Show that $p_\phi = l_z$ (4.1)

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2\theta \, \dot{\phi}$$

$$l_z = (\vec{r} \times \vec{p})_z$$

$$= x p_y - y p_x$$

$$= x m \dot{y} - y m \dot{x}$$

$$= m [x\dot{y} - y\dot{x}]$$

Now: $x = r \sin\theta \cos\phi$
$y = r \sin\theta \sin\phi$

$$\rightarrow \dot{x} = \dot{r} \sin\theta \cos\phi + r\cos\theta \cos\phi \, \dot{\theta} - r\sin\theta \sin\phi \, \dot{\phi}$$
$$\dot{y} = \dot{r} \sin\theta \sin\phi + r\cos\theta \sin\phi \, \dot{\theta} + r\sin\theta\cos\phi \, \dot{\phi}$$

Thus,

$$l_z = m [r\sin\theta \cos\phi (\dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi \, \dot{\theta} + r\sin\theta\cos\phi \, \dot{\phi})$$
$$- r\sin\theta\sin\phi (\dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi \, \dot{\theta} - r\sin\theta\sin\phi \, \dot{\phi})]$$

$$= m r\sin\theta [\cancel{\dot{r}\sin\theta\sin\phi\cos\phi} + \cancel{r\cos\theta\sin\phi\cos\phi\dot{\theta}} + r\sin\theta\cos^2\phi \, \dot{\phi}$$
$$- \cancel{\dot{r}\sin\theta\sin\phi\cos\phi} - \cancel{r\cos\theta\sin\phi\cos\phi\dot{\theta}} + r\sin\theta\sin^2\phi \, \dot{\phi}]$$

$$= mr^2\sin^2\theta (\cos^2\phi + \sin^2\phi) \dot{\phi}$$

$$= mr^2 \sin^2\theta \, \dot{\phi}$$

$$= p_\phi$$

Equivalent one-body problem: 4.2

$$\mathcal{L} = \frac{1}{2} m_1 (\dot{r}_1^{\,2} + r_1^2 \dot{\phi}_1^{\,2}) + \frac{1}{2} m_2 (\dot{r}_2^{\,2} + r_2^2 \dot{\phi}_2^{\,2}) + \frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$$

substitutions: $r_1 = \frac{m_2}{M} r$, $r_2 = \frac{m_1}{M} r$, $m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$

$\phi_1 = \phi$, $\phi_2 = \pi + \phi$, $\vec{r} = \vec{r}_1 - \vec{r}_2$

$M = m_1 + m_2$, $\mu = \frac{m_1 m_2}{(m_1 + m_2)} = \frac{m_1 m_2}{M}$ (reduced mass)

$\rightarrow$ $\dot{r}_1 = \frac{m_2}{M} \dot{r}$, $\dot{r}_2 = \frac{m_1}{M} \dot{r}$

$\dot{\phi}_1 = \dot{\phi}$, $\dot{\phi}_2 = \dot{\phi}$

$$\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|} = \frac{G}{|\vec{r}|} \mu M = \frac{G \mu M}{r}$$

$$\begin{aligned}
T &= \frac{1}{2} m_1 (\dot{r}_1^{\,2} + r_1^2 \dot{\phi}_1^{\,2}) + \frac{1}{2} m_2 (\dot{r}_2^{\,2} + r_2^2 \dot{\phi}_2^{\,2}) \\
&= \frac{1}{2} m_1 \left(\frac{m_2^2}{M^2} \dot{r}^2 + \frac{m_2^2}{M^2} r^2 \dot{\phi}^2 \right) + \frac{1}{2} m_2 \left(\frac{m_1^2}{M^2} \dot{r}^2 + \frac{m_1^2}{M^2} r^2 \dot{\phi}^2 \right) \\
&= \frac{1}{2} \left(\frac{m_1 m_2}{M^2} \right) \Big[\underbrace{(m_2 + m_1)}_{M} \dot{r}^2 + \underbrace{(m_1 + m_2)}_{M} r^2 \dot{\phi}^2 \Big] \\
&= \frac{1}{2} \left(\frac{m_1 m_2}{M} \right) [\dot{r}^2 + r^2 \dot{\phi}^2] \\
&= \frac{1}{2} \mu [\dot{r}^2 + r^2 \dot{\phi}^2]
\end{aligned}$$

Thus, $\boxed{\mathcal{L} = \frac{1}{2} \mu [\dot{r}^2 + r^2 \dot{\phi}^2] + \frac{G \mu M}{r}}$

Exer. 4.2 Comparing 2-d and 3-d Lagrangians

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r)$$

r: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\phi}^2 + \frac{\partial U}{\partial r} = 0$$

$$\boxed{m\ddot{r} - mr\dot{\phi}^2 + \frac{\partial U}{\partial r} = 0}$$

ϕ: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \cancel{\frac{\partial L}{\partial \phi}} = 0$

$$\underset{0}{}$$

Thus, $\frac{\partial L}{\partial \dot{\phi}} = \boxed{mr^2\dot{\phi} = \text{const}}$

3-d Lagrangian:

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2) - U(r)$$

r: $\boxed{0} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r}$

$$\boxed{= m\ddot{r} - mr\dot{\theta} - mr\sin^2\theta\,\dot{\phi}^2 + \frac{\partial U}{\partial r}}$$

θ: $0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$

$$= \frac{d}{dt}(mr^2\dot{\theta}) - mr^2 \sin\theta\cos\theta\,\dot{\phi}^2$$

$$\boxed{0 = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} - mr^2\sin\theta\cos\theta\,\dot{\phi}^2}$$

$\underline{\phi}$:

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \cancel{\frac{\partial L}{\partial\phi}}^{\,0}$$

Thus, $\dfrac{\partial L}{\partial\dot{\phi}} = \boxed{mr^2\sin^2\theta\,\dot{\phi} = \text{const}}$

Take these last three equations and set $\theta = \pi/2$:

$$\theta = \pi/2 \rightarrow \sin\theta = 1, \quad \cos\theta = 0$$
$$\dot{\theta} = 0, \quad \ddot{\theta} = 0$$

$$\rightarrow \quad \boxed{0 = m\ddot{r} - mr\dot{\phi}^2 + \frac{\partial U}{\partial r}}$$

$$0 = \cancel{2mr\dot{r}\,0}_{\,0} + \cancel{mr^2 0}_{\,0} - mr^2\cdot 1\cdot \cancel{0}\cdot\dot{\phi}^2$$

$$\boxed{0 = 0}$$

$$\boxed{mr^2\dot{\phi} = \text{const}}$$

These are the same equations as those for the 2-d Lagrangian.

Exercise 4.3

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = \vec{0}$$

Thus, $\vec{r}_2 = \vec{r}_1 - \vec{r}$

$$\rightarrow \quad m_1 \vec{r}_1 + m_2 (\vec{r}_1 - \vec{r}) = \vec{0}$$

$$(m_1 + m_2)\vec{r}_1 - m_2 \vec{r} = \vec{0}$$

$$\boxed{\vec{r}_1 = \frac{m_2}{m_1 + m_2} \vec{r}}$$

Also $\vec{r}_1 = \vec{r}_2 + \vec{r}$

So $\vec{r}_2 + \vec{r} = \frac{m_2}{m_1 + m_2} \vec{r}$

$$\boxed{\vec{r}_2 = \left(\frac{m_2}{m_1 + m_2} - 1 \right) \vec{r} = \left(\frac{-m_1}{m_1 + m_2} \right) \vec{r}}$$

Also:

$$\dot{\vec{r}}_1 = \frac{m_2}{m_1 + m_2} \dot{\vec{r}} \quad , \quad \dot{\vec{r}}_2 = -\frac{m_1}{m_1 + m_2} \dot{\vec{r}}$$

$$T = \frac{1}{2}\left(m_1 |\dot{\vec{r}}_1|^2 + m_2 |\dot{\vec{r}}_2|^2 \right)$$

$$= \frac{1}{2}\left(m_1 \frac{m_2^2}{(m_1 + m_2)^2} |\dot{\vec{r}}|^2 + m_2 \frac{m_1^2}{(m_1 + m_2)^2} |\dot{\vec{r}}|^2 \right)$$

$$= \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)^2} (m_2 + m_1) |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) |\dot{\vec{r}}|^2 = \boxed{\frac{1}{2} \mu |\dot{\vec{r}}|^2}$$

$$U = -\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$$

$$= \frac{-G m_1 m_2}{r}$$

$$= -G\,\mu\frac{(m_1+m_2)}{r}$$

$$= \boxed{\frac{-G\,\mu M}{r}}$$

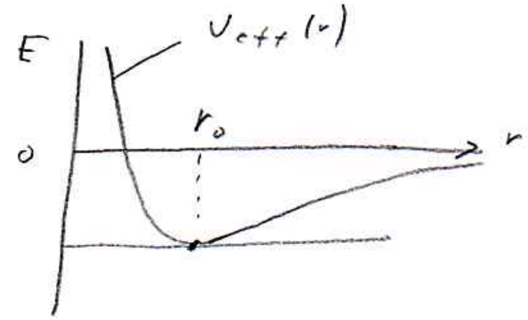
Thus, $\boxed{L = \frac{1}{2}\mu|\dot{\vec{r}}|^2 + \frac{G M \mu}{r}}$

Radius of circular orbit: (4.3)

$$E = \frac{1}{2}\cancel{\mu \dot{r}^2} + \frac{\ell^2}{2\mu r^2} - \frac{GM\mu}{r}$$

$\dot{r} = 0$ for circular orbit $(r = r_0)$

$$E = \frac{\ell^2}{2\mu r_0^2} - \frac{GM\mu}{r_0}$$ where r_0 = minimum of potential



$$U_{eff}(r) = \frac{\ell^2}{2\mu r^2} - \frac{GM\mu}{r}$$

$$0 = \frac{dU_{eff}}{dr} = \left(-\frac{\ell^2}{\mu r^3} + \frac{GM\mu}{r^2}\right)\Bigg|_{r=r_0}$$

$$= \frac{-\ell^2}{\mu r_0^3} + \frac{GM\mu}{r_0^2}$$

Thus, $$\frac{\ell^2}{\mu r_0^3} = \frac{GM\mu}{r_0^2}$$

$$\boxed{r_0 = \frac{\ell^2}{GM\mu^2}}$$

Verify ellipse in Cartesian coords! (4.4)

$0 \le e < 1$: $\quad r = \dfrac{\alpha}{1+e\cos\phi}$

convert to (x,y):

$$r(1+e\cos\phi) = \alpha$$

$$\sqrt{x^2+y^2} + ex = \alpha$$

$$\sqrt{x^2+y^2} = \alpha - ex$$

$$x^2+y^2 = \alpha^2 + e^2 x - 2\alpha e X$$

$$x^2(1-e^2) + 2\alpha e x + y^2 = \alpha^2$$

$$(1-e^2)\left[x^2 + \left(\frac{2\alpha e}{1-e^2}\right)x\right] + y^2 = \alpha^2$$

$$(1-e^2)\left[\left(x + \left(\frac{\alpha e}{1-e^2}\right)\right)^2 - \frac{\alpha^2 e^2}{(1-e^2)^2}\right] + y^2 = \alpha^2$$

$$(1-e^2)(x-x_0)^2 - \frac{\alpha^2 e^2}{(1-e^2)} + y^2 = \alpha^2 \quad , \quad \text{where } x_0 = \frac{-\alpha e}{1-e^2} = ae$$

$$(1-e^2)(x-x_0)^2 + y^2 = \alpha^2 + \frac{\alpha^2 e^2}{1-e^2}$$

$$(1-e^2)(x-x_0)^2 + y^2 = \frac{\alpha^2(1-\cancel{e^2}) + \cancel{\alpha^2 e^2}}{1-e^2} = \left(\frac{\alpha^2}{1-e^2}\right)$$

$$\frac{(x-x_0)^2}{\left[\frac{\alpha^2}{(1-e^2)^2}\right]} + \frac{y^2}{\left(\frac{\alpha^2}{1-e^2}\right)} = 1$$

$$\boxed{\frac{(x-x_0)^2}{A^2} + \frac{y^2}{B^2} = 1}$$

Ellipse

where $x_0 = \dfrac{-\alpha e}{1-e^2} = ae$

$A = \dfrac{\alpha}{1-e^2} = a$

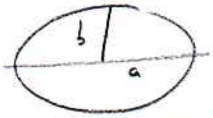
$B = \dfrac{\alpha}{\sqrt{1-e^2}} = b$

Kepler's 3rd law: (4.5)

$$\frac{dA}{dt} = \frac{l}{2\mu} = \text{const}$$

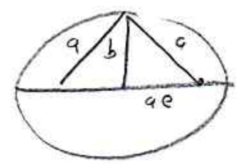
$$\rightarrow \int dA = \int \frac{l}{2\mu}\, dt$$

$$A = \frac{l}{2\mu} P \quad , \quad P: \text{period}$$



$A = \pi a b$ (for ellipse)

Thus, $\boxed{\pi a b = \left(\frac{l}{2\mu}\right) P}$



$$b^2 = a^2(1-e^2)$$
$$\rightarrow b = a\sqrt{1-e^2}$$

$$\rightarrow \quad \pi a^2 \sqrt{1-e^2} = \frac{l}{2\mu} P$$

Square: $\pi^2 a^4 \cancel{(1-e^2)} = \frac{l^2}{4\mu^2} P^2$

$$= \frac{G M \cancel{\mu^2} a}{4 \cancel{\mu^2}} P^2$$

$$= \frac{GM}{4} a \cancel{(1-e^2)} P^2$$

$$\rightarrow \boxed{\frac{P^2}{a^3} = \frac{4\pi^2}{GM}}$$

where $M = m_1 + m_2$ (total mass

Rewrite: $GM = \frac{4\pi^2}{P^2} a^3$

$= \omega^2 a^3$

Now: $\omega = \frac{2\pi}{P}$

$\omega^2 = \frac{4\pi^2}{P^2}$

$$\rightarrow \boxed{GM = \omega^2 a^3}$$

Parabolic and hyperbolic orbits $(e=1,\ e>1)$ 4.6

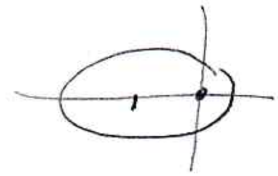
$e=1$: $\quad r(1+\cos\phi)=\alpha$

$$\sqrt{x^2+y^2}+x=\alpha$$

$$\sqrt{x^2+y^2}=\alpha-x$$

$$\cancel{x^2}+y^2=\alpha^2+\cancel{x^2}-2\alpha x$$

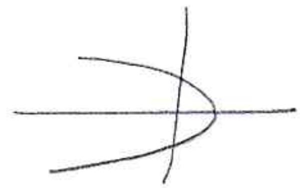
$$y^2=\alpha^2-2\alpha x$$



$$\rightarrow\quad 2\alpha x=\alpha^2-y^2$$

$$\boxed{x=\frac{\alpha}{2}-\frac{y^2}{2\alpha}}$$

parabola



$e>1$: $\quad r(1+e\cos\phi)=\alpha$

$$\sqrt{x^2+y^2}+ex=\alpha$$

$$x^2+y^2=\alpha^2+e^2x^2-2\alpha ex$$

$$-\alpha^2=(e^2-1)x^2-2\alpha ex-y^2$$

$$=(e^2-1)\left[x^2-\left(\frac{2\alpha e}{e^2-1}\right)x\right]-y^2$$

$$=(e^2-1)\left[\left(x-\frac{\alpha e}{e^2-1}\right)^2-\frac{\alpha^2e^2}{(e^2-1)^2}\right]-y^2$$

$x_0\equiv\dfrac{e\alpha}{e^2-1}$

$$=(e^2-1)(x-x_0)^2-\frac{\alpha^2e^2}{(e^2-1)}-y^2$$

$$\rightarrow\quad -\alpha^2+\frac{\alpha^2e^2}{e^2-1}=(e^2-1)(x-x_0)^2-y^2$$

$$\frac{-\alpha^2(\cancel{e^2}-1)+\cancel{\alpha^2e^2}}{e^2-1}=(e^2-1)(x-x_0)^2-y^2$$

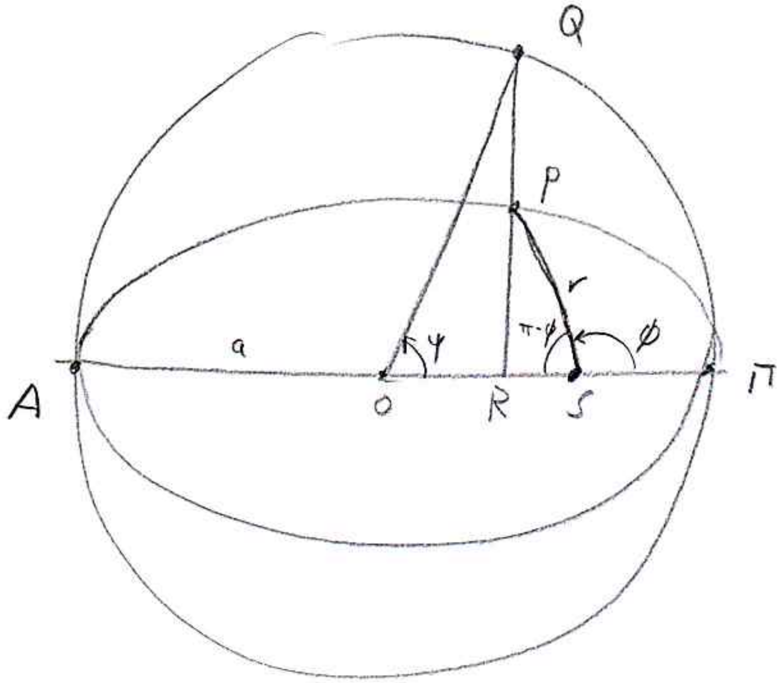
$$\frac{\alpha^2}{e^2-1}=(e^2-1)(x-x_0)^2-y^2$$

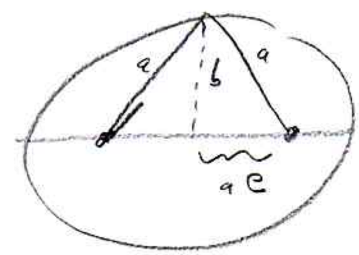
$$\boxed{1 = \frac{(x-x_0)^2}{\left(\frac{\alpha^2}{(e^2-1)^2}\right)} - \frac{y^2}{\left(\frac{\alpha^2}{e^2-1}\right)}}$$ (hyperbola)

$$= \frac{(x-x_0)^2}{A^2} - \frac{y^2}{B^2}$$

where $A = \frac{\alpha}{e^2-1}$, $B = \frac{\alpha}{\sqrt{e^2-1}}$, $x_0 = \frac{\alpha e}{e^2-1}$

Relating $\overline{PR}$ and $\overline{QR}$ for circle and ellipse (4.7)





$a^2 = b^2 + a^2e^2$

$a^2(1-e^2) = b^2$

$b = a\sqrt{1-e^2}$

$$P : (P_x, P_y) = (ae + r\cos\phi,\ r\sin\phi)$$

<u>Pf:</u> $$r\cos(\pi-\phi) = r(\cos\pi\cos\phi + \cancel{\sin\pi}^{0}\sin\phi) = -r\cos\phi$$

$$r\sin(\pi-\phi) = r(\cancel{\sin\pi}^{0}\cos\phi - \cos\pi\sin\phi) = r\sin\phi$$

$$Q : (Q_x, Q_y) = (a\cos\psi,\ a\sin\psi)$$

$$P_x = Q_x \rightarrow \quad ae + r\cos\phi = a\cos\psi$$

$$\boxed{r\cos\phi = a\cos\psi - ae}$$

Want to relate P_y to Q_y:

$$Q_x^2 + Q_y^2 = a^2 \rightarrow \quad Q_y = \sqrt{a^2 - Q_x^2} = \sqrt{a^2 - P_x^2}$$

$$\left(\frac{P_x}{a}\right)^2 + \left(\frac{P_y}{b}\right)^2 = 1$$

$$\rightarrow \left(\frac{P_x}{a}\right)^2 = 1 - \left(\frac{P_y}{b}\right)^2$$

$$P_x^{\,2} = a^2\left[1 - \left(\frac{P_y}{b}\right)^2\right]$$

Thus

$$Q_y = \sqrt{a^2 - a^2\left[1 - \left(\frac{P_y}{b}\right)^2\right]}$$

$$= \sqrt{\cancel{a^2} - \cancel{a^2} + \frac{a^2}{b^2}P_y^2}$$

$$= P_y\left(\frac{a}{b}\right)$$

so $$\boxed{\frac{P_y}{Q_y} = \frac{b}{a} = \sqrt{1-e^2}}$$

Thus, $$P_y = r\sin\phi = Q_y\,\frac{b}{a}$$

$$= a\sin\psi\sqrt{1-e^2}$$

so $$\boxed{r\sin\phi = a\sin\psi\sqrt{1-e^2}}$$

Problem: Using tangent $\frac{1}{2}$-angle formulas to relate ϕ and ψ (4.8) ①

Start with

$$r\cos\phi = a\cos\psi - ae \qquad (1)$$
$$r\sin\phi = a\sin\psi\sqrt{1-e^2} \qquad (2)$$

