqrt{-c}} \arcsin\left(\frac{2cx+b}{\sqrt{-\Delta}}\right)$$

$c < 0, \quad \Delta < 0$

where
$$R = a + bx + cx^2$$
$$\Delta = 4ac - b^2$$
$$-\Delta = b^2 - 4ac$$

---

suppose: $c = -1,\ b = 0,\ a = \frac{1}{b^2}$ $\rightarrow$ $\Delta = 4ac - \cancel{b^2} = -\frac{4}{b^2}$, $\sqrt{-\Delta} = \frac{2}{b}$

$$\int \frac{du}{\left(\frac{1}{b^2} - u^2\right)^{\frac{1}{2}}} = -\frac{1}{\sqrt{+1}} \arcsin\left(\frac{-2u}{\left(\frac{2}{b}\right)}\right) = -\arcsin(-bu) + \text{const}$$

---

p. 96: (2.264 #5)

$$\int \frac{dx}{\sqrt{R^3}} = \frac{2(2cx+b)}{\Delta\sqrt{R}}$$

- For Schwarzschild
$$d\tau^2 = -\left(1 - \frac{2M}{r}\right) dt^2 \tag{160}$$
for fixed (r, θ, ϕ).

- Thus, the periods between emission and reception at $r = r_A$ and $r = r_B$ are related by
$$\frac{\Delta\tau_B}{\Delta\tau_A} = \left(\frac{1 - 2M/r_B}{1 - 2M/r_A}\right)^{1/2} \tag{161}$$

- The frequencies ($\omega_{A,B} := 2\pi/\Delta\tau_{A,B}$) are thus related by
$$\frac{\omega_B}{\omega_A} = \left(\frac{1 - 2M/r_A}{1 - 2M/r_B}\right)^{1/2} \tag{162}$$

- Hence, taking $r_B \to \infty$, $\omega_B = \omega_\infty$, $r_A = R$, $\omega_A = \omega$ yields
$$\omega_\infty = \omega\sqrt{1 - \frac{2M}{R}} \tag{163}$$
This is the gravitational redshift formula for Schwarzschild spacetime.

- The above expression reduces to the weak-field result in the limit of large R:
$$\omega_\infty \approx \omega\left(1 - \frac{M}{R}\right) \tag{164}$$

7.3 Particle motion in Newtonian gravity

- For a spherically symmetric source of attraction $V = -GMm/r$, total energy
$$E = \frac{1}{2}m|\vec{v}|^2 + V(r) \tag{165}$$
and angular momentum
$$\vec{L} = \vec{r} \times \vec{p} \tag{166}$$
are conserved.

- Using spherical symmetry to restrict to motion in the equatorial plane ($\theta = \pi/2$):
$$L = |\vec{r} \times \vec{p}| = mr^2\frac{d\phi}{dt} \tag{167}$$

- In terms of L, the total total energy can be written as
$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V_{\text{eff}}(r) \tag{168}$$
where the *effective potential* is given by
$$V_{\text{eff}}(r) := -\frac{GMm}{r} + \frac{1}{2}\frac{L^2}{mr^2} \tag{169}$$

- Exercise: Prove the above.

- The effective potential $V_{\rm eff}(r)$ goes to zero (from below) as $r \to \infty$; it goes to ∞ at $r = 0$; and it has a minimum at $r = r_{\rm min} := L^2/GMm^2$. The minimum value of the effective potential is also the minimum allowed value of the energy given by $E_{\rm min} := -G^2M^2m^3/2L^2$.

- Exercise: Prove the above statements.

- There are three types of trajectories depending on the value of E:

  (i) $E = E_{\rm min}$: A stable circular orbit at $r = r_{\rm min}$.

  (ii) $E_{\rm min} < E < 0$: Bound orbits between turning points $r = r_1$ and $r = r_2$ ($r_1 < r_2$). These bound orbits are actually ellipses (discussed in more detail later).

  (iii) $E \geq 0$: Scattering orbits that come in from $r = \infty$, make a closest approach at $r = r_1$, and then return to ∞. ($E = 0$ is a parabola, $E > 0$ are hyperbolae.)

- Exercise: Show that for the circular orbit at $r = r_{\rm min}$, Kepler's 3rd law holds in the form

$$\Omega^2 = \frac{GM}{r_{\rm min}^3} \tag{170}$$

  where $\Omega := d\phi/dt$ (which is constant for a circular orbit).

- To prove that the trajectories correspond to circles, ellipses, parabolas, etc., we need to find $\phi(r)$. Hence, we need to integrate

$$\frac{d\phi}{dr} = \frac{d\phi}{dt}\frac{dt}{dr} = \frac{L}{mr^2}\left[\frac{2}{m}(E - V_{\rm eff}(r))\right]^{-1/2} \tag{171}$$

- The integral is most simply done by making a change of variables to $u := 1/r$. Using (for $a < 0$)

$$\int \frac{du}{\sqrt{au^2 + bu + c}} = \frac{-1}{\sqrt{-a}}\sin^{-1}\left[\frac{2au + b}{\sqrt{b^2 - 4ac}}\right] \tag{172}$$

  one can show that

$$\phi(r) = \phi_0 + \sin^{-1}\left[\frac{2au + b}{\sqrt{b^2 - 4ac}}\right] \tag{173}$$

  where

$$a = \frac{-L^2}{m^2}\,, \quad b = 2GM\,, \quad c = \frac{2E}{m}\,, \quad u = \frac{1}{r} \tag{174}$$

- Exercise: Prove the above.

- Taking ϕ_0 to be the value of $\phi(r)$ at closest approach, one eventually finds

$$r(\phi) = \frac{\alpha}{1 + \epsilon\cos\phi} \tag{175}$$

  where

$$\alpha = \frac{L^2}{GMm^2}\,, \quad \epsilon = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2} \tag{176}$$

- Exercise: Prove the above.

- The equation

$$r(\phi) = \frac{\alpha}{1 + \epsilon \cos\phi} \tag{177}$$

corresponds to a *conic section*—i.e., a cut of a right circular cone by a plane with slope ϵ relative to horizontal. $\epsilon = 0$ corresponds to a circle; $0 < \epsilon < 1$ corresponds to ellipses; $\epsilon = 1$ corresponds to a parabola; and $\epsilon > 1$ corresponds to hyperbolae.

- For fixed L, the allowed values of ϵ are determined by the allowed values for E. In particular, $E = E_{\min}$ corresponds to $\epsilon = 0$; $E_{\min} < E < 0$ corresponds to $0 < \epsilon < 1$; $E = 0$ corresponds to $\epsilon = 1$; $E > 0$ corresponds to $\epsilon > 1$.

- For an ellipse, ϵ is the *eccentricity* and 2α is the *latus rectum.* (In terms of the semi-major axis a of an ellipse, a, α and ϵ are related by $\alpha = a(1-\epsilon^2)$.)

- Exercise: Explicitly show that, for an ellipse with eccentricity ϵ and latus rectum 2α, the radial distance r from a point P on the ellipse to the focus satisfies Eq. (177), where ϕ is the angle between the line connecting the focal point to P and the semi-major axis.

7.4 Particle motion in Schwarzschild spacetime

- For Schwarzschild, the energy (at infinity) per unit particle rest mass

$$e := \frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = -\frac{1}{m} g_{t\alpha} p^\alpha \tag{178}$$

and angular momentum per unit particle rest mass

$$\ell := \frac{L}{m} = r^2 \sin^2\theta \frac{d\phi}{d\tau} = \frac{1}{m} g_{\phi\alpha} p^\alpha \tag{179}$$

are conserved.