(i) Squaring and adding yields

$$\boxed{r = a(1-e\cos\psi)}$$

(ii) Using (i) and (1) $\Rightarrow$

$$\boxed{\cos\phi = \frac{\cos\psi - e}{1-e\cos\psi}}$$

This last equation can be written in a more symmetric form by using the tan half angle formula

$$\cos\phi = \frac{1-\tan^2\left(\frac{\phi}{2}\right)}{1+\tan^2\left(\frac{\phi}{2}\right)}$$

Thus

$$\frac{1-\tan^2\left(\frac{\phi}{2}\right)}{1+\tan^2\left(\frac{\phi}{2}\right)} = \frac{\left(\frac{1-\tan^2\left(\frac{\psi}{2}\right)}{1+\tan^2\left(\frac{\psi}{2}\right)}\right) - e}{1 - e\left(\frac{1-\tan^2\left(\frac{\psi}{2}\right)}{1+\tan^2\left(\frac{\psi}{2}\right)}\right)}$$

Let $y \equiv \tan^2\left(\frac{\phi}{2}\right)$, $x \equiv \tan^2\left(\frac{\psi}{2}\right)$

Then

$$\frac{1-y}{1+y} = \frac{\frac{1-x}{1+x} - e}{1-e\left(\frac{1-x}{1+x}\right)} = \frac{(1-x) - e(1+x)}{1+x - e(1-x)}$$

$$= \frac{(1-e) - (1+e)x}{(1-e) + (1+e)x}$$

Thus,

$$(1-y)\left((1-e)+(1+e)x\right) = (1+y)\left((1-e)-(1+e)x\right)$$

$$(1-y)(1-e)+(1-y)(1+e)x = (1+y)(1-e) - (1+y)(1+e)x$$

$$-y(1-e) + (1+e)x = y(1-e) - (1+e)x$$

$$\cancel{2}(1+e)x = \cancel{2}y(1-e)$$

$$\rightarrow \quad y = \left(\frac{1+e}{1-e}\right)x$$

$$\boxed{\tan\left(\frac{\phi}{2}\right) = \sqrt{\frac{1+e}{1-e}}\;\tan\left(\frac{\psi}{2}\right)}$$

(4.9)

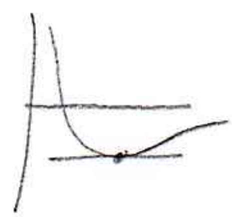
~~Prob 3.7~~ : Virial theorem for circular orbit

Circular orbit : $r = r_0$

$$E = \left(\frac{1}{2} \mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{GM\mu}{r} \right) \Bigg|_{r=r_0}$$

$$= \underbrace{\frac{\ell^2}{2\mu r_0^2}}_{KE} - \underbrace{\frac{GM\mu}{r_0}}_{PE}$$

For circular orbit $E = U_{eff,min}$



$\ell = \mu r^2 \dot{\phi}$

$$\rightarrow \frac{\ell^2}{2\mu r_0^2} = \frac{\mu^2 r_0^4 \dot{\phi}^2}{2\mu r_0^2} = \frac{1}{2} \mu r_0^2 \dot{\phi}^2$$

angular velocity

$\dot{\phi} = \omega = \frac{2\pi}{P}$

So $E = \frac{1}{2} \mu r_0^2 \frac{4\pi^2}{P^2} - \frac{GM\mu}{r_0}$

$$T = \frac{1}{2} \mu r_0^2 \frac{4\pi^2}{P^2} = \frac{1}{2} \mu r_0^2 \frac{GM}{r_0^3} = \frac{1}{2} \mu \frac{GM}{r_0}$$

Using Kepler's 3rd law

so $\boxed{T = -\frac{1}{2} U}$

$\Lambda(u)$ in terms of the potential (4.10)

$$\Lambda(u) = -\frac{\mu}{l^2 u^2} F\left(\frac{1}{u}\right)$$

$$= -\frac{\mu}{l^2 u^2}\left(-\frac{dU(r)}{dr}\right)$$

$$= \frac{\mu}{l^2 u^2} \frac{du}{dr} \frac{dU\left(\frac{1}{u}\right)}{du}$$

$$= -\frac{\mu}{l^2} \frac{dU\left(\frac{1}{u}\right)}{du}$$

$u = \frac{1}{r}$

$\frac{du}{dr} = -\frac{1}{r^2} = -u^2$

(1)

Problem: 4.11 show $\beta^2 > 0$ follows from $\left.\frac{d^2U}{dr^2}\right|_{r_0} > -\frac{3}{r_0}\left.\frac{dU}{dr}\right|_{r_0}$

~~$1 - \Lambda'(u_0) > 0$~~

proof: $\beta^2 = 1 - \left.\frac{d\Lambda}{du}\right|_{u_0}$, $\Lambda(u) = \frac{-\mu}{\ell^2 u^2} F\left(\frac{1}{u}\right)$

$$= 1 + \frac{\mu}{\ell^2} \frac{d}{du}\left(\frac{1}{u^2} F\left(\frac{1}{u}\right)\right)\Big|_{u_0}$$

$$= 1 + \frac{\mu}{\ell^2}\left[-\frac{2}{u^3} F\left(\frac{1}{u}\right) + \frac{1}{u^2} F'\left(\frac{1}{u}\right)\left(\frac{-1}{u^2}\right)\right]\Big|_{u_0}$$

$$= 1 + \frac{\mu}{\ell^2}\left[-\frac{2}{u_0^3} F\left(\frac{1}{u_0}\right) - \frac{1}{u_0^4} F'\left(\frac{1}{u_0}\right)\right]$$

Now: $\left.\frac{d^2U}{dr^2}\right|_{r_0} > -\frac{3}{r_0}\left.\frac{dU}{dr}\right|_{r_0}$

$$RHS = \frac{3}{r_0} F(r_0)$$
$$= 3u_0 F\left(\frac{1}{u_0}\right)$$

$\frac{d}{dr}\left(\frac{1}{r}\right) = -\frac{1}{r^2} = -u^2$

so $\frac{du}{dr} = -u^2$

$$LHS = \frac{d}{dr}\left(\frac{dU}{dr}\right)\Big|_{r_0}$$

$$= -\frac{d}{dr}\left(F(r)\right)\Big|_{r_0}$$

$$= -\frac{du}{dr}\left(\frac{d}{du} F\left(\frac{1}{u}\right)\right)\Big|_{u_0}$$

$$= +u_0^2 F'\left(\frac{1}{u_0}\right)\frac{-1}{u_0^2}$$

$$= -F'\left(\frac{1}{u_0}\right)$$

Thus $-F'\left(\frac{1}{u_0}\right) > 3u_0 F\left(\frac{1}{u_0}\right)$ ~~$\Rightarrow F'\left(\frac{1}{u_0}\right) < -3u_0 F\left(\frac{1}{u_0}\right)$~~

$$0 > F'\left(\frac{1}{u_0}\right) + 3u_0 F\left(\frac{1}{u_0}\right)$$

Return to

$$\beta^2 = 1 + \frac{m}{\ell^2}\left[-\frac{2}{u_0^3} F\left(\frac{1}{u_0}\right) - \frac{1}{u_0^4} F'\left(\frac{1}{u_0}\right)\right]$$

$$= 1 + \frac{m}{\ell^2}\left(\frac{1}{u_0^4}\right)\left[-F'\left(\frac{1}{u_0}\right) - 2u_0 F\left(\frac{1}{u_0}\right)\right]$$

$$> 1 + \frac{m}{\ell^2}\left(\frac{1}{u_0^4}\right)\left[3u_0 F\left(\frac{1}{u_0}\right) - 2u_0 F\left(\frac{1}{u_0}\right)\right]$$

$$= 1 + \frac{m}{\ell^2}\frac{1}{u_0^4} u_0 F\left(\frac{1}{u_0}\right)$$

$$= 1 + \frac{m}{\ell^2 u_0^3} F\left(\frac{1}{u_0}\right)$$

Recall:

$$\frac{dV}{dr_0} = \frac{\ell^2}{m r_0^3} \quad \text{(at extremum)}$$

$$-F(r_0) = \frac{\ell^2}{m r_0^3}$$

$$-F\left(\frac{1}{u_0}\right) = \frac{\ell^2 u_0^3}{m}$$

Thus, $\boxed{F\left(\frac{1}{u_0}\right) = -\frac{\ell^2 u_0^3}{m}}$

$$\rightarrow \quad \beta^2 > 1 + \frac{m}{\ell^2 u_0^3} F\left(\frac{1}{u_0}\right) = 1 - 1 = 0$$

or $\boxed{\beta^2 > 0}$

Problem 4.12 3-d harmonic oscillator potential

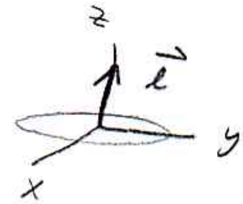
$$U(r) = \frac{kr^2}{2}, \qquad \vec{F}(r) = -\frac{\partial U}{\partial r}\hat{r} = -kr\hat{r}$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta} + r^2\sin^2\theta\,\dot{\phi}^2) - U(r)$$

No ϕ-dependence $\rightarrow$ $p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi} = \text{const} = l_z$

Choose coords so ~~th~~ that z is intially aligned with $\vec{l}$.

Then $l_z = l \longrightarrow \vec{l} = \text{const}$



$\rightarrow$ motion in x-y plane $(\theta = \pi/2)$:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}) - U(r)\Big|_{z=0}$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2+y^2)$$

EL equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a} = 0, \quad q^a = (x,y)$

$$m\ddot{x} + kx = 0 \rightarrow \ddot{x} = -\frac{k}{m}x$$

$$m\ddot{y} + ky = 0 \rightarrow \ddot{y} = -\frac{k}{m}y$$

soln: $x(t) = A\sin(\omega t) + B\cos(\omega t), \quad \omega = \sqrt{\frac{k}{m}}$

$y(t) = C\sin(\omega t) + D\cos(\omega t)$

ICs: $x(0) = \boxed{a = B}$

$\dot{x}(0) = [A\omega\cos(\omega t) - B\omega\sin(\omega t)]\Big|_{t=0}$

$= A\omega$

$= 0 \rightarrow \boxed{A = 0}$

$y(0) = \boxed{0 = D}$

$\dot{y}(0) = [C\omega\cos(\omega t) - D\omega\sin(\omega t)]\Big|_{t=0}$

$= C\omega$

$= b\omega \rightarrow \boxed{C = b}$

Thus, $x(t) = a\cos(\omega t)$
$y(t) = b\sin(\omega t)$

$$\rightarrow \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2(\omega t) + \sin^2(\omega t)$$
$$= 1$$

which is the equation of an ellipse centered at the origin.

Problem: Elliptical orbits from Runge-Lenz Vector

4.13 $\vec{A} = \vec{p} \times \vec{\ell} - k\mu \hat{r}$

NOTE: $\vec{A} \cdot \vec{\ell} = \cancel{(\vec{p} \times \vec{\ell}) \cdot \vec{\ell}} \to 0 - k\mu \cancel{\hat{r} \times \vec{\ell}} \to 0$

$= 0$ (since orbit is in the plane)

$$\vec{r} \cdot \vec{A} = \vec{r} \cdot [\vec{p} \times \vec{\ell} - k\mu \hat{r}]$$
$$= \vec{r} \cdot (\vec{p} \times \vec{\ell}) - k\mu \, \vec{r} \cdot \hat{r}$$
$$= \vec{\ell} \cdot (\vec{r} \times \vec{p}) - k\mu r$$
$$= \vec{\ell} \cdot \vec{\ell} - k\mu r$$
$$= \ell^2 - k\mu r$$

Recall $\alpha \equiv \dfrac{\ell^2}{k\mu}$ ~~$A\alpha$~~

Thus, $\vec{r} \cdot \vec{A} = k\mu \left(\dfrac{\ell^2}{k\mu} - r \right) = k\mu(\alpha - r)$

$$\boxed{\vec{r} \cdot \vec{A} = k\mu(\alpha - r)}$$

Defining: $\vec{r} \cdot \vec{A} = rA\cos\phi$

$\to$ $rA\cos\phi = k\mu\alpha - k\mu r$

$r(A\cos\phi + \mu k) = k\mu\alpha$

$$\boxed{r = \frac{\cancel{k\mu}\,\alpha}{\cancel{k\mu}\left(1 + \frac{A}{k\mu}\cos\phi\right)} = \frac{\alpha}{(1 + e\cos\phi)}}$$

provided $\dfrac{A}{k\mu} = e$

Problem: 4.14

Show $\frac{d\vec{A}}{dt} = 0$ for the Laplace-Runge-Lenz vector

---

Proof: $\vec{A} = \vec{p} \times \vec{\ell} - k\mu\hat{r}$

$$\frac{d\vec{A}}{dt} = \dot{\vec{p}} \times \vec{\ell} + \cancel{\vec{p} \times \dot{\vec{\ell}}}^{\to 0} - k\mu \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

$$= -\frac{k}{r^2}\hat{r} \times \vec{\ell} - k\mu\left[-\frac{1}{r^2}\dot{r}\vec{r} + \frac{1}{r}\dot{\vec{r}}\right]$$

$$= -\frac{k}{r^2}\hat{r} \times [\vec{r} \times \vec{p}] + \frac{k\mu}{r}\dot{r}\hat{r} - \frac{k\mu}{r}\dot{\vec{r}}$$

Now: $\hat{r} \times (\vec{r} \times \vec{p}) = \vec{r}(\hat{r}\cdot\vec{p}) - \vec{p}(\hat{r}\cdot\vec{r})$

$$= \vec{r}(\hat{r}\cdot\mu\dot{\vec{r}}) - \vec{p}\,r$$

$$= \mu\hat{r}(\vec{r}\cdot\dot{\vec{r}}) - \mu\dot{\vec{r}}\,r$$

NOTE: $\frac{d}{dt}(\vec{r}\cdot\vec{r}) = 2\vec{r}\cdot\dot{\vec{r}} = \frac{dr^2}{dt} = 2r\dot{r}$

so $\vec{r}\cdot\dot{\vec{r}} = r\dot{r}$

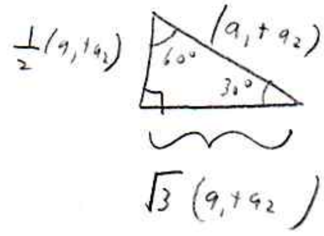
Thus, $\hat{r} \times (\vec{r} \times \vec{p}) = \mu\hat{r}r\dot{r} - \mu r\dot{\vec{r}}$

$$\rightarrow \frac{d\vec{A}}{dt} = -\frac{k}{r^2}\left(\mu\hat{r}r\dot{r} - \mu r\dot{\vec{r}}\right) + \frac{k\mu}{r}\dot{r}\hat{r} - \frac{k\mu}{r}\dot{\vec{r}}$$

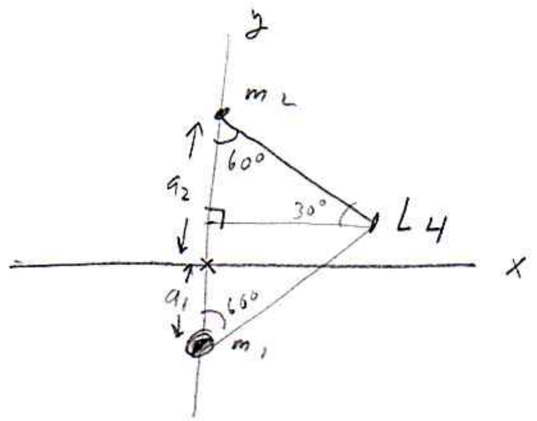
$$= \cancel{-\frac{k\mu}{r}\hat{r}\dot{r}} + \cancel{\frac{k\mu}{r}\dot{\vec{r}}} + \cancel{\frac{k\mu}{r}\dot{r}\hat{r}} - \cancel{\frac{k\mu}{r}\dot{\vec{r}}}$$

$$= \boxed{0}$$

Exercise 4.15



$$a_2 - \frac{1}{2}(a_1+a_2) = \frac{1}{2}(a_2-a_1)$$



$$L_4 : \left(\sqrt{3}(a_1+a_2),\ \frac{1}{2}(a_2-a_1)\right)$$

$$L_5 : \left(-\sqrt{3}(a_1+a_2),\ \frac{1}{2}(a_2-a_1)\right)$$

origin at COM

so $m_1 a_1 = m_2 a_2$

$m_1 : (0, -a_1)$

$m_2 : (0, +a_2)$

$$U_R = -\frac{1}{2} m\omega^2 (x^2+y^2) - \frac{Gm_1 m}{r_1} - \frac{Gm_2 m}{r_2}$$

$$= -\frac{1}{2} m\omega^2 (x^2+y^2) - \frac{Gm_1 m}{\sqrt{x^2+(y+a_1)^2}} - \frac{Gm_2 m}{\sqrt{x^2+(y-a_2)^2}}$$

$$\frac{\partial U_R}{\partial x} = -m\omega^2 x - \frac{Gm_1 m}{(\ )^{3/2}}\left(\frac{-1}{2}\right)2x - \frac{Gm_2 m}{(\ )^{3/2}}\left(-\frac{1}{2}\right)2x$$

$$= -m\omega^2 x + \frac{Gm_1 m x}{r_1^3} + \frac{Gm_2 m x}{r_2^3}$$

$$= mx\left[-\omega^2 + \frac{Gm_1}{r_1^3} + \frac{Gm_2}{r_2^3}\right]$$

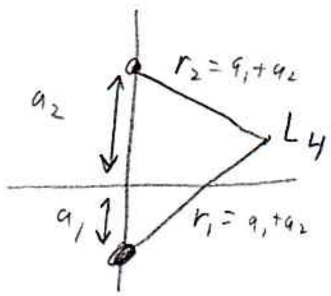
$$\frac{\partial U_R}{\partial y} = -m\omega^2 y - \frac{Gm_1 m}{(\ )^{3/2}}\left(-\frac{1}{2}\right)2(y+a_1) - \frac{Gm_2 m}{(\ )^{3/2}}\left(-\frac{1}{2}\right)2(y-a_2)$$

$$= -m\omega^2 y + \frac{Gm_1 m}{r_1^3}(y+a_1) + \frac{Gm_2 m}{r_2^3}(y-a_2)$$

Check if L_4, L_5 satisfy these equations

(1) $$0 = \frac{\partial U_R}{\partial x} = m x \left[-\omega^2 + \frac{G m_1}{r_1^3} + \frac{G m_2}{r_2^3} \right]$$

(2) $$0 = \frac{\partial U_R}{\partial y} = -m\omega^2 y + \frac{G m_1 m}{r_1^3}(y+a_1) + \frac{G m_2 m}{r_2^3}(y-a_2)$$



(1) $$\text{RHS} = \pm m\sqrt{3}\,(a_1+a_2)\left[-\omega^2 + \frac{G m_1}{(a_1+a_2)^3} + \frac{G m_2}{(a_1+a_2)^3} \right]$$

$$= \pm m\sqrt{3}(a_1+a_2)\left[-\omega^2 + \underbrace{\frac{G(m_1+m_2)}{(a_1+a_2)^3}}_{=\omega^2} \right]$$

$$= \boxed{0} \checkmark$$

(2) $$\text{RHS} = -m\omega^2 \frac{1}{2}(a_2-a_1) + \frac{G m_1 m}{(a_1+a_2)^3}\frac{1}{2}(a_1+a_2) + \frac{G m_2 m}{(a_1+a_2)^3}\left(-\frac{1}{2}\right)(a_1+a_2)$$

$$= -\frac{m}{2}\left[\omega^2(a_2-a_1) - \frac{G m_1}{(a_1+a_2)^2} + \frac{G m_2}{(a_1+a_2)^2} \right]$$

$$= -\frac{m}{2}\left[\frac{G(m_1+m_2)}{(a_1+a_2)^3}(a_2-a_1) - \frac{G(m_1-m_2)}{(a_1+a_2)^2} \right]$$

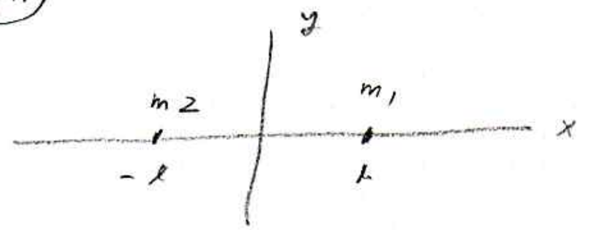
$$= \frac{-mG}{2(a_1+a_2)^3}\left[(m_1+m_2)(a_2-a_1) - (m_1-m_2)(a_1+a_2) \right]$$

$$= \frac{-mG}{2(a_1+a_2)^3}\left[\cancel{m_1 a_2} - m_1 a_1 + m_2 a_2 - \cancel{m_2 a_1} - m_1 a_1 - \cancel{m_1 a_2} + \cancel{m_2 a_1} + m_2 a_2 \right]$$

$$= \frac{-mG}{\cancel{2}(a_1+a_2)}\cancel{2}(m_2 a_2 - m_1 a_1) = \boxed{0} \checkmark$$ — since COM at origin

Problem: Elliptic coords for motion in 2-d plane about two fixed masses

(4.1)



Elliptic coordinates (ξ, η):

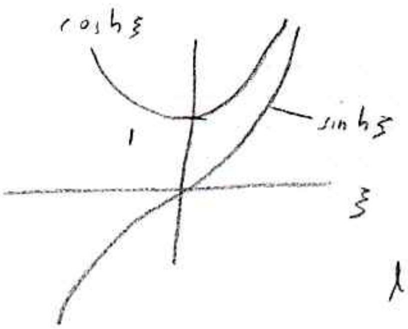
$$x = l \cosh\xi \cos\eta$$
$$y = l \sinh\xi \sin\eta$$

Divide by $l\cosh\xi$, $l\sinh\xi$:

$$\left(\frac{x}{l\cosh\xi}\right)^2 + \left(\frac{y}{l\sinh\xi}\right)^2 = \cos^2\eta + \sin^2\eta = 1$$

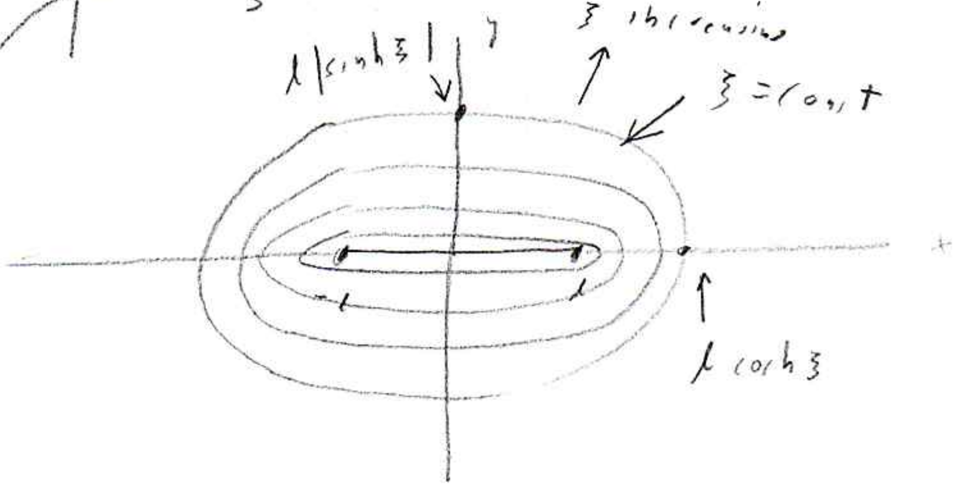
For fixed ξ, these are ellipses  with focal points $(-l,0)$, $(l,0)$ and

$$a = l\cosh\xi \quad \text{(semi-major axis)}$$
$$b = l|\sinh\xi| \quad \text{(semi-minor axis)}$$



$a \geq l$ ($= l$ when $\xi = 0$)

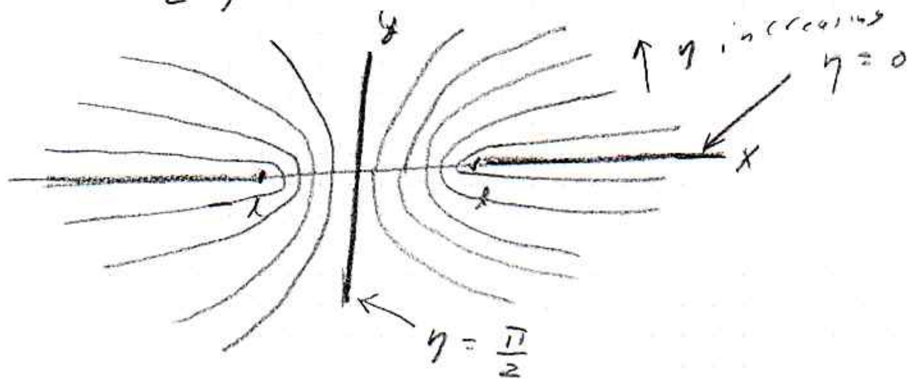
$0 \leq b < \infty$ ($= 0$ when $\xi = 0$)

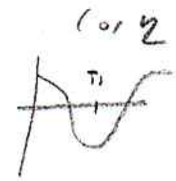


Divide by $l\cos\eta$, $l\sin\eta$:

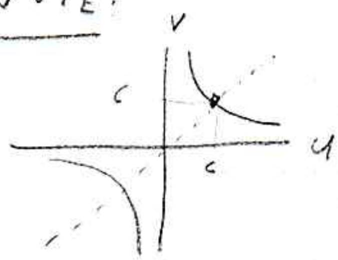
$$\left(\frac{x}{l\cos\eta}\right)^2 - \left(\frac{y}{l\sin\eta}\right)^2 = \cosh^2\xi - \sinh^2\xi = 1$$

For fixed η, these are hyperbolae



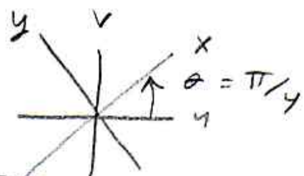


NOTE:



$$uv = c^2$$

rotate coordinates



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} c^2 &= uv \\ &= (\cos\theta\, x - \sin\theta\, y)(\sin\theta\, x + \cos\theta\, y) \\ &= \sin\theta\cos\theta\, x^2 - \sin\theta\cos\theta\, y^2 + xy(\cos^2\theta - \sin^2\theta) \\ &= \frac{1}{2}\sin 2\theta\,(x^2 - y^2) + xy\cos(2\theta) \end{aligned}$$

Take: $\theta = \pi/4 = 45^\circ \rightarrow \cos(2\theta) = \cos\frac{\pi}{2} = 0$

$\sin(2\theta) = \sin\frac{\pi}{2} = 1$

$$\rightarrow c^2 = \frac{1}{2}(x^2 - y^2) \rightarrow x^2 - y^2 = 2c^2$$

$$\left(\frac{x}{\sqrt{2}c}\right)^2 - \left(\frac{y}{\sqrt{2}c}\right)^2 = 1$$

Turns out that ξ and η are orthogonal coords:

Proof:

$$\underbrace{\frac{\partial}{\partial \xi}}_{\text{coord basis vector}} = \frac{\partial x}{\partial \xi}\left(\frac{\partial}{\partial x}\right) + \frac{\partial y}{\partial \xi}\left(\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\left(\frac{\partial}{\partial x}\right) + \frac{\partial y}{\partial \eta}\left(\frac{\partial}{\partial y}\right)$$

Now: $x = l \cosh\xi \cos\eta$
$y = l \sinh\xi \sin\eta$

$$\frac{\partial x}{\partial \xi} = l \sinh\xi \cos\eta$$

$$\frac{\partial y}{\partial \xi} = l \cosh\xi \sin\eta$$

$$\frac{\partial x}{\partial \eta} = -l \cosh\xi \sin\eta$$

$$\frac{\partial y}{\partial \eta} = l \sinh\xi \cos\eta$$

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{array}{c|cc} & \xi & \eta \rightarrow \\ x & l\sinh\xi\cos\eta & -l\cosh\xi\sin\eta \\ y & l\cosh\xi\sin\eta & l\sinh\xi\cos\eta \\ \downarrow & & \end{array}$$

So:

$$\frac{\partial}{\partial \xi} = l\sinh\xi\cos\eta\left(\frac{\partial}{\partial x}\right) + l\cosh\xi\sin\eta\left(\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \eta} = -l\cosh\xi\sin\eta\left(\frac{\partial}{\partial x}\right) + l\sinh\xi\cos\eta\left(\frac{\partial}{\partial y}\right)$$

$$\rightarrow \left(\frac{\partial}{\partial \xi}\right)\cdot\left(\frac{\partial}{\partial \eta}\right) = -l^2 \sinh\xi\cosh\xi\cos\eta\sin\eta \overbrace{\left(\frac{\partial}{\partial x}\right)\cdot\left(\frac{\partial}{\partial x}\right)}^{=1}$$

$$+ l^2 \sinh\xi\cosh\xi\sin\eta\cos\eta \overbrace{\left(\frac{\partial}{\partial y}\right)\cdot\left(\frac{\partial}{\partial y}\right)}^{=1}$$

$$+ 0 \underbrace{\cancel{\left(\frac{\partial}{\partial x}\right)\cdot\left(\frac{\partial}{\partial y}\right)}}_{0} + 0 \underbrace{\cancel{\left(\frac{\partial}{\partial y}\right)\cdot\left(\frac{\partial}{\partial x}\right)}}_{0}$$

$$= \boxed{0} \checkmark$$

Normalization:

$$\left(\frac{\partial}{\partial\xi}\right)\cdot\left(\frac{\partial}{\partial\xi}\right) = l^2\sinh^2\xi\cos^2\eta + l^2\cosh^2\xi\sin^2\eta$$

$$\left(\frac{\partial}{\partial\eta}\right)\cdot\left(\frac{\partial}{\partial\eta}\right) = l^2\cosh^2\xi\sin^2\eta + l^2\sinh^2\xi\cos^2\eta$$

same

volume elt:
(area)

$$dx\wedge dy = \left(l\sinh\xi\cos\eta\,d\xi - l\cosh\xi\sin\eta\,d\eta\right)\wedge$$
$$\left(l\cosh\xi\sin\eta\,d\xi + l\sinh\xi\cos\eta\,d\eta\right)$$

$$= \circ\,\underbrace{d\xi\wedge d\xi}_{0} + \circ\,\underbrace{d\eta\wedge d\eta}_{0}$$
$$+ l^2\sinh^2\xi\cos^2\eta\,d\xi\wedge d\eta$$
$$- l^2\cosh^2\xi\sin^2\eta\,d\eta\wedge d\xi$$

$$= \left(l^2\sinh^2\xi\cos^2\eta + l^2\cosh^2\xi\sin^2\eta\right)d\xi\wedge d\eta$$

Kinetic energy in elliptic coords:

$$T = \frac{1}{2}mv^2$$
$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$
$$= \frac{1}{2}m\Big[\left(l\sinh\xi\cos\eta\,\dot{\xi} - l\cosh\xi\sin\eta\,\dot{\eta}\right)^2$$
$$+ \left(l\cosh\xi\sin\eta\,\dot{\xi} + l\sinh\xi\cos\eta\,\dot{\eta}\right)^2\Big]$$

$$= \frac{1}{2}m\Big[\left(l^2\sinh^2\xi\cos^2\eta + l^2\cosh^2\xi\sin^2\eta\right)\left(\dot{\xi}^2 + \dot{\eta}^2\right)$$
$$\underbrace{-2l^2\sinh\xi\cosh\xi\cos\eta\sin\eta\,\dot{\xi}\dot{\eta} + 2l^2\cosh\xi\sinh\xi\sin\eta\cos\eta\,\dot{\xi}\dot{\eta}}_{0}\Big]$$

The expression

$$l^2 \sinh^2\xi \cos^2\eta + l^2 \cosh^2\xi \sin^2\eta$$

shows up often. [Jacobian of transformation between (x,y) and (ξ,η)]

Simplify:

$$l^2 \sinh^2\xi \cos^2\eta + l^2 \cosh^2\xi \sin^2\eta$$

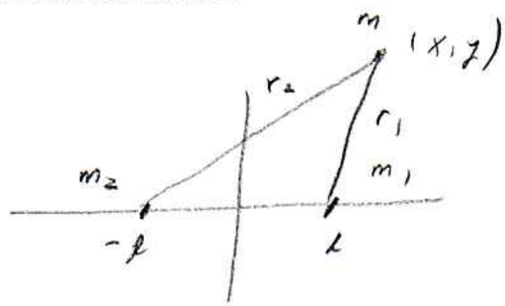
$$= l^2 \sinh^2\xi \cos^2\eta + l^2 \cosh^2\xi (1-\cos^2\eta)$$

$$= l^2 \cosh^2\xi - l^2 \cosh^2\xi \cos^2\eta + l^2 \sinh^2\xi \cos^2\eta$$

$$= l^2 \cosh^2\xi - l^2 \cos^2\eta \underbrace{[\cosh^2\xi - \sinh^2\xi]}_{=1}$$

$$= l^2 (\cosh^2\xi - \cos^2\eta)$$

Gravitational PE:



$$U = -\frac{G m_1 m}{r_1} - \frac{G m_2 m}{r_2}$$

$$= -Gm\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right)$$

Now: $r_1^2 = (x-l)^2 + y^2$

$$= x^2 + l^2 - 2xl + y^2$$

$$= x^2 + y^2 + l(l-2x)$$

$$r_2^2 = (x+l)^2 + y^2$$

$$= x^2 + l^2 + 2xl + y^2$$

$$= x^2 + y^2 + l(l+2x)$$

Express in elliptic coords:

$$x = l\cosh\xi\cos\eta \quad , \quad y = l\sinh\xi\sin\eta$$

$$\rightarrow (x-l)^2 + y^2 = l^2(\cosh\xi\cos\eta - 1)^2 + l^2\sinh^2\xi\sin^2\eta$$

$$= l^2\left[\cosh^2\xi\cos^2\eta + 1 - 2\cosh\xi\cos\eta + \sinh^2\xi\sin^2\eta\right]$$

$$= l^2\left[\cosh^2\xi\cos^2\eta + 1 - 2\cosh\xi\cos\eta + \sinh^2\xi(1-\cos^2\eta)\right]$$

$$= l^2\left[(\cosh^2\xi - \sinh^2\xi)\cos^2\eta + 1 + \sinh^2\xi - 2\cosh\xi\cos\eta\right]$$

$$= l^2\left[\cos^2\eta + \cosh^2\xi - 2\cosh\xi\cos\eta\right]$$

$$= l^2\left[(\cosh\xi - \cos\eta)^2\right]$$

Also:

$$(x+l)^2 + y^2 = l^2(\cosh\xi\cos\eta + 1)^2 + l^2\sinh^2\xi\sin^2\eta$$

...

$$= l^2\left[(\cosh\xi + \cos\eta)^2\right]$$

so

$$U(\xi,\eta) = -Gm\left[\frac{m_1}{l(\cosh\xi - \cos\eta)} + \frac{m_2}{l(\cosh\xi + \cos\eta)}\right]$$

$$= -\frac{Gmm_1}{l}\left(\frac{1}{\cosh\xi - \cos\eta}\right) - \frac{Gmm_2}{l}\left(\frac{1}{\cosh\xi + \cos\eta}\right)$$

$$= \frac{U_1}{\cosh\xi - \cos\eta} + \frac{U_2}{\cosh\xi + \cos\eta}$$

where $U_1 \equiv -\frac{Gmm_1}{l}$, $U_2 \equiv -\frac{Gm_2 m}{l}$ is the gravitational potential energy of m at the origin $(0,0)$

Components of gravitational force in $\{\hat{\xi}, \hat{\eta}\}$ basis:

$$\vec{F} = -\vec{\nabla} U$$

Recall: For general orthogonal coords (u, v)

$$d\vec{s} = du \overrightarrow{\left(\frac{\partial}{\partial u}\right)} + dv \overrightarrow{\left(\frac{\partial}{\partial v}\right)}$$
$$= du\, f \hat{u} + dv\, g \hat{v}$$

$$d\vec{s} \cdot \vec{\nabla}\phi = \frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv$$
$$= du\, f\, \hat{u} \cdot \vec{\nabla}\phi + dv\, g\, \hat{v} \cdot \vec{\nabla}\phi$$

so
$$\hat{u} \cdot \vec{\nabla}\phi = \frac{1}{f} \frac{\partial \phi}{\partial u}$$
$$\hat{v} \cdot \vec{\nabla}\phi = \frac{1}{g} \frac{\partial \phi}{\partial v}$$

Now $\overrightarrow{\left(\frac{\partial}{\partial \xi}\right)} = f \hat{\xi}$, $\overrightarrow{\left(\frac{\partial}{\partial \eta}\right)} = g \hat{\eta}$

where $f = g = \sqrt{\text{Jacobian of transformation}}$
$$= l \left(\cosh^2 \xi - \cos^2 \eta\right)^{\frac{1}{2}}$$

Thus,

$$\hat{\xi} \cdot \vec{F}_g = -\hat{\xi} \cdot \vec{\nabla} U$$
$$= -\frac{1}{l\left(\cosh^2 \xi - \cos^2 \eta\right)^{\frac{1}{2}}} \frac{\partial U}{\partial \xi}$$
$$= -\frac{1}{l\left(\cosh^2 \xi - \cos^2 \eta\right)^{\frac{1}{2}}} \left[\frac{-U_1}{\left(\cosh \xi - \cos \eta\right)^2} \sinh \xi \right.$$
$$\left. - \frac{U_2}{\left(\cosh \xi + \cos \eta\right)^2} \sinh \xi \right]$$

$$\hat{\eta} \cdot \vec{F}_g = - \hat{\eta} \cdot \vec{\nabla} U$$

$$= - \frac{1}{l(\cosh^2 \xi - \cos^2 \eta)^{\frac{1}{2}}} \frac{\partial U}{\partial \eta}$$

$$= \frac{-1}{l(\cosh^2 \xi - \cos^2 \eta)^{\frac{1}{2}}} \left[\frac{-U_1}{(\cosh \xi - \cos \eta)^2} \sin \eta \right.$$

$$\left. - \frac{U_2}{(\cosh \xi + \cos \eta)^2} (-\sin \eta) \right]$$

$$= \frac{-1}{l(\cosh^2 \xi - \cos^2 \eta)^{\frac{1}{2}}} \left[- \frac{U_1 \sin \eta}{(\cosh \xi - \cos \eta)^2} \right.$$

$$\left. + \frac{U_2 \sin \eta}{(\cosh \xi + \cos \eta)^2} \right]$$

Prob: Differential equation in terms of $U(1/u)$: Prob 4.2

$$\beta^2 = 1 - \frac{d\Lambda}{du}\Big|_{u_0}$$

$$\rightarrow \quad 3 - \beta^2 = \frac{u_0}{F(1/u_0)} \frac{dF(1/u)}{du}\Big|_{u_0}$$

a) Now: $F = -\frac{dU(r)}{dr} = -\frac{du}{dr}\frac{dU(1/u)}{du}$

$$u = \frac{1}{r}, \quad \frac{du}{dr} = -\frac{1}{r^2} = -u^2$$

$$\rightarrow \quad \boxed{F\left(\tfrac{1}{u}\right) = u^2 \frac{dU(\frac{1}{u})}{du}}$$

$$\Lambda(u) = \frac{-\mu}{l^2u^2} F\left(\tfrac{1}{u}\right) = -\frac{\mu}{l^2}\frac{dU(\frac{1}{u})}{du}$$

$$\frac{d\Lambda}{du} = \frac{2\mu}{l^2u^3}F\left(\tfrac{1}{u}\right) - \frac{\mu}{l^2u^2}\frac{dF(\frac{1}{u})}{du}$$

$$\frac{d\Lambda}{du} = \frac{2\mu}{l^2u}\frac{dU}{du} - \frac{\mu}{l^2u^2}\frac{d}{du}\left(u^2\frac{dU}{du}\right)$$

$$= \cancel{\frac{2\mu}{l^2u}\frac{dU}{du}} - \frac{\mu}{l^2u^2}\left[\cancel{2u\frac{dU}{du}} + u^2\frac{d^2U}{du^2}\right]$$

$$= -\frac{\mu}{l^2}\frac{d^2U}{du^2}$$

Thus,

$$\beta^2 = 1 - \frac{d\Delta}{du}\Big|_{u_0}$$

$$= 1 + \left[\frac{\mu}{\lambda^2}\frac{d^2U}{du^2}\right]\Big|_{u_0}$$

Circularity condition:

$$\boxed{\begin{aligned}\frac{\lambda^2}{\mu} &= -\frac{1}{u_0^3}F\left(\frac{1}{u_0}\right)\\ &= -\frac{1}{u_0}\frac{dU}{du}\Big|_{u_0}\end{aligned}}$$