- Since the Schwarzschild geometry is spherically symmetric, the trajectory will be in a 2-d plane. Taking this to be the equatorial plane ($\theta = \pi/2$),

$$\ell = r^2 \frac{d\phi}{d\tau} \tag{180}$$

- Exercise: Using the above conserved quantities, show that

$$-1 = \mathbf{u} \cdot \mathbf{u} = g_{\alpha\beta} u^\alpha u^\beta \tag{181}$$

is equivalent to

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) \tag{182}$$

where

$$\mathcal{E} := \frac{e^2 - 1}{2} \tag{183}$$

and

$$V_{\text{eff}}(r) := -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} \tag{184}$$

- NOTE: The last term $-M\ell^2/r^3$ in the effective potential is what's responsible for the differences between Newtonian gravity and GR for particle motion in a spherically symmetric potential—e.g., the perihelion precession of Mercury, existence of plunge orbits, etc.

- The shape of the effective potential $V_{\text{eff}}(r)$ depends on the value of ℓ^2/M^2.

- If $\ell^2/M^2 > 12$, the effective potential has two extrema corresponding to a local maximum and local minimum. The extreme values of r are given by

$$r = r_{\substack{\text{min}\\ \text{max}}} := \frac{\ell^2}{2M}\left[1 \pm \sqrt{1 - \frac{12M^2}{\ell^2}}\right] . \tag{185}$$

- <u>Exercise</u>: Prove the above statements. (Hint: The local maximum and minimum are determined by setting $dV_{\text{eff}}(r)/dr = 0$.)

- If $\ell^2/M^2 = 12$, the effective potential has a point of inflection at $r_{\text{min}} = r_{\text{max}} = 6M =: r_{\text{ISCO}}$. This is the radius of the Innermost Stable Circular Orbit, hence the acronym ISCO.

- If $\ell^2/M^2 < 12$, the effective potential has no local maxima or minima.

- For $\ell^2/M^2 > 12$, there are five types of trajectories depending on the value of $\mathcal{E}$:

  (i) $\mathcal{E} = \mathcal{E}_{\text{min}}$: A stable circular orbit at $r = r_{\text{min}}$.

  (ii) $\mathcal{E}_{\text{min}} < \mathcal{E} < 0$: Bound orbits between turning points $r = r_1$ and $r = r_2$ ($r_1 < r_2$). Unlike the case for Newtonian gravity, these bound orbits are not closed; they are ellipses whose turning points precess (more about this later).

  (iii) $0 \le \mathcal{E} < \mathcal{E}_{\text{max}}$: Scattering orbits that come in from $r = \infty$, make a closest approach at $r = r_1$, and then return to ∞. (The particle can actually orbit around the centre of curvature a number of times before returning to ∞.)

  (iv) $\mathcal{E} = \mathcal{E}_{\text{max}}$: An unstable circular orbit at $r = r_{\text{max}}$. Unstable circular orbits do not exist for Newtonian gravity.

  (v) $\mathcal{E} > \mathcal{E}_{\text{max}}$: A plunge orbit in which the particle comes in from $r = \infty$ and eventually hits the surface of the star or falls inside the event horizon ($r = 2M$) of a black hole. These plunge orbits do not exist for Newtonian gravity.

- <u>Exercise</u>: Show that for the stable circular orbit at $r = r_{\text{min}}$, Kepler's 3rd law holds in the form

$$\Omega^2 = \frac{M}{r_{\text{min}}^3} \tag{186}$$

  where $\Omega := d\phi/dt$ (note the derivative with respect to t not τ).

- Bound orbits precess. If we define

$$\delta\phi_{\text{prec}} := \Delta\phi - 2\pi \tag{187}$$

  where $\Delta\phi$ is calculated from one turning point $r = r_1$ back to $r = r_1$, one can show that the first-order correction to the Newtonian result ($\delta\phi_{\text{prec}} = 0$) is

$$\delta\phi_{\text{prec}} \approx \frac{6\pi G}{c^2}\frac{M}{a(1-\epsilon^2)} \tag{188}$$

- The largest precession occurs for small a and large ϵ—i.e., Mercury in our solar system. Substituting the values for Mercury gives $\delta\phi_{\rm prec} \approx 43$ arcseconds per century, in agreement with observation. This retrodiction was one of the major triumphs of GR.