Thus,
$$\beta^2 = 1 + \left[\frac{\mu}{\lambda^2}\frac{d^2U}{du^2}\right]\Big|_{u_0}$$

$$= 1 - \frac{u_0\left(\frac{d^2U}{du^2}\right)_{u_0}}{\left(\frac{dU}{du}\right)_{u_0}}$$

$$\rightarrow \cancel{\beta^2\frac{dU}{du} = u_0\frac{d^2U}{du^2}}$$

$$\cancel{u_0\frac{d^2U}{du^2}\Big|_{u_0} + \beta^2\frac{dU}{du}\Big|_{u_0}} = 0$$

D~~efine G(u)~~

$$(1-\beta^2)\left(\frac{dU}{du}\right)_{u_0} = u_0\left(\frac{d^2U}{du^2}\right)_{u_0}$$

$$\boxed{0 = u_0\left(\frac{d^2U}{du^2}\right)_{u_0} - (1-\beta^2)\left(\frac{dU}{du}\right)_{u_0}}$$

D. E. u:

$$u \frac{d^2U}{du^2} + (1-\beta^2)\frac{dU}{du} = 0$$

Define $G(u) = \frac{dU}{du}$:

$$\boxed{u \frac{dG}{du} - (1-\beta^2)\, G(u) = 0}$$

Transformation: $u = e^t$ —— $\frac{du}{dt} = e^t = u$

$$u \frac{dG(u)}{du} = u \frac{dt}{du} \frac{dG(e^t)}{dt}$$ so $u \frac{dt}{du} = 1$

$$= \frac{dG(e^t)}{dt}$$

Thus, $$\frac{dG}{dt} + (1-\beta^2)\, G = 0$$

$$\frac{dG}{G} = (1-\beta^2)\, dt$$

$$\ln G = (1-\beta^2)\, t + \text{const}$$

$$G(t) = A\, e^{(1-\beta^2)t}$$

$$\boxed{G(u) = A\, u^{1-\beta^2}}$$

$$\frac{dU}{du} = G(u) = A u^{(1-\beta^2)} \rightarrow \boxed{U(u) = \frac{A}{2-\beta^2} u^{2-\beta^2} + B}$$

$$\boxed{F\left(\frac{1}{u}\right) = u^2 \frac{dU}{du} = A\, u^{3-\beta^2}}$$

Problem 4.3 3rd-order calculation

$$\eta = \eta_0 + \eta_1 \cos(\beta\phi) + \eta_2 \cos(2\beta\phi) + \eta_3 \cos(3\beta\phi)$$

$$\underset{3^{rd}}{\eta_3} \ll \underset{2^{nd}}{\eta_2, \eta_0} \ll \underset{1^{st}}{\eta_1}$$

Ignore 4th order terms and higher

Keep: $\eta_1,\ \eta_1^2,\ \eta_1^3$

$\eta_1\eta_0,\ \eta_1\eta_2$

η_3

$$\rightarrow \eta^2 = \eta_1^2 \cos^2(\beta\phi) + 2\eta_0\eta_1 \cos(\beta\phi) + 2\eta_2\eta_1 \cos(2\beta\phi)\cos(\beta\phi)$$

$$= \eta_1^2 \tfrac{1}{2}\left[1 + \cos(2\beta\phi)\right] + 2\eta_0\eta_1\cos(\beta\phi) \cancel{+ 2\eta_2\eta_1\cos(2\beta\phi)}$$

$$+ \cancel{2}\eta_1\eta_2 \cancel{\tfrac{1}{2}}\left[\cos(3\beta\phi) + \cos(\beta\phi)\right]$$

(Using $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$)

$$= \boxed{\tfrac{1}{2}\eta_1^2 + \left(2\eta_0\eta_1 + \eta_1\eta_2\right)\cos(\beta\phi) + \tfrac{1}{2}\eta_1^2\cos(2\beta\phi) + \eta_1\eta_2\cos(3\beta\phi)}$$

$$\Rightarrow \eta^3 = \eta_1^3 \cos^3(\beta\phi)$$

$$= \eta_1^3 \cos(\beta\phi) \frac{1}{2} \left[1 + \cos(2\beta\phi) \right]$$

$$= \frac{1}{2} \eta_1^3 \cos(\beta\phi) + \frac{1}{2} \eta_1^3 \underbrace{\cos(\beta\phi)\cos(2\beta\phi)}_{\frac{1}{2}\left[\cos(3\beta\phi) + \cos(\beta\phi)\right]}$$

$$= \frac{1}{2} \eta_1^3 \cos(\beta\phi) + \frac{1}{4} \eta_1^3 \cos(3\beta\phi) + \frac{1}{4} \eta_1^3 \cos(\beta\phi)$$

$$= \boxed{\frac{3}{4} \eta_1^3 \cos(\beta\phi) + \frac{1}{4} \eta_1^3 \cos(3\beta\phi)}$$

Differential equation:

$$(u_0 + \eta)'' + u_0 + \eta = \Lambda(u_0 + \eta)$$

$$\underbrace{u_0''}_{0} + \eta'' + u_0 + \eta = \Lambda(u_0) + \eta \frac{d\Lambda}{du}\Big|_{u_0} + \frac{1}{2}\eta^2 \frac{d^2\Lambda}{du^2}\Big|_{u_0} + \frac{1}{3!}\eta^3 \frac{d^3\Lambda}{du^2}\Big|_{u_0}$$

$$\boxed{\eta'' + \eta = \eta \frac{d\Lambda}{du}\Big|_{u_0} + \frac{1}{2}\eta^2 \frac{d^2\Lambda}{du^2}\Big|_{u_0} + \frac{1}{6}\eta^3 \frac{d^3\Lambda}{du^2}\Big|_{u_0}}$$

$$\eta = \eta_0 + \eta_1 \cos(\beta\phi) + \eta_2 \cos(2\beta\phi) + \eta_3 \cos(3\beta\phi)$$

$$\Rightarrow \eta'' = -\beta^2 \eta_1 \cos(\beta\phi) - 4\beta^2 \eta_2 \cos(2\beta\phi) - 9\beta^2 \eta_3 \cos(3\beta\phi)$$

Thus,

$$\boxed{\begin{aligned} LHS &= \eta'' + \eta \\ &= \eta_0 + \eta_1(1-\beta^2)\cos(\beta\phi) + \eta_2(1-4\beta^2)\cos(2\beta\phi) \\ &\quad + \eta_3(1-9\beta^2)\cos(3\beta\phi) \end{aligned}}$$

~~RHS~~ =

To evaluate RHS, need

$$\boxed{\left.\frac{d\Lambda}{du}\right|_{u_0} = 1-\beta^2}$$

$$\boxed{\left.\frac{d^2\Lambda}{du^2}\right|_{u_0} = \frac{-\beta^2(1-\beta^2)}{u_0}}$$

Now: $\Lambda(u) = u_0^{\beta^2} u^{1-\beta^2}$

$$\rightarrow \frac{d\Lambda}{du} = (1-\beta^2) u_0^{\beta^2} u^{-\beta^2}$$

$$\frac{d^2\Lambda}{du^2} = -\beta^2(1-\beta^2) u_0^{\beta^2} u^{-\beta^2-1}$$

$$\frac{d^3\Lambda}{du^3} = +\beta^2(1-\beta^2)(1+\beta^2) u_0^{\beta^2} u^{-\beta^2-2}$$

$$\rightarrow \boxed{\left.\frac{d^3\Lambda}{du^3}\right|_{u_0} = \frac{\beta^2(1-\beta^2)(1+\beta^2)}{u_0^2}}$$

$$RHS = \eta(1-\beta^2) + \frac{1}{2}\eta^2 \frac{(1-\beta^2)(1-\beta^2)}{u_0} + \frac{1}{6}\eta^3 \frac{\beta^2(1-\beta^2)(1+\beta^2)}{u_0^2}$$

$$= \left(\eta_0 + \eta_1 \cos(\beta\phi) + \eta_2 \cos(2\beta\phi) + \eta_3 \cos(3\beta\phi)\right)(1-\beta^2)$$

$$- \frac{1}{2}\frac{\beta^2(1-\beta^2)}{u_0}\left[\frac{1}{2}\eta_1^2 + (2\eta_0\eta_1 + \eta_1\eta_2)\cos(\beta\phi) + \frac{1}{2}\eta_1^2\cos(2\beta\phi) + \eta_1\eta_2\cos(3\beta\phi)\right]$$

$$+ \frac{1}{6}\frac{\beta^2(1-\beta^2)(1+\beta^2)}{u_0^2}\left[\frac{3}{4}\eta_1^3\cos(\beta\phi) + \frac{1}{4}\eta_1^3\cos(3\beta\phi)\right]$$

$$\boxed{\begin{aligned} = &\left[\eta_0(1-\beta^2) - \frac{1}{4}\frac{\beta^2(1-\beta^2)}{u_0}\eta_1^2\right] \\ &+ \left[\eta_1(1-\beta^2) - \frac{1}{2}\frac{\beta^2(1-\beta^2)}{u_0}(2\eta_0\eta_1 + \eta_1\eta_2) + \frac{1}{8}\frac{\beta^2(1-\beta^2)(1+\beta^2)}{u_0^2}\eta_1^3\right]\cos(\beta\phi) \\ &+ \left[\eta_2(1-\beta^2) - \frac{1}{4}\frac{\beta^2(1-\beta^2)}{u_0}\eta_1^2\right]\cos(2\beta\phi) \\ &+ \left[\eta_3(1-\beta^2) - \frac{1}{2}\frac{\beta^2(1-\beta^2)}{u_0}\eta_1\eta_2 + \frac{1}{24}\frac{\beta^2(1-\beta^2)(1+\beta^2)}{u_0^2}\eta_1^3\right]\cos(3\beta\phi)\end{aligned}}$$

Equate coeffs of $\cos(n\beta\phi)$ terms:

(1) $$\eta_0 = \eta_0 (1-\beta^2) - \frac{1}{4}\frac{\beta^2(1-\beta^2)}{u_0}\eta_1^2$$

$$\rightarrow \boxed{\eta_0 = -\frac{1}{4}\frac{(1-\beta^2)}{u_0}\eta_1^2}$$ (same as before)

---

(2) $$\cancel{\eta_1(1-\beta^2)} = \cancel{\eta_1(1-\beta^2)} - \frac{1}{2}\frac{\beta^2(1-\beta^2)}{u_0}(2\eta_0\eta_1 + \eta_1\eta_2)$$
$$+ \frac{1}{8}\frac{\beta^2(1-\beta^2)(1+\beta^2)}{u_0^2}\eta_1^3$$

[continued on next page]

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(3) $$\eta_2(1-4\beta^2) = \eta_2(1-\beta^2) - \frac{1}{4}\frac{\beta^2(1-\beta^2)\eta_1^2}{u_0}$$

$$\boxed{\eta_2 = +\frac{1}{12}\frac{(1-\beta^2)}{u_0}\eta_1^2}$$ (same as before)

---

(4) $$\eta_3(1-9\beta^2) = \eta_3(1-\beta^2) - \frac{1}{2}\frac{\beta^2(1-\beta^2)}{u_0}\eta_1\eta_2$$
$$+ \frac{1}{24}\frac{\beta^2(1-\beta^2)(1+\beta^2)}{u_0^2}\eta_1^3$$

$$\eta_3 = \frac{1}{16}\frac{(1-\beta^2)}{u_0}\eta_1\eta_2 - \frac{1}{192}\frac{(1-\beta^2)(1+\beta^2)}{u_0^2}\eta_1^3$$

24
8
192

So $\eta_3 = \frac{1}{16}\frac{(1-\beta^2)}{u_0}\eta_1 \boxed{\frac{1}{12}\frac{(1-\beta^2)}{u_0}\eta_1^2} - \frac{1}{192}\frac{(1-\beta^2)(1+\beta^2)}{u_0^2}\eta_1^3$

$$= \frac{1}{192}\frac{(1-\beta^2)}{u_0^2}\eta_1^3\left[\underbrace{(1-\beta^2)-(1+\beta^2)}_{-2\beta^2}\right]$$

$$\boxed{= \frac{-1}{96}\frac{\beta^2(1-\beta^2)}{u_0^2}\eta_1^3}$$

$8\cdot 12 = 96$

---

(2) continued:

$$0 = -\frac{1}{2}\frac{\beta^2(1-\beta^2)}{u_0}\left(2\eta_0\eta_1 + \eta_1\eta_2\right) + \frac{1}{8}\frac{\beta^2(1-\beta^2)(1+\beta^2)}{u_0^2}\eta_1^3$$

$$= \frac{\beta^2(1-\beta^2)}{u_0}\left[-\eta_0\eta_1 - \frac{1}{2}\eta_1\eta_2 + \frac{1}{8}(1+\beta^2)\frac{1}{u_0}\eta_1^3\right]$$

$$= \frac{\beta^2(1-\beta^2)}{u_0}\left[\frac{1}{4}\frac{(1-\beta^2)}{u_0}\eta_1^3 - \frac{1}{2}\eta_1\frac{1}{12}\frac{(1-\beta^2)}{u_0}\eta_1^3 + \frac{1}{8}(1+\beta^2)\frac{1}{u_0}\eta_1^3\right]$$

$$= \frac{\beta^2(1-\beta^2)}{u_0^2}\eta_1^3\left[\underbrace{\frac{1}{4}(1-\beta^2) - \frac{1}{24}(1-\beta^2) + \frac{1}{8}(1+\beta^2)}\right]$$

$$= \frac{1}{4} - \frac{1}{24} + \frac{1}{8} - \beta^2\left(\frac{1}{4} - \frac{1}{24} - \frac{1}{8}\right)$$

$$= \left(\frac{6-1+3}{24}\right) - \beta^2\left(\frac{6-1-3}{24}\right)$$

$$= \frac{1}{3} - \beta^2\frac{1}{12}$$

$$= \frac{1}{12}(4-\beta^2)$$

So

$$\boxed{0 = \frac{\beta^2(1-\beta^2)(4-\beta^2)}{12 u_0^2} \eta_1^3}$$

$\rightarrow \quad \beta = 0 \quad , \quad \beta = 1 \quad , \quad \beta = 2$

Recall: $\quad F(r) = \dfrac{A}{r^{3-\beta^2}}$

So: $\beta = 0 \rightarrow F(r) = \dfrac{A}{r^3}$

$\beta = 1 \rightarrow F(r) = \dfrac{A}{r^2}$ (inverse-square-law)

$\beta = 2 \rightarrow F(r) = Ar$ (harmonic oscillator)

$\beta = 0$: $\quad F(r) = \dfrac{A}{r^3} \quad , \quad U(r) = -\int_\infty^r F(r)\, dr$

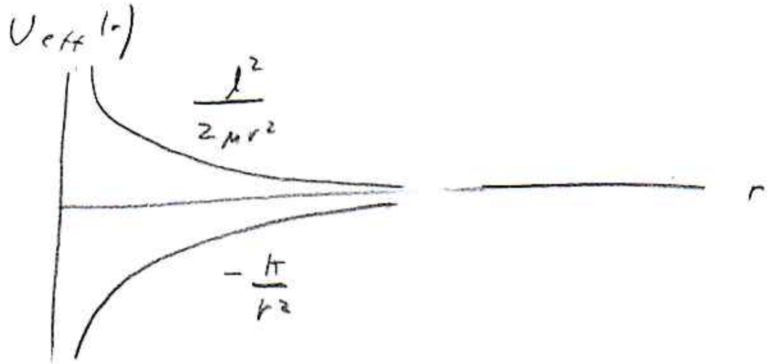
$$= -\int_\infty^r \frac{A}{r^3}\, dr$$

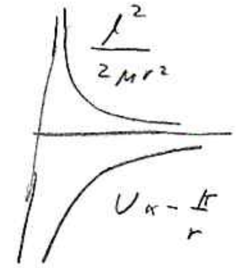
$$= +\frac{A}{2} \frac{1}{r^2} \Big|_\infty^r$$

$$= \frac{A}{2} \frac{1}{r^2}$$

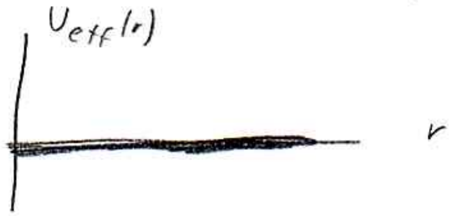
$$\rightarrow \boxed{U(r) \propto \frac{1}{r^2}}$$

So $U_{eff}(r) = \frac{-H}{r^2} + \frac{l^2}{2\mu r^2} = \frac{1}{r^2}\left(-H + \frac{l^2}{2\mu}\right)$



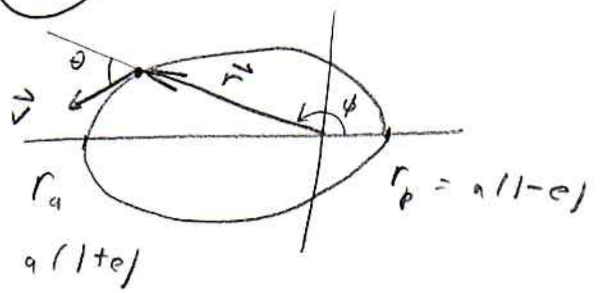


So only solution has $\frac{l^2}{2\mu} - H = 0$ so



Problem: Expression for apapsis (1)

4.4



Cons. of angular momentum:

$$|\vec{\ell}| = |\mu \vec{r} \times \vec{v}| = \mu r v \sin\theta$$
$$= \mu r_a v_a$$
$$= \mu r_p v_p$$

(since $\theta = \frac{\pi}{2}$ at either apapsis or periapsis)

Conservation of Energy:

$$E = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r}$$
$$= \frac{1}{2}\mu v_a^2 - \frac{GM\mu}{r_a}$$

substitute for v_a using $v_a = \left(\frac{r}{r_a}\right) v \sin\theta$

$$\rightarrow \frac{1}{2}\mu v^2 - \frac{GM\mu}{r} = \frac{1}{2}\mu\left(\frac{r}{r_a}\right)^2 v^2 \sin^2\theta - \frac{GM\mu}{r_a}$$

$$\cancel{\left(\frac{1}{2}\mu v^2 - \frac{GM\mu}{r}\right) r_a^2 = \frac{1}{2}\mu r^2 v^2 \sin^2\theta - GM\mu r_a}$$

$$\cancel{\left(\frac{1}{2}\mu v^2 - \frac{GM\mu}{r}\right) r_a^2 + GM\mu r_a - \frac{1}{2}\mu r^2 v^2 \sin^2\theta} = 0$$

$$\left(\frac{1}{r_a}\right)^2\left(\frac{1}{2}\mu r^2v^2\sin^2\theta\right) - \frac{1}{r_a}GM\mu - \left(\frac{1}{2}\mu v^2 - \frac{GM\mu}{r}\right) = 0$$

solve for $\frac{1}{r_a}$:

$$\frac{1}{r_a} = \frac{GM\mu \pm \sqrt{(-GM\mu)^2 + 4\left(\frac{1}{2}\mu r^2v^2\sin^2\theta\right)\left(\frac{1}{2}\mu v^2 - \frac{GM\mu}{r}\right)}}{2\left(\frac{1}{2}\mu r^2v^2\sin^2\theta\right)}$$

$$= \frac{GM\mu \pm \sqrt{G^2M^2\mu^2 + 2\mu r^2v^2\sin^2\theta\left(\frac{1}{2}\mu v^2 - \frac{GM\mu}{r}\right)}}{\mu r^2v^2\sin^2\theta}$$

$$= \frac{GM\mu\left[1 \pm \sqrt{1 + \frac{2\mu r^2v^2\sin^2\theta}{G^2M^2\mu^2}\left(\frac{1}{2}\mu v^2 - \frac{GM\mu}{r}\right)}\right]}{\mu r^2v^2\sin^2\theta}$$

$$= \frac{GM}{r^2v^2\sin^2\theta}\left[1 \pm \sqrt{1 + \frac{r^2v^4\sin^2\theta}{G^2M^2} - \frac{2rv^2\sin^2\theta}{GM}}\right]$$

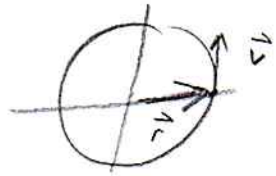
r_a is large, so take $-\sqrt{\ }$
r_p is small, so take $+\sqrt{\ }$

Thus, $$r_{a,p} = \frac{r^2v^2\sin^2\theta}{GM}\left[1 \mp \sqrt{1 + \frac{r^2v^4\sin^2\theta}{G^2M^2} - \frac{2rv^2\sin^2\theta}{GM}}\right]$$

b) Circular orbit

$r = a$ (always)

$\vec{r} \perp \vec{v}$: $\theta = \frac{\pi}{2}$ $\rightarrow \sin\theta = 1$



Conservation of energy equation:

$$\frac{v^2}{Gm} = \frac{2}{r} - \frac{1}{a} \rightarrow \boxed{\underbrace{\frac{rv^2}{Gm}}_{=1} = 2 - \frac{r}{a}}$$

Thus

$$RHS = \frac{r^2v^2}{Gm}\left[1 - \sqrt{1 - \frac{2rv^2}{Gm} + \frac{r^2v^4}{G^2m^2}}\right]^{-1}$$

$$= r\underbrace{\left(\frac{rv^2}{Gm}\right)}_{1}\left[1 - \sqrt{\underbrace{\cancel{1-2+(1)^2}}_{0}}\right]^{-1}$$

$$= r$$

$$LHS = r_a = a = r$$

Virial theorem:

$$\overline{T} = -\frac{1}{2}\overline{U} \qquad \text{inverse-square-law force}$$

$$\cancel{\frac{1}{2}}\mu v^2 = -\cancel{\frac{1}{2}}\left(-\frac{GM\mu}{r}\right)$$

$$v^2 = \frac{GM}{r}$$

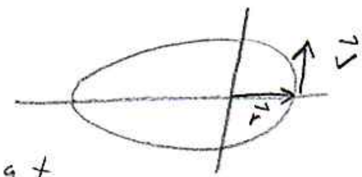
or $\frac{rv^2}{Gm} = 1$ (Which we already found from conservation of energy equation with $r=a$)

④

c) Periapsis:

$r = r_p = a(1-e)$

$v = v_p$ (note $\vec{r}$, $\vec{v}$ $\perp$ at periapsis)



$$RHS = \frac{r_p^2 v_p^2}{GM}\left[1 - \sqrt{1 - \frac{2r_p v_p^2}{GM} + \frac{r_p^2 v_p^4}{G^2M^2}}\,\right]^{-1}$$

$$= \frac{r_p^2 v_p^2}{GM}\left[1 - \left|1 - \frac{r_p v_p^2}{GM}\right|\right]^{-1}$$

$$= \frac{r_p^2 v_p^2}{GM}\left[2 - \frac{r_p v_p^2}{GM}\right]^{-1} \leftarrow \left(\text{see below for } \left|1-\frac{r_p v_p^2}{GM}\right| = \frac{r_p v_p^2}{GM} - 1\right)$$

$$= r_p \left(\frac{GM}{r_p v_p^2}\right)^{-1}\left[2 - \frac{r_p v_p^2}{GM}\right]^{-1}$$

$$= r_p^2 \left[\frac{2GM}{r_p v_p^2} - 1\right]^{-1}$$

NOTE: $\dfrac{v^2}{GM} = \dfrac{2}{r} - \dfrac{1}{a} \rightarrow \dfrac{rv^2}{GM} = 2 - \dfrac{r}{a}$

At periapsis: $\dfrac{r_p v^2}{GM} = 2 - \dfrac{a(1-e)}{a} = 1+e > 1$

Thus, $\left|1 - \dfrac{r_p v_p^2}{GM}\right| = \dfrac{r_p v_p^2}{GM} - 1$

Problem: Lunar injection orbit ①

(4.5)

Recall: $E = -\frac{GM_M}{2a}$

a) Initially, $r = a_i = r_p$ (circular orbit)

$$E_i = -\frac{GM_M}{2r_p}$$

After translunar injection:

$$E_f = -\frac{GM_M}{2a}, \quad r_p = a(1-e) = 6.71 \times 10^6 \text{ m}$$
$$r_a = a(1+e) = 4 \times 10^8 \text{ m}$$

Thus, $a = \frac{1}{2}(r_p + r_a) = 2.03 \times 10^8$ m

$$e = \frac{r_a - r_p}{2a} = \frac{r_a - r_p}{r_a + r_p} = \boxed{0.967}$$

Initial velocity

$$-\frac{GM_M}{2r_p} = -\frac{GM_M}{r_p} + \frac{1}{2}\mu v_i^2$$

$$\frac{GM}{\cancel{2}r_p} = \frac{1}{\cancel{2}} v_i^2$$

$$\rightarrow \boxed{v_i = \sqrt{\frac{GM}{r_p}}}$$

Final velocity ②

$$-\frac{GM\cancel{m}}{2a} = -\frac{GM\cancel{m}}{r_p} + \frac{1}{2}\cancel{m}v_f^2$$

$$GM\left(\frac{1}{r_p} - \frac{1}{2a}\right) = \frac{1}{2}v_f^2$$

$$\boxed{v_f = \sqrt{2GM\left(\frac{1}{r_p} - \frac{1}{2a}\right)}} = \sqrt{\frac{GM(2a-r_p)}{r_p a}}$$

numerical values

$G = 6.67 \times 10^{-11}$ mks
$M = 5.97 \times 10^{24}$ kg

$\rightarrow v_i = 7.4 \times 10^3$ m/s
$v_f = 1.04 \times 10^4$ m/s
$\Delta v \equiv v_f - v_i = 3.1 \times 10^3$ m/s

Alternative calculation:

From previous problem:

$$r_a = r_p\left[\frac{2GM}{r_p v_p^2} - 1\right]^{-1}$$

where p, a stand for perigee and apogee

$$\rightarrow \quad \frac{2GM}{r_p v_p^2} - 1 = \frac{r_p}{r_a}$$

$$\frac{2GM}{r_p v_p^2} = 1 + \frac{r_p}{r_a}$$

$$\rightarrow \quad v_p = \sqrt{\frac{2GM}{r_p\left(1+\frac{r_p}{r_a}\right)}}$$

$$= \sqrt{\frac{2GM}{\frac{r_p}{r_a}(r_a+r_p)}}$$

$$= \sqrt{\frac{\cancel{2}GM}{\frac{r_p}{r_a}\cancel{2}a}}$$

$$= \sqrt{\frac{GM}{r_p \cdot a}\, r_a}$$

$$= \sqrt{\frac{GM}{r_p\, a}(2a-r_p)}$$

$r_p = a(1-e)$
$r_a = a(1+e)$
$r_p + r_a = 2a$
$r_a = 2a - r_p$

which agrees with what we found for v_f.

b) $\dfrac{P^2}{a^3} = \dfrac{4\pi^2}{GM}$ — total mass of Earth + spacecraft $\approx M_{Earth}$

time to apogee $= \frac{1}{2} P$

$$= \frac{1}{2}\sqrt{\frac{4\pi^2 a^3}{GM}}$$

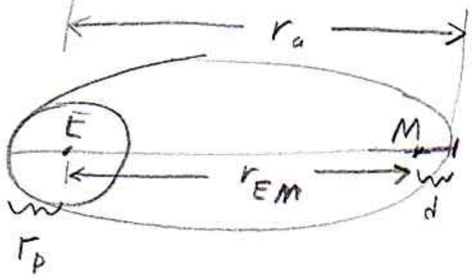
$$= \frac{1}{\cancel{2}}\cancel{2}\pi\sqrt{\frac{a^3}{GM}}$$

$$= \pi\sqrt{\frac{a^3}{GM}}$$

$$= 4.7\times10^5\,s\left(\frac{1\,hr}{3600\,s}\right)\left(\frac{day}{24\,hr}\right)$$

$$= \boxed{5.28\ days}$$

c)



$$r_p + r_{Em} + d = 2a$$

$$\rightarrow \quad d = 2a - r_p - r_{Em}$$

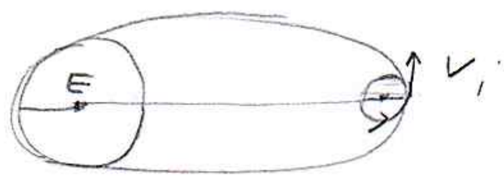
$$= 2(2.03 \times 10^8 \text{ m}) - 6.71 \times 10^6 - 3.84 \times 10^8 \text{ m}$$

$$= 4 \times 10^8 \text{ m} - 3.84 \times 10^8 \text{ m}$$

$$= 0.16 \times 10^8 \text{ m}$$

$$= \boxed{1.6 \times 10^7 \text{ m}}$$

d)



want to place space craft in circular orbit around the moon at height d from center of moon

$$E_f = -\frac{G M_{moon} \cdot m}{2d}$$

Final velocity

$$-\frac{G M_{moon} m}{2d} = -\frac{G M_{moon} m}{d} + \frac{1}{2} m v_f^2$$

$$v_f = \sqrt{\frac{G M_{moon}}{d}}$$

Initial velocity (at apogee of elliptical orbit around Earth

$$-\frac{GM_E \cancel{m}}{2a} = -\frac{GM_E \cancel{m}}{r_a} + \frac{1}{2}\cancel{m} v_a^2$$

$$GM_E\left(\frac{1}{r_a} - \frac{1}{2a}\right) = \frac{1}{2} v_a^2$$

$$v_a = \sqrt{2GM_E\left(\frac{1}{r_a} - \frac{1}{2a}\right)}$$

$$= \sqrt{GM_E \frac{(2a - r_a)}{r_a \cdot a}}$$

Numerical values:

$$v_f = 553 \text{ m/s}$$

$$v_a = 181 \text{ m/s}$$

$$\Delta v \equiv v_f - v_a = \boxed{372 \text{ m/s}}$$

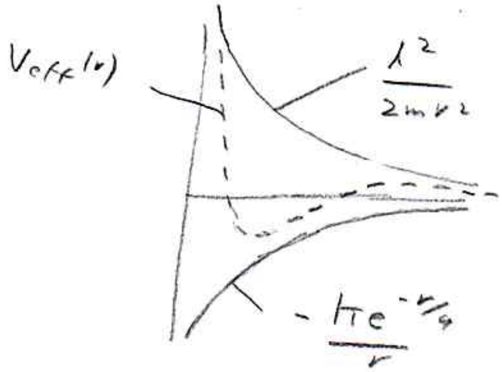
so one has to actually increase the speed of the spacecraft to keep it in a circular orbit around the moon.

```
% script for lunar injection problem

% constants
G = 6.67e-11; % MKS
ME = 5.97e24; % kg
MM = 7.35e22; % kg
rEM = 3.84e8; % m

% perigee and apogee for translunar injection orbit
rp = 6.71e6; % m
ra = 4e8; % m

a = 0.5*(rp+ra);
e = (ra-rp)/(ra+rp);

fprintf('a = %g m, e = %1.3f\n', a, e)

% calculate initial and final velocities (at perigee)
vi = sqrt(G*ME/rp);
vp = sqrt(G*ME*(2*a-rp)/(rp*a));
dv = vp-vi;

% alternative caculation of vp
%vp = sqrt(2*G*ME/(rp*(1+rp/ra)));

fprintf('vi = %g m/s, vp = %g m/s, deltavee = %g m/s\n', vi, vp, dv)

% calculate time to apogee
P = sqrt(4*pi^2*a^3/(G*ME)); % kepler's 3rd law (ignoring mass of
spacecraft)
Ta = 0.5*P;
Ta_days = Ta*(1/3600)*(1/24);

fprintf('time to apogee = %1.3f days\n', Ta_days)

% calculate initial and final velocities (at apogee) for lunar orbit
insertion
d = ra - rEM;
vf = sqrt(G*MM/d);
va = sqrt(G*ME*(2*a-ra)/(ra*a));
dv = vf-va;

% alternative caculation of va
%va = sqrt(G*ME*(1-e)/(a*(1+e)))

fprintf('va = %g m/s, vf = %g m/s, deltavee = %g m/s\n', va, vf, dv)
```

Problem 4.6 Yukawa potential

$$U(r) = -\frac{k e^{-r/a}}{r}$$

$$U_{eff}(r) = -\frac{k e^{-r/a}}{r} + \frac{l^2}{2mr^2}$$



(a) For $r \to \infty$: $e^{-r/a} \to 0$ faster than $\frac{1}{r^n}$ for any n

Thus, $U_{eff}(r) \simeq \frac{l^2}{2mr^2} > 0$ as $r \to \infty$

For $r \to 0$: $e^{-r/a} \to 1$. Also, $\frac{1}{r^2} \to +\infty$ faster than $-\frac{1}{r} \to -\infty$.

Thus, $V_{eff}(r) \simeq \frac{l^2}{2mr^2} \to +\infty$ as $r \to 0$.

(b) For circular orbits:

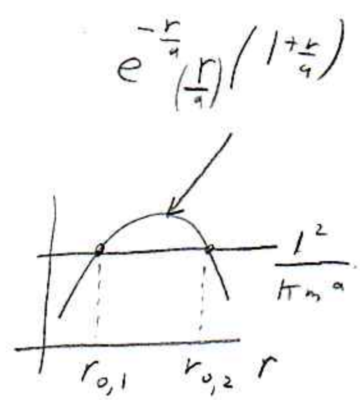
$$0 = \frac{dU_{eff}}{dr}\Big|_{r=r_0}$$

$$= \left[+ \frac{ke^{-r/a}}{r^2} + \frac{k}{ar} e^{-r/a} - \frac{l^2}{mr^3} \right]\Big|_{r=r_0}$$

$$= \left[\frac{ke^{-r/a}}{r^2}\left(1 + \frac{r}{a}\right) - \frac{l^2}{mr^3} \right]\Big|_{r=r_0}$$

Thus,
$$\frac{l^2}{mr_0^3} = \frac{ke^{-r_0/a}}{r_0^2}\left(1 + \frac{r_0}{a}\right)$$

$$\underbrace{\frac{l^2}{kma} = e^{-r_0/a}\left(\frac{r_0}{a}\right)\left(1 + \frac{r_0}{a}\right)}_{\text{implicit equation for } r_0 \text{ that can be solved graphically}}$$



Maximum value of ℓ occurs at maximum of $f(x) = e^{-x}(x+x^2)$, where $x = \left(\frac{r}{a}\right)$.

Thus,
$$f'(x) = -e^{-x}(x+x^2) + e^{-x}(1+2x)$$
$$= e^{-x}[1+2x-x-x^2]$$
$$= e^{-x}[1+x-x^2]$$

$f'(x) = 0$ iff
$$x^2 - x - 1 = 0$$
$$x = \frac{1 \pm \sqrt{1-4\cdot 1\cdot(-1)}}{2}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$

Thus, $\boxed{r_{0,\pm} = a\left(\frac{1\pm\sqrt{5}}{2}\right)}$ $= \begin{cases} 1.618a & (+ \text{ sign}) \\ -0.618a & (- \text{ sign}) \end{cases}$

Check to see which of these corresponds to a maximum:

$$f''(x) = -e^{-x}[1+x-x^2] + e^{-x}[1-2x]$$
$$= -e^{-x}[\cancel{1}+x-x^2 - \cancel{1}+2x]$$
$$= -e^{-x}[3x-x^2]$$
$$= -\underbrace{x}_{}\underbrace{e^{-x}}_{>0}\underbrace{[3-x]}_{>0 \text{ for both } r_{0,\pm}}$$

>0 for $+$ sign
<0 for $-$ sign

Thus, $f''(x) < 0$ for $x = \frac{1+\sqrt{5}}{2}$

So only $\boxed{r_0 = a\left(\frac{1+\sqrt{5}}{2}\right)}$ corresponds to maximum value of ℓ

(c) ~~B~~ Does $r = r_0$ correspond to a stable circular orbit? (3)

$$\frac{dU_{eff}}{dr} = \frac{k e^{-r/a}}{r^2}\left(1 + \frac{r}{a}\right) - \frac{l^2}{mr^3}$$

$$= \frac{k}{a^2} \frac{e^{-r/a}}{\left(\frac{r}{a}\right)^2}\left(1 + \left(\frac{r}{a}\right)\right) - \frac{l^2}{ma^3\left(\frac{r}{a}\right)^3}$$

$$= \frac{k}{a^2} \frac{e^{-x}}{x^2}(1+x) - \frac{l^2}{ma^3x^3}$$

$$\frac{d^2U_{eff}}{dr^2} = \frac{1}{a}\frac{d}{dx}\left[\frac{dU_{eff}}{dr}\right]$$

$\frac{d}{dr} = \frac{dx}{dr}\frac{d}{dx}$

$x = \frac{r}{a}$

$\frac{dx}{dr} = \frac{1}{a}$

$$= \frac{1}{a}\left[\frac{k}{a^2}\left(\frac{-e^{-x}}{x^2}\right)(1+x) - 2\frac{k}{a^2}\frac{e^{-x}}{x^3}(1+x) + \frac{k}{a^2}\frac{e^{-x}}{x^2} + \frac{3l^2}{ma^3x^4}\right]$$

$$= \frac{1}{a}\left[\left(\frac{k}{a^2}\right)\frac{e^{-x}}{x^3}\left(-x(1+x) - 2(1+x) + x\right) + \frac{3l^2}{ma^3x^4}\right]$$

$$= \frac{1}{a}\left[\left(\frac{k}{a^2}\right)\frac{e^{-x}}{x^3}\left(-\cancel{x} - x^2 - 2 - 2x + \cancel{x}\right) + \frac{3l^2}{ma^3x^4}\right]$$

$$= \frac{1}{a}\frac{1}{x^4}\left[-\left(\frac{k}{a^2}\right)e^{-x}(x^2+2x+2)x + \frac{3l^2}{ma^3}\right]$$

stability requires that $\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} > 0$ (+)

Recall: $\frac{\ell^2}{Kma} = e^{-\frac{r_0}{a}}\left(\frac{r_0}{a}\right)\left(1+\frac{r_0}{a}\right)$ has max value at $r_0 = a\left(\frac{1+\sqrt{5}}{2}\right) = 1.618a$

$$\left.\frac{d^2 U_{eff}}{dr^2}\right|_{r=r_0=a\left(\frac{1+\sqrt{5}}{2}\right)} = \frac{K}{a^3}\frac{1}{x^4}\left[-e^{-x}(x^2+2x+2)x + \frac{3\ell^2}{Kma}\right]$$

$$= \frac{K}{a^3}\frac{1}{x^4}\left[-e^{-x}(x^2+2x+2)\underline{\underline{x}} + 3e^{-x}\underline{\underline{x}}(1+x)\right]$$