- Radial infall from rest at $r = \infty$ has $\mathcal{E} = 0$, $\ell = 0$, $\theta =$ const, and $\phi =$ const. The effective potential simplifies to $V_{\rm eff}(r) = -M/r$ and the radial equation can be written as
$$\frac{d\tau}{dr} = -\sqrt{\frac{r}{2M}} \tag{189}$$
It has solution
$$\tau - \tau_* = -\frac{2}{3}\frac{1}{\sqrt{2M}} r^{3/2} \tag{190}$$
where $\tau = \tau_*$ corresponds to $r = 0$.

- Note that it take a *finite* amount of proper time to fall from any finite r (e.g., $r = 10M$) to $r = 2M$.

- The same radial infall, described in terms of the Schwarzschild coordinate time t, satisfies the differential equation
$$\frac{dt}{dr} = \frac{dt}{d\tau}\frac{d\tau}{dr} = -\sqrt{\frac{r}{2M}}\left(1 - \frac{2M}{r}\right)^{-1} \tag{191}$$

- This has solution
$$t - t_* = 2M\left[-\frac{2}{3}\left(\frac{r}{2M}\right)^{3/2} - 2\left(\frac{r}{2M}\right)^{1/2} + \log\left|\frac{\sqrt{r/2M}+1}{\sqrt{r/2M}-1}\right|\right] \tag{192}$$
where $t = t_*$ corresponds to $r = 0$.

- <u>Exercise</u>: Prove the above.

- Note that, contrary to what we found for proper time, it takes an *infinite* amount of coordinate time to fall from any finite r (e.g., $r = 10M$) to $r = 2M$.

7.5 Light rays in Schwarzschild spacetime

- For light rays in Schwarzschild spacetime, the energy (at infinity)
$$e := \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} = -g_{t\alpha}p^\alpha \tag{193}$$
and angular momentum
$$\ell := r^2 \sin^2\theta \frac{d\phi}{d\lambda} = g_{\phi\alpha}p^\alpha \tag{194}$$
are conserved, where $p^\alpha := dx^\alpha/d\lambda$ with λ an affine parameter for the motion.

- Spherical symmetry implies motion in a 2-d plane. Taking this to be the equatorial plane ($\theta = \pi/2$),
$$\ell = r^2\frac{d\phi}{d\lambda} \tag{195}$$

- Exercise: Using the above conserved quantities, show that

$$0 = \mathbf{u} \cdot \mathbf{u} = g_{\alpha\beta} u^\alpha u^\beta \tag{196}$$

implies

$$\frac{1}{b^2} = \frac{1}{\ell^2} \left(\frac{dr}{d\lambda} \right)^2 + W_{\text{eff}}(r) \tag{197}$$

where

$$b^2 := \frac{\ell^2}{e^2} \tag{198}$$

and

$$W_{\text{eff}}(r) := \frac{1}{r^2} \left(1 - \frac{2M}{r} \right) \tag{199}$$

- NOTE: The motion depends only on the ratio ℓ/e, since a different choice of affine parameter changes the values of ℓ and e, but not their ratio.

- b has the interpretation of an *impact parameter*—it is the perpendicular distance (at large r) between the direction of the light ray and a parallel line passing through the centre of curvature. In other words, $\ell = bp$ at large r, where p is the magnitude of the linear momentum.

- The shape of the effective potential $W_{\text{eff}}(r)$ is independent of ℓ.