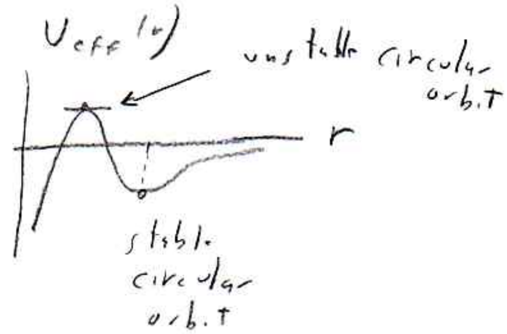
$$= \underbrace{\frac{K}{a^3}\frac{1}{x^3}e^{-x}}_{>0}\underbrace{\left[-x^2-2x-2+3+3x\right]}_{\underbrace{[-x^2+x+1]}_{\text{but this } = 0 \text{ at } x=\left(\frac{1+\sqrt{5}}{2}\right)}}$$

$$= 0$$

so $\boxed{\left.\frac{d^2 U_{eff}}{dr^2}\right|_{r_0=a\left(\frac{1+\sqrt{5}}{2}\right)} = 0}$ $\Rightarrow$ $r_0 = a\left(\frac{1+\sqrt{5}}{2}\right)$ does not correspond to a stable circular orbit

Problem 4.7 GR effective potential

$$U_{eff}(r) = \frac{l^2}{2mr^2}\left(1 - \frac{2GM}{rc^2}\right) - \frac{GMm}{r}$$



a) stable circular orbit

$$0 = \frac{dU_{eff}}{dr}$$

$$= \frac{-\cancel{2}l^2}{\cancel{2}mr^3} + 3\frac{l^2GM}{mc^2}\frac{1}{r^4} + \frac{GMm}{r^2}$$

$$= \frac{1}{r^4}\left[GMmr^2 - \frac{l^2}{m}r + \frac{3l^2GM}{mc^2}\right]$$

$$= \frac{GMm}{r^4}\left[r^2 - \frac{l^2}{GMm^2}r + \frac{3l^2}{m^2c^2}\right]$$

solve quadratic equation:

$$r = \frac{\frac{l^2}{GMm^2} \pm \sqrt{\frac{l^4}{G^2M^2m^4} - 4\left(\frac{3l^2}{m^2c^2}\right)}}{2}$$

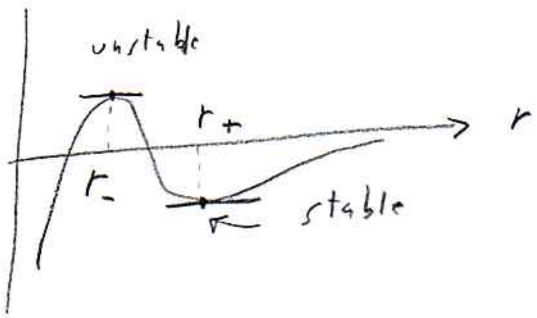
$$= \frac{1}{2}\left[\frac{l^2}{GMm^2} \pm \frac{l^2}{GMm^2}\sqrt{1 - 4\left(\frac{3l^2}{m^2c^2}\right)\frac{G^2M^2m^4}{l^4}}\right]$$

$$= \frac{1}{2}\frac{l^2}{GMm^2}\left[1 \pm \sqrt{1 - 12\cdot\frac{G^2M^2m^2}{l^2c^2}}\right]$$

Thus,

$$r_+ = \frac{1}{2}\left(\frac{l}{m}\right)^2 \frac{1}{GM}\left[1 + \sqrt{1 - 12\frac{G^2M^2}{c^2(l/m)^2}}\right]$$

$$r_- = \frac{1}{2}\left(\frac{l}{m}\right)^2 \frac{1}{GM}\left[1 - \sqrt{1 - 12\frac{G^2M^2}{c^2\left(\frac{l}{m}\right)^2}}\right]$$



b) minimum value of l for stable circular orbit

$$1 - 12\frac{G^2M^2}{c^2\left(\frac{l}{m}\right)^2} = 0$$

$$c^2\left(\frac{l}{m}\right)^2 = 12\,G^2M^2$$

$$\longrightarrow \boxed{\left(\frac{l}{m}\right)\Bigg|_{min} = \sqrt{\frac{12\,G^2M^2}{c^2}}} = \sqrt{12}\,\frac{GM}{c}$$

c) radius for minimum stable circular orbit:

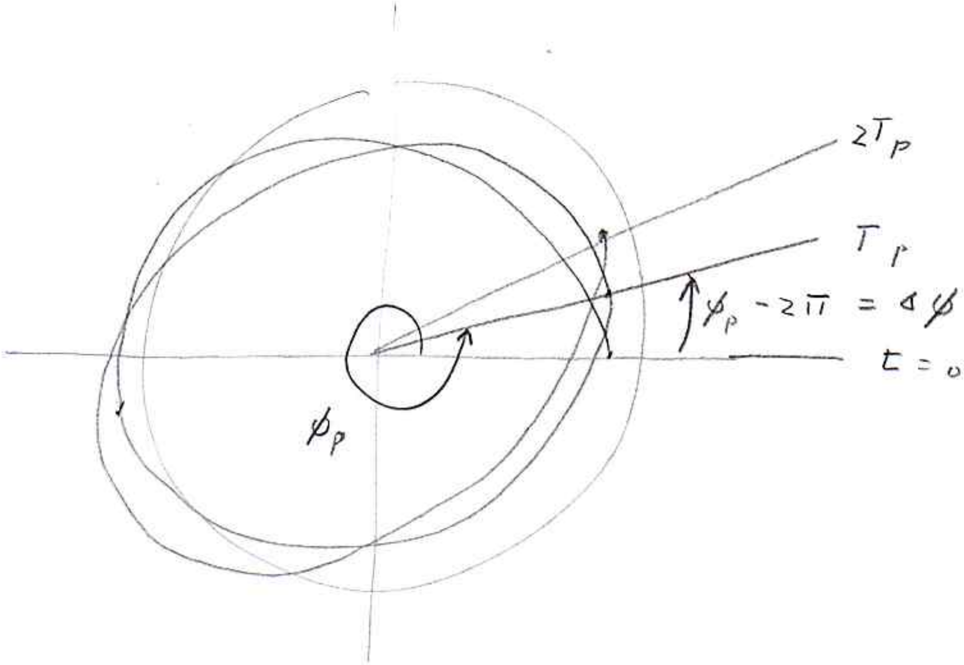
$$r_+\Big|_{min} = \frac{1}{2}\left(\frac{12\,G^2M^2}{c^2}\right)\frac{1}{GM}$$

$$= 6\,\frac{GM}{c^2}$$

$$= 3\,R_{schwarzschild}$$

Problem: **Precession rate**

(4.8)



Time from periapse to periapse

$$T_p = \frac{\phi_p}{\omega_0} \quad , \qquad \omega_0 \equiv \frac{d\phi}{dt} = \frac{l}{\mu r_0^2} \quad \text{(for circular orbit)}$$

Now:
$$\Delta\phi = \phi_p - 2\pi = \frac{2\pi}{\beta} - 2\pi = 2\pi\left(\frac{1}{\beta} - 1\right)$$

So:
$$\text{precession rate} = \frac{\Delta\phi}{T_p}$$
$$= \frac{2\pi\left(\frac{1}{\beta}-1\right)}{\phi_p}$$
$$= \frac{\cancel{2\pi}\left(\frac{1}{\beta}-1\right)\omega_0}{\cancel{2\pi}/\beta}$$
$$= (1-\beta)\,\omega_0$$
$$= \boxed{(1-\beta)\frac{l}{\mu r_0^2}}$$

Problem 4.9 closed orbits for Yukawa potential

$$U(r) = -\frac{k e^{-r/a}}{r}, \quad r = \frac{1}{u}$$

$$U(\tfrac{1}{u}) = -ku\, e^{-\frac{1}{ua}}$$

a) $$\beta^2 = 1 - \frac{d\Lambda}{du}\bigg|_{u_0}$$

$$\Lambda(u) = -\frac{\mu}{l^2}\frac{dU(\frac{1}{u})}{du}$$

$$= -\frac{\mu}{l^2}\frac{d}{du}\left[-ku\, e^{-\frac{1}{ua}}\right]$$

$$= -\frac{\mu}{l^2}\left[-ke^{-\frac{1}{ua}} - kue^{-\frac{1}{ua}}\left(\frac{1}{u^2 a}\right)\right]$$

$$= +\frac{\mu}{l^2} ke^{-\frac{1}{ua}}\left[1 + \frac{1}{ua}\right]$$

$$\rightarrow \quad \beta^2 = 1 - \frac{d}{du}\left[\frac{\mu}{l^2}k\, e^{-\frac{1}{ua}}\left(1 + \frac{1}{ua}\right)\right]\bigg|_{u_0}$$

$$= 1 - \frac{\mu k}{l^2}\left(\frac{1}{u^2 a}\right)e^{-\frac{1}{ua}}\left(1 + \frac{1}{ua}\right)\bigg|_{u_0}$$

$$- \frac{\mu k}{l^2} e^{-\frac{1}{ua}}\left(-\frac{1}{u^2 a}\right)\bigg|_{u_0}$$

$$= 1 - \frac{\mu k}{l^2} e^{-\frac{1}{u_0 a}}\left(\frac{1}{u_0^2 a}\right)\left[\cancel{1} + \frac{1}{u_0 a} - \cancel{1}\right]$$

$$= 1 - \frac{\mu k}{l^2}\frac{1}{u_0^3 a^2} e^{-\frac{1}{u_0 a}}$$

Thus, $$\boxed{\beta = \left[1 - \frac{\mu k}{l^2}\frac{1}{u_0^3 a^2} e^{-\frac{1}{u_0 a}}\right]^{1/2}}$$

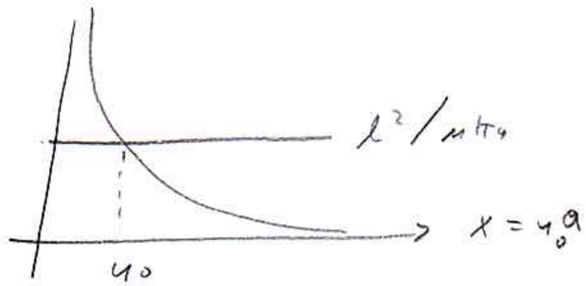
(2)

b) $\underline{\beta = 0}$:

$$0 = \left[1 - \frac{\mu k}{\ell^2} \frac{1}{u_0^3 a^2} e^{-\frac{1}{u_0 a}} \right]^{1/2}$$

$$\rightarrow \quad 1 = \frac{\mu k}{\ell^2} \frac{1}{u_0^3 a^2} e^{-\frac{1}{u_0 a}}$$

$$\boxed{\frac{\ell^2}{\mu k a} = \frac{1}{(u_0 a)^3} e^{-\frac{1}{u_0 a}}}$$ — Transcendental equation for u_0



c) Closed after two orbits : $2\beta = 1 \rightarrow \beta = \frac{1}{2}$

$$\frac{1}{2} = \left[1 - \frac{\mu k}{\ell^2} \frac{1}{u_0^3 a^2} e^{-\frac{1}{u_0 a}} \right]^{\frac{1}{2}}$$

$$\frac{1}{4} = 1 - \frac{\mu k}{\ell^2} \frac{1}{u_0^3 a^2} e^{-\frac{1}{u_0 a}}$$

$$\rightarrow \quad \frac{3}{4} = \frac{\mu k}{\ell^2} \frac{1}{u_0^3 a^2} e^{-\frac{1}{u_0 a}}$$

$$\boxed{\frac{3}{4} \frac{\ell^2}{\mu k a} = \frac{1}{(u_0 a)^3} e^{-\frac{1}{u_0 a}}}$$ — similar transcendental equation

Problem 4.10 GR perihelion precession

$$U(r) = -\frac{GM\mu}{r} - \frac{l^2 GM}{\mu c^2 r^3}$$

(a) Precession rate:

$$\omega_p = (1-\beta)\frac{l}{\mu r_0^2}$$

where $\beta^2 = 1 - \left.\frac{d\Lambda}{du}\right|_{u_0}$

$$\Lambda(u) = -\frac{\mu}{l^2}\frac{dU(\frac{1}{u})}{du}$$

In terms of $u = \frac{1}{r}$

$$U(\tfrac{1}{u}) = -GM\mu u - \frac{l^2 GM}{\mu c^2}u^3$$

$$\frac{dU(\frac{1}{u})}{du} = -GM\mu - \frac{3l^2 GM}{\mu c^2}u^2$$

$$\rightarrow \Lambda(u) = -\frac{\mu}{l^2}\left[-GM\mu - \frac{3l^2 GM}{\mu c^2}u^2\right]$$

$$= \frac{GM\mu^2}{l^2}\left[1 + \frac{3l^2 u^2}{\mu^2 c^2}\right]$$

$$\rightarrow \frac{d\Lambda}{du} = \frac{GM\cancel{\mu^2}}{\cancel{l^2}}\left(\frac{6\cancel{l^2}u}{\cancel{\mu^2}c^2}\right)$$

$$= +6\frac{GM}{c^2}u$$

Thus,
$$\beta^2 = 1 - \left.\frac{d\Lambda}{du}\right|_{u_0}$$
$$= 1 - 6\frac{GM}{c^2}u_0$$
$$= 1 - 6\frac{GM}{r_0 c^2}$$

$$\rightarrow \quad \beta = \sqrt{1 - \frac{6GM}{r_0 c^2}}$$

$$\omega_p = \left(1 - \sqrt{1 - \frac{6GM}{r_0 c^2}}\right)\frac{\ell}{\mu r_0^2}$$

Now: recall the $\ell = \mu r_0^2 \dot{\phi} = \mu r_0^2 \frac{d\phi}{dt}$

$$\rightarrow \quad dt = \frac{\mu r_0^2}{\ell} d\phi$$

$$\boxed{P_{orb} = \frac{2\pi \mu r_0^2}{\ell}} \rightarrow \frac{\ell}{\mu r_0^2} = \frac{2\pi}{P_{orb}}$$

Thus,
$$\boxed{\omega_p = \left(1 - \sqrt{1 - 6\frac{GM}{r_0 c^2}}\right)\frac{2\pi}{P_{orb}}}$$

(b) For Mercury, $M = M_\odot + m_{mercury}$
$$\approx M_\odot$$
$$= 2 \times 10^{30}\ \text{kg}$$

$P_{orb} = 88$ day
$a = 5.8 \times 10^{10}$ m
$= r_0$

$$\rightarrow \quad \omega_p = \frac{2\pi}{88 \text{ days}} \left(1 - \sqrt{1 - \underbrace{6\frac{GM}{r_0 c^2}}_{1.533 \times 10^{-7}}} \right)$$

$$\cong \frac{2\pi}{88 \text{ days}} \left(\cancel{1} - \left(\cancel{1} - \frac{1}{2}(1.533 \times 10^{-7}) \right) \right)$$

$$= \frac{2\pi}{88 \text{ days}} \frac{1}{2} \, 1.533 \times 10^{-7} \left(\frac{365 \text{ days}}{\text{yr}} \right) \left(\frac{100 \text{ yr}}{\text{century}} \right)$$

$$= 2 \times 10^{-4} \frac{\cancel{\text{rad}}}{\text{century}} \left(\frac{360^\circ}{2\pi \cancel{\text{rad}}} \right) \left(\frac{60 \cancel{\text{arc min}}}{\cancel{\text{degree}}} \right) \left(\frac{60 \text{ arc sec}}{\cancel{\text{arc min}}} \right)$$

$$= 41 \text{ arc sec/century}$$

(close to 43 arc sec/century)

Prob 4.11 Plummer potential

a) $$\rho = \frac{3Ma^2}{4\pi\,(r^2+a^2)^{5/2}}$$

$$\int_{\text{all space}} \rho\, dV = \int_0^\infty r^2 dr \underbrace{\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{4\pi} \frac{3Ma^2}{4\pi (r^2+a^2)^{5/2}}$$

$$= \frac{3Ma^2}{\cancel{4\pi}} \cdot \cancel{4\pi} \int_0^\infty dr\, r^2 \frac{1}{(r^2+a^2)^{5/2}}$$

$$= 3Ma^2 \int_0^\infty \frac{r^2 dr}{(r^2+a^2)^{5/2}}$$

Let: $r = a\tan\theta$

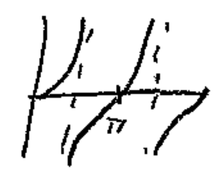
$$r^2+a^2 = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

$$dr = a\, d(\tan\theta)$$

$$= \frac{a}{\cos^2\theta}\, d\theta = a\sec^2\theta\, d\theta$$

$r=0,\ \theta=0$

$r=\infty,\ \theta=\pi/2$



$$\Rightarrow \int_{\text{all space}} \rho\, dV = 3Ma^2 \int_0^{\pi/2} \frac{a^2\tan^2\theta\;\; a\sec^2\theta\, d\theta}{a^5\sec^5\theta}$$

$$= 3M \int_0^{\pi/2} \frac{\tan^2\theta}{\sec^3\theta}\, d\theta$$

$$= 3M \int_0^{\pi/2} \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^3\theta\, d\theta$$

$$= 3M \int_0^{\pi/2} \sin^2\theta \cos\theta \, d\theta$$

$$= 3M \int_0^{\pi/2} \sin^2\theta \, d(\sin\theta)$$

$$= \cancel{3} M \frac{1}{\cancel{3}} \sin^3\theta \Big|_0^{\pi/2}$$

$$= \boxed{M} \quad (\text{total mass})$$

b) Determine mass within radius a:

$$\int_0^a r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \; \rho = \ldots \quad (\text{as above})$$

$$= M \sin^3\theta \Big|_0^{\pi/4}$$

$r = a$
$a \tan\theta = a$
$\tan\theta = 1$
$\theta = \pi/4$

$$= M \left(\frac{1}{\sqrt{2}}\right)^3$$

$\sin\frac{\pi}{4} = .707 = \frac{1}{\sqrt{2}}$

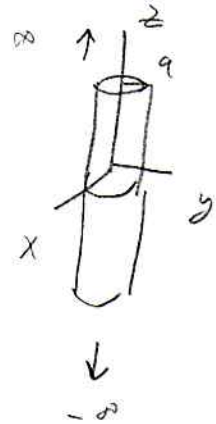
$$= \boxed{\frac{M}{2^{3/2}}}$$

c) $$4\pi \Phi \rho r^2 dr = \underbrace{\overline{U}_v(r)\, dV = -2\overline{T}_v(r)\, dV}_{\text{virial theorem}} = -4\pi\rho\sigma_v^2 r^2 dr$$

Thus, $$\boxed{\sigma_v^2(r) = -\Phi(r) = +\frac{GM}{\sqrt{r^2+a^2}}}$$

Prob 4.12 2-D projection of cluster

a)



$$dV = \rho\, d\rho\, d\phi\, dz$$

$$I = \int \rho\, dV = \int_0^a \rho\, d\rho \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \; \frac{3Ma^2}{4\pi(\rho^2+z^2+a^2)^{5/2}}$$

inf long cylinder radius a

mass density

$$= \frac{3Ma^2}{4\pi} \cdot 2\pi \int_0^a \rho\, d\rho \int_{-\infty}^{\infty} dz \; \frac{1}{(\rho^2+z^2+a^2)^{5/2}}$$

$$= \frac{3}{2} M a^2 \int_0^a \rho\, d\rho \int_{-\infty}^{\infty} \frac{dz}{(z^2+(\rho^2+a^2))^{5/2}}$$

Try: Let $z = (\rho^2+a^2)^{\frac{1}{2}} \tan\theta$

$$dz = (\rho^2+a^2)^{\frac{1}{2}}\, d(\tan\theta) = (\rho^2+a^2)^{\frac{1}{2}} \sec^2\theta\, d\theta$$

$$(z^2+(\rho^2+a^2))^{5/2} = \left((\rho^2+a^2)(1+\tan^2\theta)\right)^{5/2}$$

$$= (\rho^2+a^2)^{5/2} \sec^5\theta$$

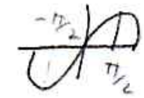
$$z = \pm\infty \leftrightarrow \theta = \pm \pi/2$$

Thus,

$$I = \frac{3}{2} M a^2 \int_0^a \rho\, d\rho \int_{-\pi/2}^{\pi/2} \frac{(\rho^2+a^2)^{\frac{1}{2}} \sec^2\theta\, d\theta}{(\rho^2+a^2)^{5/2} \sec^5\theta}$$

$$= \frac{3}{2} M a^2 \int_0^a \frac{\rho\, d\rho}{(\rho^2+a^2)^2} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sec^3\theta}$$

Now.
$$\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sec^3\theta} = \int_{-\pi/2}^{\pi/2} \cos^3\theta \, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} (1-\sin^2\theta)\cos\theta \, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} (1-\sin^2\theta)\, d(\sin\theta)$$
$$= \left[\sin\theta - \frac{\sin^3\theta}{3}\right]\Bigg|_{-\pi/2}^{\pi/2}$$



$$= \left(1-\frac{1}{3}\right) - \left(-1 - \frac{(-1)}{3}\right)$$
$$= \frac{2}{3} + \frac{2}{3}$$
$$= \boxed{\frac{4}{3}}$$

Thus,
$$I = \frac{3}{2} M a^2 \cdot \frac{4}{3} \int_0^a \frac{\rho \, d\rho}{(\rho^2+a^2)^2}$$

Let: $\rho = a\tan\theta$ $\qquad \rho = 0, a \rightarrow \theta = 0, \frac{\pi}{4}$
$$d\rho = a \, d(\tan\theta) = a\sec^2\theta \, d\theta$$
$$(\rho^2+a^2)^2 = \left[a^2(1+\tan^2\theta)\right]^2 = a^4\sec^4\theta$$

$$\rightarrow \int_0^a \frac{\rho \, d\rho}{(\rho^2+a^2)^2} = \int_0^{\pi/4} \frac{a\tan\theta \; a\sec^2\theta \, d\theta}{a^4\sec^4\theta}$$

$$= \frac{1}{a^2} \int_0^{\pi/4} \frac{\tan\theta}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{a^2} \int_0^{\pi/4} \frac{\sin\theta}{\cos\theta} \cdot \cos^2\theta\, d\theta$$

$$= \frac{1}{a^2} \int_0^{\pi/4} \sin\theta \cos\theta\, d\theta$$

$$= \frac{1}{a^2} \int_0^{\pi/4} \sin\theta\, d(\sin\theta)$$

$$= \frac{1}{a^2} \left. \frac{\sin^2\theta}{2} \right|_0^{\pi/4}$$

$$= \frac{1}{a^2} \frac{1}{2} \left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \boxed{\frac{1}{4a^2}}$$

$\frac{1}{2} \sin 2\theta \, d\theta$

$\frac{1}{2} \cdot -\frac{1}{2} \cos 2\theta \Big|_0^{\pi/4} = \left(-\frac{1}{2}(0-1)\right)\frac{1}{2}$

$= \frac{1}{2} \cdot \frac{1}{2}$

Thus,

$$I = \frac{3}{2} M \cancel{a^2} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \frac{1}{\cancel{4} \cancel{a^2}}$$

$$= \boxed{\frac{M}{2}}$$

b) $\sigma_V^2(r) = -\Phi(r) = \dfrac{GM}{\sqrt{r^2+a^2}}$ (From Prob 4.11, part (c))

Assume: $3\sigma_r^2 = \sigma_V^2(0) = \dfrac{GM}{a}$ (a constant)

$$\rightarrow \boxed{M = \frac{3\sigma_r^2 a}{G}}$$

Now: $\sigma_V^2(0)$ overestimates the velocity dispersion

Thus, σ_r^2 is overestimated $\rightarrow$ M is overestimated.

Poisson brackets of Runge-Lenz Vector Prob (4.13) ①

$$\vec{A} = \vec{p} \times \vec{l} - k\mu \hat{r}$$
$$= \vec{p} \times \vec{l} - GM\mu^2 \left(\frac{\vec{r}}{r}\right)$$

Recall: $\{l_i, l_j\} = \epsilon_{ijk} l_k$

where $\vec{l} = \vec{r} \times \vec{p}$

$$\{l_i, p_j\} = \{\epsilon_{imn} x_m p_n, p_j\}$$
$$= \epsilon_{imn} x_m \underbrace{\{p_n, p_j\}}_{0} + \epsilon_{imn} p_n \{x_m, p_j\}$$
$$= \epsilon_{imn} p_n \delta_{mj}$$
$$= \epsilon_{ijn} p_n$$
$$= \boxed{\epsilon_{ijk} p_k}$$

$$\{l_i, x_j\} = \{\epsilon_{imn} x_m p_n, x_j\}$$
$$= \epsilon_{imn} p_n \underbrace{\{x_m, x_j\}}_{0} + \epsilon_{imn} x_m \{p_n, x_j\}$$
$$= -\epsilon_{imn} x_m \delta_{jn}$$
$$= -\epsilon_{imj} x_m$$
$$= \boxed{\epsilon_{ijk} x_k}$$

$$\{l_i, r\} = \{l_i, \delta_{jk} x_j x_k\}$$
$$= \delta_{jk} x_j \{l_i, x_k\} + \delta_{jk} x_k \{l_i, x_j\}$$
$$= \delta_{jk} x_j \epsilon_{ikl} x_l + \delta_{jk} x_k \epsilon_{ijl} x_l$$

②

$$= \epsilon_{ij\cancel{\ell}} x_j x_\ell + \epsilon_{ik\ell} \cancel{x_k x_\ell}$$

$$= \boxed{0}$$

$$\{ l_i , A_j \} = \{ l_i , \epsilon_{jk\ell} p_k l_\ell - k\mu \frac{x_j}{r} \}$$

$$= \epsilon_{jk\ell} \{ l_i , p_k \} l_\ell + \epsilon_{jk\ell} \{ l_i , l_\ell \} p_k$$

$$- \frac{k\mu}{r} \{ l_i , x_j \} - k\mu x_j \{ l_i , \tfrac{1}{r} \}$$

$$= \epsilon_{jk\ell} \epsilon_{ikm} p_m l_\ell + \epsilon_{jk\ell} \epsilon_{i\ell m} l_m p_k$$

$$- \frac{k\mu}{r} \epsilon_{ijk} x_k + \frac{1}{r^2} k\mu x_j \cancel{\{ l_i , r \}}$$

$$= (\delta_{ji} \delta_{\ell m} - \delta_{jm} \delta_{\ell i}) p_m l_\ell + (\delta_{jm} \delta_{ki} - \delta_{ji} \delta_{km}) l_m p_k$$

$$- \frac{k\mu}{r} \epsilon_{ijk} x_k$$

$$= \cancel{\delta_{ji} \vec{p} \cdot \vec{l}} - p_j l_i + l_j p_i - \cancel{\delta_{ji} \vec{p} \cdot \vec{l}} - \frac{k\mu}{r} \epsilon_{ijk} x_k$$

$$= p_i l_j - l_i p_j - \frac{k\mu}{r} \epsilon_{ijk} x_k$$

$$= \epsilon_{ijk} (\vec{p} \times \vec{l})_k - \epsilon_{ijk} \frac{k\mu x_k}{r}$$

$$= \epsilon_{ijk} [(\vec{p} \times \vec{l})_k - \frac{k\mu}{r} x_k]$$

$$= \boxed{\epsilon_{ijk} A_k}$$

$$\{A_i, x_j\} = \{\epsilon_{ikl}\, p_k l_l - k\mu \frac{x_i}{r},\ x_j\}$$

$$= \epsilon_{ikl}\{p_k, x_j\} l_l + \epsilon_{ikl}\{l_l, x_j\} p_k - k\mu \underbrace{\{\frac{x_i}{r}, x_j\}}_{0}$$

$$= -\epsilon_{ikl}\,\delta_{jk}\, l_l + \epsilon_{ikl}\,\epsilon_{ljm}\, x_m p_k$$

$$= -\epsilon_{ijl}\, l_l + (\delta_{ij}\delta_{km} - \delta_{im}\delta_{kj})\, x_m p_k$$

$$= \boxed{-\epsilon_{ijl}\, l_l + \delta_{ij}\, \vec{p}\cdot\vec{r} - x_i p_j}$$

$$= -\epsilon_{ijl}\,\epsilon_{lmn}\, x_m p_n + \delta_{ij}\, \vec{p}\cdot\vec{r} - x_i p_j$$

$$= -(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\, x_m p_n + \delta_{ij}\, \vec{p}\cdot\vec{r} - x_i p_j$$

$$= -x_i p_j + x_j p_i + \delta_{ij}\, \vec{p}\cdot\vec{r} - x_i p_j$$

$$= \boxed{-2x_i p_j + x_j p_i + \delta_{ij}\, \vec{p}\cdot\vec{r}}$$

$$\{A_i, p_j\} = \{\epsilon_{ikl}\, p_k l_l - k\mu \frac{x_i}{r},\ p_j\} \qquad (r^2)^{\frac{1}{2}}$$

$$= \epsilon_{ikl}\, p_k \{l_l, p_j\} - \frac{k\mu}{r}\{x_i, p_j\} + \frac{k\mu}{r^2} x_i \{r, p_j\}$$

$$= \epsilon_{ikl}\, p_k\, \epsilon_{ljm}\, p_m - \frac{k\mu}{r}\delta_{ij} + \frac{k\mu}{r^2} x_i \frac{1}{\cancel{2}}(r^2)^{-\frac{1}{2}} \underbrace{\{x_k x_k, p_j\}}_{=2x_k\delta_{kj} = \cancel{2}x_j}$$

$$= \epsilon_{ikl}\,\epsilon_{jml}\, p_k p_m - \frac{k\mu}{r}\delta_{ij} + \frac{k\mu}{r^3} x_i x_j$$

$$= (\delta_{ij}\delta_{km} - \delta_{im}\delta_{kj})\, p_k p_m - \frac{k\mu}{r}\delta_{ij} + \frac{k\mu}{r^3} x_i x_j$$

$$= \delta_{ij}\, p^2 - p_i p_j - \frac{k\mu}{r}\delta_{ij} + \frac{k\mu}{r^3} x_i x_j$$

$$= \boxed{p^2\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right) - \frac{k\mu}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)}$$

$$\{A_i, r\} = \{A_i, (r^2)^{\frac{1}{2}}\}$$

$$= \tfrac{1}{2}(r^2)^{-\frac{1}{2}} \{A_i, \overbrace{x_j x_j}^{r^2}\}$$

$$= \frac{1}{\cancel{2}r} \cancel{2} x_j \{A_i, x_j\}$$

$$= \frac{x_j}{r}\left[-2x_i p_j + x_j p_i + \delta_{ij}\,\vec{p}\cdot\vec{r}\right]$$

$$= \frac{1}{r}\left[-2(\vec{p}\cdot\vec{r})x_i + r^2 p_i + x_i(\vec{p}\cdot\vec{r})\right]$$

$$= \boxed{\frac{1}{r}\left[r^2 p_i - (\vec{p}\cdot\vec{r})x_i\right]}$$

NOTE.

$$(\vec{p}\times\vec{l})_i = \epsilon_{ijk}\, p_j\, l_k$$

$$= \epsilon_{ijk}\, p_j\, \epsilon_{kmn}\, x_m p_n$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\, x_m p_j p_n$$

$$= \boxed{x_i p^2 - p_i(\vec{p}\cdot\vec{r})}$$

$$H = \frac{p^2}{2\mu} - \frac{k}{r}$$

$$\{H, A_i\} = -\{A_i, H\}$$

<u>Now:</u>

$$\{A_i, H\} = \{A_i, \frac{p^2}{2\mu} - \frac{k}{r}\}$$

$$= \frac{1}{\cancel{2}\mu} \cancel{2} p_j \{A_i, p_j\} + \frac{k}{r^2}\{A_i, r\}$$

$$= \frac{p_j}{\mu}\left[p^2\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right) - \frac{k\mu}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)\right]$$

$$+ \frac{k}{r^2}\frac{1}{r}\left[r^2 p_i - (\vec{p}\cdot\vec{r})x_i\right]$$

$$= \frac{1}{\mu}\left[p^2\left(p_i - p_i \frac{p^2}{p^2}\right) - \frac{k\mu}{r}\left(\cancel{p_i} - x_i \frac{\vec{p}\cdot\vec{r}}{r^2}\right)\right]$$

$$+ \frac{k}{r^3}\left[\cancel{r^2 p_i} - (\vec{p}\cdot\vec{r})x_i\right]$$

$$= 0$$

Thus, $\boxed{\{H, A_i\} = 0}$

$$\{A_i, A_j\} = \{A_i, \epsilon_{jkl}\, p_k l_l - \frac{k\mu}{r} x_j\}$$

$$= \epsilon_{jkl}\{A_i, p_k\} l_l + \epsilon_{jkl}\, p_k \{A_i, l_l\} - k\mu \{A_i, \tfrac{1}{r}\} x_j - \frac{k\mu}{r}\{A_i, x_j\}$$

$$= \epsilon_{jkl}\, l_l \left[p^2 \left(\delta_{ik} - \frac{p_i p_k}{p^2} \right) - \frac{k\mu}{r} \left(\delta_{ik} - \frac{x_i x_k}{r^2} \right) \right]$$

$$- p_k \epsilon_{jkl} \epsilon_{lim} A_m + \frac{k\mu}{r^2} x_j \{A_i, r\}$$

$$- \frac{k\mu}{r} \left[-2 x_i p_j + x_j p_i + \delta_{ij}\, \vec{p}\cdot\vec{r} \right]$$

$$= \epsilon_{jkl}\, l_l \left[p^2 \left(\qquad \right) - \frac{k\mu}{r} \left(\qquad \right) \right] \quad (1)$$

$$- p_k (\delta_{ji}\delta_{km} - \delta_{jm}\delta_{ki}) \left((\vec{p}\times\vec{l})_m - \frac{k\mu}{r} x_m \right) \quad (2)$$

$$+ \frac{k\mu}{r^2} x_j \frac{1}{r} \left[r^2 p_i - (\vec{p}\cdot\vec{r}) x_i \right] \quad (3)$$

$$- \frac{k\mu}{r} \left[-2 x_i p_j + x_j p_i + \delta_{ij}\, \vec{p}\cdot\vec{r} \right] \quad (4)$$

$$① = \epsilon_{jkl}\, l_l \left[p^2 (\quad) - \frac{k\mu}{r}(\quad) \right]$$

$$= (x_j p_k - x_k p_j)\left[p^2\left(\delta_{ik} - \frac{p_i p_k}{p^2}\right) - \frac{k\mu}{r}\left(\delta_{ik} - \frac{x_i x_k}{r^2}\right)\right]$$

$$= p^2 \cancel{(x_j p_i - x_j p_i)} - \frac{k\mu}{r}\left(x_j p_i - x_i x_j \frac{\vec{p}\cdot\vec{r}}{r^2}\right)$$

$$- p^2\left(x_i p_j - \frac{p_i p_j\, \vec{p}\cdot\vec{r}}{p^2}\right) + \frac{k\mu}{r}\cancel{(x_i p_j - x_i p_j)}$$

$$= -\frac{k\mu}{r}\left(\cancel{x_j} p_i - \cancel{(\vec{p}\cdot\vec{r})\left(\frac{x_i x_j}{r^2}\right)}\right) \to (a)$$

$$- p^2\left(x_i p_j - \cancel{(\vec{p}\cdot\vec{r})\left(\frac{p_i p_j}{p^2}\right)}\right) \to (b)$$

$$② = -(\delta_{ij}\, p_m - \delta_{jm}\, p_i)\left((\vec{p}\times\vec{l})_m - \frac{k\mu}{r} x_m\right)$$

$$= -\left\{ \delta_{ij}\, \underbrace{\vec{p}\cdot(\vec{p}\times\vec{l})}_{0} - \delta_{ij}\frac{k\mu}{r}(\vec{p}\cdot\vec{r}) - p_i(\vec{p}\times\vec{l})_j + \frac{k\mu}{r} p_i x_j \right\}$$

$$= \cancel{\delta_{ij}\frac{k\mu}{r}(\vec{p}\cdot\vec{r})} \to (c) + \left(p_i x_j p^2 - \cancel{p_i p_j (\vec{p}\cdot\vec{r})} \to (b)\right) \boxed{-\frac{k\mu}{r} p_i x_j}$$

$$③ = \frac{k\mu}{r} p_i \cancel{x_j} - \cancel{\frac{k\mu}{r^3}(\vec{p}\cdot\vec{r})\, x_i x_j} \to (a)$$

$$④ = \boxed{+2\frac{k\mu}{r} x_i p_j} \boxed{-\frac{k\mu}{r} x_j p_i} - \cancel{\frac{k\mu}{r}\delta_{ij}\, \vec{p}\cdot\vec{r}} \to (c)$$

Thus,

$$\{\ _i, A_j\} = 2\frac{k\mu}{r}(x_i p_j - x_j p_i) - p^2 (x_i p_j - x_j p_i)$$

$$= -2\mu (x_i p_j - x_j p_i)\left[-\frac{k}{r} + \frac{p^2}{2\mu}\right]$$

Thus,

$$\{A_i, A_j\} = -2\mu\left[\frac{p^2}{2\mu} - \frac{k}{r}\right] \epsilon_{ijk}\, l_k$$

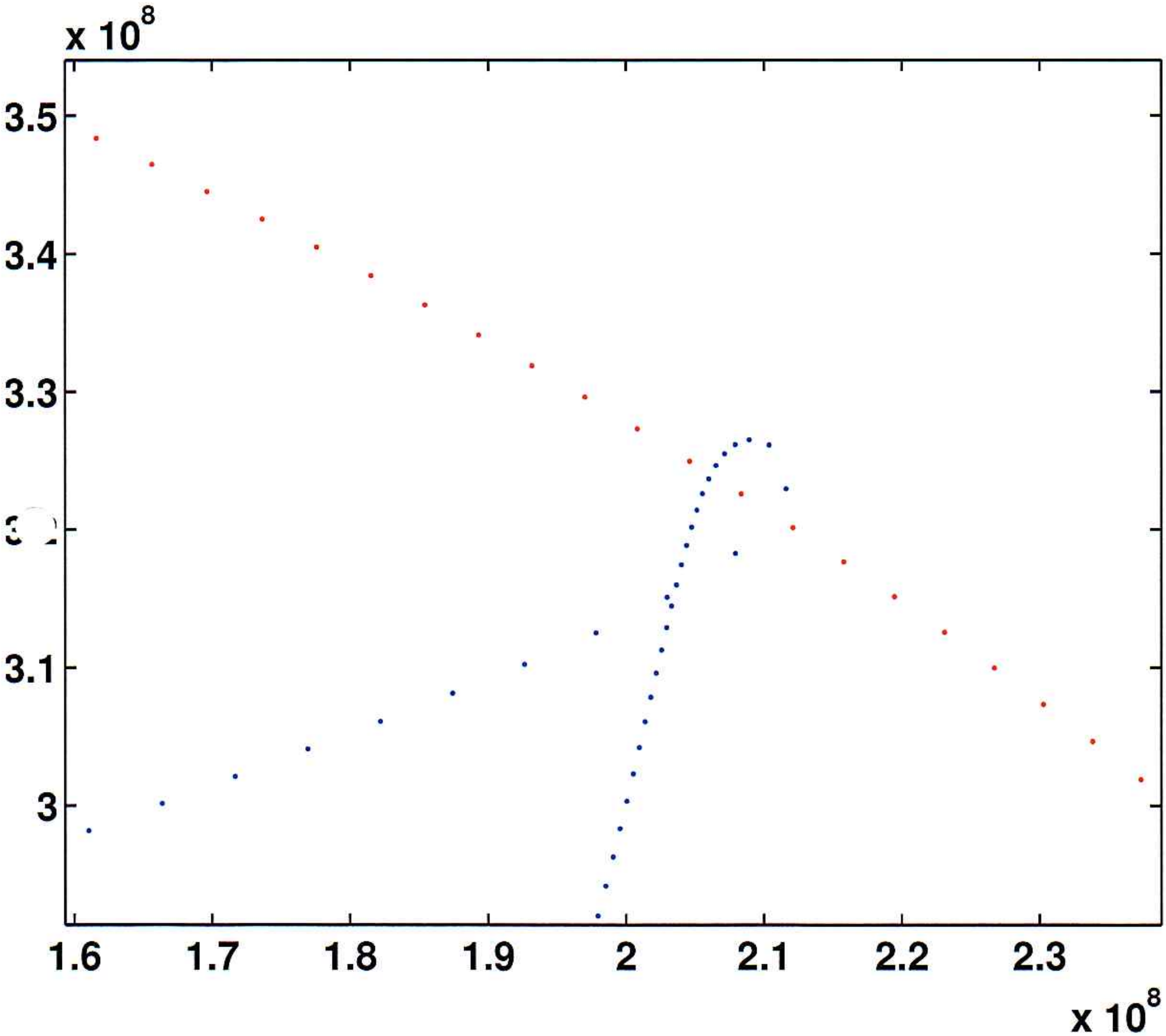
$$= 2\mu |E| \, \epsilon_{ijk}\, l_k$$

where $E \equiv -\left(\frac{p^2}{2\mu} - \frac{k}{r}\right) \equiv$ conserved energy