- The effective potential has one extremum corresponding to a local maximum at $r = r_{\text{max}} = 3M$. The value of $W_{\text{eff}}(r)$ at $r = 3M$ is

$$W_{\text{max}} := W_{\text{eff}}(r = 3M) = \frac{1}{27M^2} \tag{200}$$

  $W_{\text{eff}}(r)$ goes to zero (from above) as $r \to \infty$; it equals 0 at $r = 2M$.

- Exercise: Prove the above.

- There are three types of trajectories depending on the value of $1/b^2$:

  (i) $0 < 1/b^2 < W_{\text{max}}$: Scattering orbits that come in from $r = \infty$, make a closest approach at $r = r_1$, and then return to ∞. (The light ray can actually orbit around the centre of curvature a number of times before returning to ∞.) Hence light is deflected by a spherically symmetric potential in GR.

  (ii) $1/b^2 = W_{\text{max}}$: A unstable circular orbit at $r = r_{\text{max}} = 3M$.

  (iii) $1/b^2 > W_{\text{max}}$: A plunge orbit in which light comes in from $r = \infty$ and eventually hits the surface of the star or falls inside the event horizon ($r = 2M$) of a black hole.

  Note that a small impact parameter b means small angular momentum ℓ, and hence a greater chance for capture by the star or black hole.

- To determine the deflection of light, one needs to integrate

$$\frac{d\phi}{dr} = \frac{d\phi}{d\lambda}\frac{d\lambda}{dr} = \pm \frac{1}{r^2} \left[\frac{1}{b^2} - W_{\text{eff}}(r) \right]^{-1/2} \tag{201}$$

- We define the deflection angle via

$$\delta\phi_{\text{def}} := \Delta\phi - \pi \tag{202}$$

where

$$\Delta\phi := 2\int_{\infty}^{r_1} \frac{d\phi}{dr}\, dr \tag{203}$$

with r_1 being the closest approach of the trajectory.

- Keeping only the leading order correction terms to the Newtonian result, we find

$$\delta\phi_{\text{def}} \approx \frac{4GM}{bc^2} \tag{204}$$

- Exercise: Prove the above.

- For light grazing the Sun (i.e., $M = M_\odot$, $b = R_\odot$), $\delta\phi_{\text{def}} \approx 1.7$ arcseconds, which was verified by Eddington in 1919 during a solar eclipse. This prediction was another major success of GR.

- The deflection of light as it passes a source of curvature also gives rise to a *time delay* effect, as compared to straight line travel time in Newtonian gravity.

- To calculate the excess time delay one needs to integrate

$$\frac{dt}{dr} = \frac{dt}{d\lambda}\frac{d\lambda}{dr} = \pm\frac{1}{b}\left(1 - \frac{2M}{r}\right)^{-1}\left[\frac{1}{b^2} - W_{\text{eff}}(r)\right]^{-1/2} \tag{205}$$

- Consider sending light or a radar signal from the Earth (past the Sun) to a reflector located at r_R, and then waiting for the reflected signal. The total travel time of the signal is given by

$$(\Delta t)_{\text{tot}} = 2t(r_\oplus, r_1) + 2t(r_R, r_1) \tag{206}$$

where r_1 is the closest approach to the Sun, $r_\oplus$ and r_R are distances of the Earth and reflector from the Sun, and $t(r, r_1)$ is the time that it takes the signal to travel from r to r_1.

- We define the excess time delay as

$$(\Delta t)_{\text{excess}} := (\Delta t)_{\text{tot}} - 2\sqrt{r_\oplus^2 - r_1^2} - 2\sqrt{r_R^2 - r_1^2} \tag{207}$$

where the last two terms give the travel time from Newtonian theory.

- If we assume $r_1 \ll r_\oplus$ and r_R, and carry out the calculation keeping only the first-order correction terms, we find

$$(\Delta t)_{\text{excess}} \approx \frac{4GM}{c^3}\left[\log\left(\frac{4r_R r_\oplus}{r_1^2}\right) + 1\right] \tag{208}$$

- Exercise: Prove the above.

- The excess time delay has also been confirmed experimentally.