Thus, if we define

$$D_i = \frac{1}{\sqrt{2\mu |E|}} A_i$$

Then $\boxed{\{D_i, D_j\} = \epsilon_{ijk}\, l_k}$


x 10^8
3.5
3.4
3.3
3.2
3.1
3
1.6
1.7
1.8
1.9
2
2.1
2.2
2.3
x 10^8

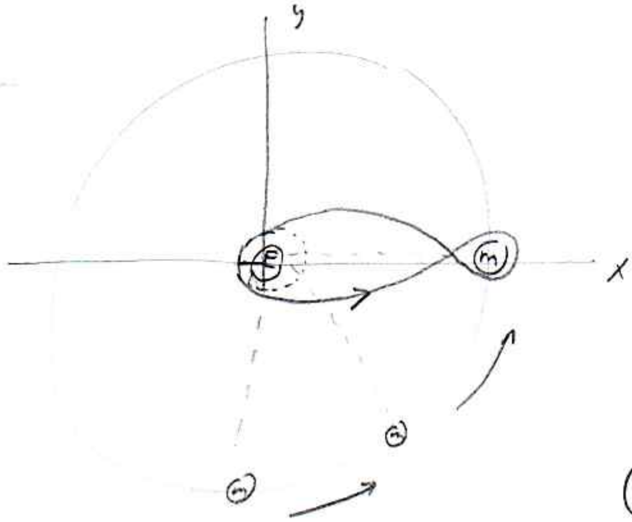
```
function freereturn(scale_angle, scale_v)
%
% script for integrating free return lunar orbit
%
% example: freereturn(.98,1.0001)
%
%%%%%%%%%%%%%%%%%%%%%%%%%
close all

% constants
G = 6.67e-11; % MKS
ME = 5.97e24; % kg
MM = 7.35e22; % kg
rEM = 3.84e8; % m
rE = 6.4e6; % m (earth radius)
rM = 1.74e6; % m (moon radius)
omega = 2*pi/(27.32*24*3600); % rad/s (lunar rotation rate)
TM = 2*pi/omega; % period of lunar rotation

% do the calculation in corotating coordinates

% NOTE: all variable initially refer to these coordinates
% but we will drop the primes

% earth at (0,0)
% moon at (rEM,0)

% perigee and apogee for simplified analysis
rp = 6.71e6; % m
%ra = 4e8; % m (a little beyond the moon's orbit)
ra = rEM + 2*rM; % m (2 moon radii beyond moon's orbit)
a = 0.5*(rp+ra);
e = (ra-rp)/(ra+rp);

% velocity and time needed to get to apogee
vp = scale_v*sqrt(2*G*ME/(rp*(1+rp/ra)));
vp = vp - rp*omega; % compensate for corotating velocity
P = sqrt(4*pi^2*a^3/(G*ME)); % kepler's 3rd law (ignoring mass of
spacecraft)
Ta = 0.5*P;
theta0 = scale_angle*(pi+ 2*pi*Ta/TM);

% initial conditions (in circular orbit around earth)
x0 = rp*cos(theta0);
y0 = rp*sin(theta0);
vx0 = -vp*sin(theta0);
vy0 = vp*cos(theta0);

% discrete times
Nt = 1e5;
```

```
t = linspace(0, P, Nt);
dt = t(2)-t(1);

% loop over discrete times
for ii=1:Nt

  if ii==1
    % initial values
    x(ii) = x0;
    y(ii) = y0;
    vx(ii) = vx0;
    vy(ii) = vy0;

  else
    % calculate accelerations at previous values
    [ax, ay] = cal_a(x(ii-1), y(ii-1), vx(ii-1), vy(ii-1));

    % calculate values at mid points
    xmid = x(ii-1) + vx(ii-1)*dt/2;
    ymid = y(ii-1) + vy(ii-1)*dt/2;
    vxmid = vx(ii-1) + ax*dt/2;
    vymid = vy(ii-1) + ay*dt/2;

    % recalculate accelerations at midpoint
    [axmid, aymid] = cal_a(xmid, ymid, vxmid, vymid);

    % update position and velocity of spacecraft
    x(ii) = x(ii-1) + vxmid*dt;
    y(ii) = y(ii-1) + vymid*dt;
    vx(ii) = vx(ii-1) + axmid*dt;
    vy(ii) = vy(ii-1) + aymid*dt;

  end

end

% plot orbit in corotating frame
figure(1)
plot(x,y)
hold on
plot(0,0,'r+',rEM,0,'r+')
axis equal

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% plot orbit in inertial frame
theta = omega*t;
xM = rEM*cos(theta);
yM = rEM*sin(theta);

for ii=1:Nt
```

```
  xi(ii) = x(ii)*cos(theta(ii)) - y(ii)*sin(theta(ii));
  yi(ii) = x(ii)*sin(theta(ii)) + y(ii)*cos(theta(ii));
end

figure(2)
plot(xi,yi)
hold on
plot(xM,yM,'r')
plot(0,0,'r+')
axis equal

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% animation in interial frame

figure()
Na = 200;
fac = ceil(Nt/Na);
for ii=1:Na
%for ii=floor(Na/2)-20:floor(Na/2)-10
  pause(.01)
  plot(0, 0, 'r+')
  hold on
  plot(xM(fac*(ii-1)+1), yM(fac*(ii-1)+1), 'r.')
  plot(xi(fac*(ii-1)+1), yi(fac*(ii-1)+1), 'b.')
  xlim([0 1.2*rEM])
  ylim([0 1.2*rEM])
  axis equal

  F(ii)=getframe;
end

fname = ['freereturn.avi'];
movie2avi(F,fname)

return
```

```
function [ax, ay] = cal_a(x, y, vx, vy)
%
% calculate x, y components of acceleration for free return lunar
orbit
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% constants
G = 6.67e-11; % MKS
ME = 5.97e24; % kg
MM = 7.35e22; % kg
rEM = 3.84e8; % m
omega = 2*pi/(27.32*24*3600); % rad/s (lunar rotation rate)
%MM=0;
%omega=0;

r = sqrt(x^2 + y^2);
rM = sqrt((x-rEM)^2 + y^2);

ax = (-G*ME/r^3 - G*MM/rM^3 + omega^2)*x + (G*MM/rM^3)*rEM +
2*omega*vy;
ay = (-G*ME/r^3 - G*MM/rM^3 + omega^2)*y - 2*omega*vx;

return
```

Lunar ~~#~~ injection:



Spacecraft initially in a circular orbit around Earth

M_E: mass of earth
M_m: mass of moon
μ: mass of spaceship

(motion in x-y plane): $\theta = \pi/2$

$$T = \frac{1}{2}\mu v^2 + \frac{1}{2} M_m v_m^2$$

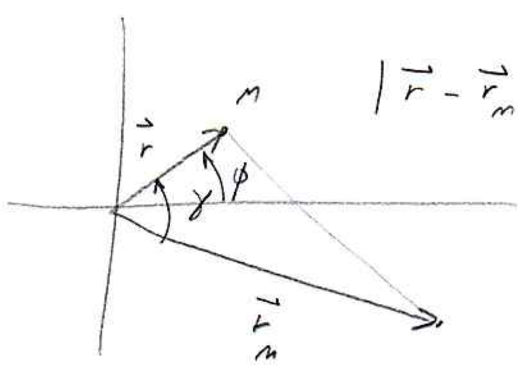
$$= \frac{1}{2}\mu(\dot r^2 + r^2\dot\phi^2) + \frac{1}{2} M_m R_{Em}^2 \omega^2 \quad \text{where } \omega = \frac{2\pi}{1\text{ month}} \approx 29\text{ day}$$

$$= \frac{1}{2}\mu(\dot r^2 + r^2\dot\phi^2) + \underbrace{\text{const}}_{\text{can ignore in Lagrangian formulation}}$$

$$U = -\frac{G M_E \mu}{r} - \frac{G M_m \mu}{|\vec r - \vec r_m|} - \underbrace{\frac{G M_E M_m}{|\vec r_E - \vec r_m|}}_{\text{const (can ignore)}}$$

$$\vec r_m = (R_{Em}\cos(\omega t + \phi_M),\ R_{Em}\sin(\omega t + \phi_M))$$

circular motion in xy plane around Earth (at least)



$$|\vec r - \vec r_m| = \sqrt{r^2 + r_{Em}^2 - 2 r r_{Em}\cos\gamma}$$

$$\gamma = \phi - (\omega t + \phi_M)$$

initial const. phase offset

$$L = T - U$$
$$= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{GM_E\mu}{r} + \frac{GM_m\mu}{|\vec{r}-\vec{r}_m|}$$

where $|\vec{r}-\vec{r}_m| = \sqrt{r^2 + r_{Em}^2 - 2rr_{Em}\cos[\phi - (\omega t + \phi_m)]}$

This last term $\rightarrow$ L depends on ϕ, t
so p_ϕ, E not conserved any more

r equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\rightarrow 0 = \frac{d}{dt}(\mu\dot{r}) + \frac{GM_E\mu}{r^2} + \frac{GM_m\mu}{|\vec{r}-\vec{r}_m|^2}\frac{\partial}{\partial r}|\vec{r}-\vec{r}_m|$$

$$= \mu\ddot{r} + \frac{GM_E\mu}{r^2} + \frac{GM_m\mu}{|\vec{r}-\vec{r}_m|^2}\frac{\partial}{\partial r}|\vec{r}-\vec{r}_m|$$

Drop the μ's:

$$\boxed{\ddot{r} = -\frac{GM_E}{r^2} - \frac{GM_m}{|\vec{r}-\vec{r}_m|^2}\frac{\partial}{\partial r}|\vec{r}-\vec{r}_m|}$$

ϕ equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$\rightarrow 0 = \frac{d}{dt}(\mu r^2\dot{\phi}) + \frac{GM_m\mu}{|\vec{r}-\vec{r}_m|^2}\frac{\partial}{\partial \phi}|\vec{r}-\vec{r}_m|$$

$$= 2\mu r\dot{r}\dot{\phi} + \mu r^2\ddot{\phi} + \frac{GM_m\mu}{|\vec{r}-\vec{r}_m|^2}\frac{\partial}{\partial \phi}|\vec{r}-\vec{r}_m|$$

Drop the μ's, divide by r^2

$$\boxed{\ddot{\phi} = -2\left(\frac{\dot{r}}{r}\right)\dot{\phi} - \frac{GM_m}{|\vec{r}-\vec{r}_m|^2}\frac{1}{r^2}\frac{\partial}{\partial \phi}|\vec{r}-\vec{r}_m|}$$

NOTE.

$$\frac{\partial}{\partial r}|\vec{r}-\vec{r}_m| = \frac{\partial}{\partial r}\left(\sqrt{r^2+r_{Em}^2-2rr_E\cos[\phi-(\omega t+\phi_m)]}\right)$$

$$= \frac{1}{\cancel{2}\sqrt{}}\left[\cancel{2}r - \cancel{2}r_E\cos[\phi-(\omega t+\phi_m)]\right]$$

$$= \frac{1}{\sqrt{}}\left(r - r_E\cos[\phi-(\omega t+\phi_m)]\right)$$

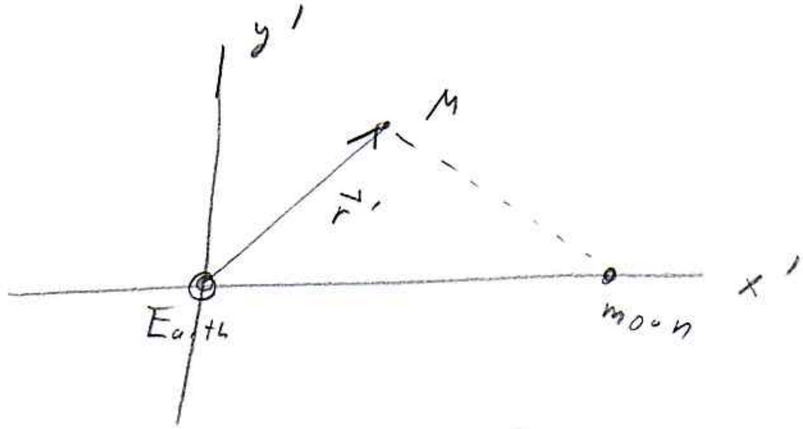
$$= \frac{1}{|\vec{r}-\vec{r}_m|}\left(r - r_E\cos[\phi-(\omega t+\phi_m)]\right)$$

$$\frac{\partial}{\partial \phi}|\vec{r}-\vec{r}_m| = \frac{\partial}{\partial \phi}\left(\sqrt{r^2+r_{Em}^2-2rr_E\cos[\phi-(\omega t+\phi_m)]}\right)$$

$$= \frac{1}{\cancel{2}\sqrt{}}\left(+\cancel{2}rr_E\sin[\phi-(\omega t+\phi_m)]\right)$$

$$= \frac{1}{|\vec{r}-\vec{r}_m|}\,rr_E\sin[\phi-(\omega t+\phi_m)]$$

Consider motion in co-rotating reference frame



$$\mu\vec{a}' = \vec{F} - \mu\cancel{\dot{\vec{\omega}}}^{\,0}\times\vec{r}' - 2\mu\,\vec{\omega}\times\vec{v}' - \mu\vec{\omega}\times(\vec{\omega}\times\vec{r}')$$
$$= \vec{F} - 2\mu\,\vec{\omega}\times\vec{v}' - \mu\,\vec{\omega}\times(\vec{\omega}\times\vec{r}')$$

where $\vec{\omega} = \omega\hat{k}$

$\vec{r}' = x'\hat{\imath} + y'\hat{\jmath}$

$\vec{v}' = \dot{x}'\hat{\imath} + \dot{y}'\hat{\jmath}$

$$\vec{\omega}\times\vec{r}' = \omega\hat{k}\times(x'\hat{\imath} + y'\hat{\jmath})$$
$$= \omega[x'\hat{\jmath} - y'\hat{\imath}]$$
$$= \omega[-y'\hat{\imath} + x'\hat{\jmath}]$$
$$\vec{\omega}\times(\vec{\omega}\times\vec{r}') = \omega^2\left(\hat{k}\times[-y'\hat{\imath} + x'\hat{\jmath}]\right)$$
$$= \omega^2(-y'\hat{\jmath} - x'\hat{\imath})$$
$$= -\omega^2(x'\hat{\imath} + y'\hat{\jmath})$$
$$= -\omega^2\vec{r}'$$
$$\vec{\omega}\times\vec{v}' = \omega\hat{k}\times[\dot{x}'\hat{\imath} + \dot{y}'\hat{\jmath}]$$
$$= \omega[\dot{x}'\hat{\jmath} - \dot{y}'\hat{\imath}]$$
$$= \omega[-\dot{y}'\hat{\imath} + \dot{x}'\hat{\jmath}]$$

$$\vec{F} = -\frac{GM_E m}{r'^3}\vec{r}' - \frac{GM_m m}{|\vec{r}' - \vec{r}_m|^3}(\vec{r}' - \vec{r}_m)$$

where $\vec{r}' - \vec{r}_m = \vec{r}' - r_{Em}\hat{i}$

Divide by m:

$$\vec{a}' = \frac{\vec{F}}{m} - 2\vec{\omega}\times\vec{v}' + \omega^2\vec{r}'$$

$$= -\frac{GM_E}{r'^3}\vec{r}' - \frac{GM_m}{|\vec{r}' - \vec{r}_m|^3}(\vec{r}' - \vec{r}_m) - 2\vec{\omega}\times\vec{v}' + \omega^2\vec{r}'$$

$$= \left(-\frac{GM_E}{r'^3} - \frac{GM_m}{|\vec{r}' - \vec{r}_m|^3} + \omega^2\right)\vec{r}' + \frac{GM_m}{|\vec{r}' - \vec{r}_m|^3} r_{Em}\hat{i}$$

$$- 2\omega\left[-\dot{y}'\hat{i} + \dot{x}'\hat{j}\right]$$

where $r' = \sqrt{x'^2 + y'^2}$

$$|\vec{r}' - \vec{r}_m| = \sqrt{(x' - r_{Em})^2 + y'^2}$$

$$\vec{r}' = x'\hat{i} + y'\hat{j}$$

Thus,

$$\boxed{\ddot{x}' = \left(-\frac{GM_E}{r'^3} - \frac{GM_m}{|\vec{r}' - \vec{r}_m|^3} + \omega^2\right)x' + \frac{GM_m}{|\vec{r}' - \vec{r}_m|^3} r_{Em} + 2\omega\dot{y}'}$$

$$\boxed{\ddot{y}' = \left(-\frac{GM_E}{r'^3} - \frac{GM_m}{|\vec{r}' - \vec{r}_m|^3} + \omega^2\right)y' - 2\omega\dot{x}'}$$

Treat as a system of 1^{st} order equations for the variables, x', y', v_x', v_y'.

$$\dot{v}_x' = \left(-\frac{GM_E}{r'^3} - \frac{GM_m}{|\vec{r}' - \vec{r}_m'|^3} + \omega^2 \right) x' + \frac{GM_m}{|\vec{r}' - \vec{r}_m|^3} r_{Em} + 2\omega v_y'$$

$$\dot{v}_y' = \left(-\frac{GM_E}{r'^3} - \frac{GM_m}{|\vec{r}' - \vec{r}_m'|^3} + \omega^2 \right) y' - 2\omega v_x'$$

$$\dot{x}' = v_x$$

$$\dot{y}' = v_y$$

Drop the primes:

$$\frac{dx}{dt} = v_x \equiv f_1(t, x, y, v_x, v_y)$$

$$\frac{dy}{dt} = v_y \equiv f_2(t, x, y, v_x, v_y)$$

$$\frac{dv_x}{dt} = \left(-\frac{GM_E}{r^3} - \frac{GM_m}{|r - r_m|^3} + \omega^2 \right) x + \frac{GM_m}{|r - r_m|^3} r_{Em} + 2\omega v_y \equiv f_3(t, x, y, v_x, v_y)$$

$$\frac{dv_y}{dt} = \left(\quad '' \quad \right) y - 2\omega v_x \equiv f_4(t, x, y, v_x, v_y)$$

~~Second order~~ solution:

$$\cancel{k_1 = \Delta t}\, v_x \big|_n$$

$$\cancel{k_2 = \Delta t}$$

$$x_{mid} = x(i-1) + v_x(i-1)\left(\frac{\Delta t}{2}\right)$$

$$y_{mid} = y(i-1) + v_y(i-1)\left(\Delta t/_2\right)$$

$$v_{x\,mid} = v_x(i-1) + a_x(i-1)\left(\Delta t/_2\right)$$

$$v_{y\,mid} = v_y(i-1) + a_y(i-1)\left(\Delta t/_2\right)$$

$$\cancel{x(i) = x(i-1) + \frac{1}{2}\left(v_x(i-1) + v_{x1}\right)} \Delta t$$

$$\cancel{y(i) = y(i-1) + \frac{1}{2}\left(v_y(i-1) + v_{y1}\right)} \Delta t$$

$$\cancel{v_x(i) = v_x(i) + \frac{1}{2}\left(a_x(i-1) + a_x\right.}\left(x_1, y_1, v_{x1}, v_{y1}\right)$$

$$x(i) = x(i-1) + \Delta t\; v_{x\,mid}$$

$$y(i) = y(i-1) + \Delta t\; v_{y\,mid}$$

$$v_x(i) = v_x(i-1) + \Delta t\; a_x\left(x_{mid}, y_{mid}, v_{x\,mid}, v_{y\,mid}\right)$$

$$v_y(i) = v_y(i-1) + \Delta t\; a_y\left(x_{mid}, y_{mid}, v_{x\,mid}, v_{y\,mid}\right)$$