Exer: 3.1

$$\vec{F} = -\vec{\nabla} U(\vec{r},t)$$

a) $L = T - U$ $\quad\Big|\quad |\dot{\vec{r}}|^2 = \dot{x}^2+\dot{y}^2+\dot{z}^2$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2 - U(\vec{r},t)$$

$$0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$= -\frac{\partial U}{\partial x} - \frac{d}{dt}(m\dot{x})$$

$$= -\frac{\partial U}{\partial x} - m\ddot{x}$$

so $m\ddot{x} = -\frac{\partial U}{\partial x}$

similarly, $m\ddot{y} = -\frac{\partial U}{\partial y}$, $m\ddot{z} = -\frac{\partial U}{\partial z}$

so $\boxed{m\ddot{\vec{r}} = -\vec{\nabla} U}$

b) In sph. polar coords

$$T = \frac{1}{2} m |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

$\underline{\partial r:}$

$$0 = \frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$$

$$= -\frac{\partial U}{\partial r} + mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\phi}^2 - \frac{d}{dt}(m\dot{r})$$

$$\rightarrow \boxed{\frac{d}{dt}(m\dot{r}) = -\frac{\partial U}{\partial r} + mr\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)}$$

$\delta\theta$:

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$$

$$= -\frac{\partial U}{\partial \theta} + m r^2 \sin\theta \cos\theta \, \dot{\phi}^2 - \frac{d}{dt}\left(m r^2 \dot{\theta}\right)$$

$$\rightarrow \boxed{\frac{d}{dt}\left(m r^2 \dot{\theta}\right) = -\frac{\partial U}{\partial \theta} + m r^2 \sin\theta \cos\theta \, \dot{\phi}^2}$$

$\delta\phi$:

$$0 = \frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)$$

$$= -\frac{\partial U}{\partial \phi} - \frac{d}{dt}\left(m r^2 \sin^2\theta \, \dot{\phi}\right)$$

$$\rightarrow \boxed{\frac{d}{dt}\left(m r^2 \sin^2\theta \, \dot{\phi}\right) = -\frac{\partial U}{\partial \phi}}$$

Exer 3.2

show $L' = L + \frac{d\Lambda}{dt}$ gives same EOMs as L
where $\Lambda = \Lambda(q,t)$

$$S' = \int_{t_1}^{t_2} dt\, L'$$

$$= \int_{t_1}^{t_2} dt \left(L + \frac{d\Lambda}{dt} \right)$$

$$= S + \int_{t_1}^{t_2} dt \frac{d\Lambda}{dt}$$

$$= S + \Lambda \Big|_{t_1}^{t_2}$$

$$\delta S' = \delta S + \delta\Lambda \Big|_{t_1}^{t_2}$$

$$= \delta S + \frac{\partial \Lambda}{\partial q} \underbrace{\cancel{\delta q}}_{0} \Big|_{t_1}^{t_2}$$

$$= \delta S$$

EL equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

Another way:

$$\frac{d\Lambda}{dt} = \frac{\partial \Lambda}{\partial q}\dot{q} + \frac{\partial \Lambda}{\partial t}$$

$$\frac{\partial}{\partial q}\left(\frac{d\Lambda}{dt}\right) = \frac{\partial^2 \Lambda}{\partial q^2}\dot{q} + \frac{\partial^2 \Lambda}{\partial q\, \partial t}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\left(\frac{d\Lambda}{dt}\right)\right) = \frac{d}{dt}\left[\frac{\partial \Lambda}{\partial q}\right]$$

$$= \frac{\partial^2 \Lambda}{\partial q^2}\dot{q} + \frac{\partial^2 \Lambda}{\partial t\, \partial t}$$

same

Thus,
$$\frac{\partial}{\partial q}\left(\frac{d\Lambda}{dt}\right) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\left(\frac{d\Lambda}{dt}\right)\right) = 0$$

3.3

Prob: Invariance of form of EL equations under a point transformation

$$q^a \to Q^a = Q^a(q, t) \quad \leftrightarrow \quad q^a = q^a(Q, t)$$

$$\dot q^a = \sum_b \frac{\partial q^a}{\partial Q^b}\dot Q^b + \frac{\partial q^a}{\partial t}$$

$$L'(Q, \dot Q, t) = L(q, \dot q, t)\Big|_{\substack{q = q(Q,t) \\ \dot q = \dot q(Q, \dot Q, t)}}$$

$$\frac{\partial L'}{\partial Q^a} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot Q^a}\right) = \sum_b \frac{\partial L}{\partial q^b}\frac{\partial q^b}{\partial Q^a} + \sum_b \frac{\partial L}{\partial \dot q^b}\frac{\partial \dot q^b}{\partial Q^a} - \frac{d}{dt}\Bigg(\sum_b \frac{\partial L}{\partial \dot q^b}\underbrace{\frac{\partial \dot q^b}{\partial \dot Q^a}}_{\frac{\partial q^b}{\partial Q^a}}\Bigg)$$

$$= \sum_b \left\{ \frac{\partial L}{\partial q^b}\frac{\partial q^b}{\partial Q^a} + \frac{\partial L}{\partial \dot q^b}\frac{\partial \dot q^b}{\partial Q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^b}\right)\frac{\partial q^b}{\partial Q^a} - \frac{\partial L}{\partial \dot q^b}\frac{d}{dt}\left(\frac{\partial q^b}{\partial Q^a}\right) \right\}$$

$$= \sum_b \frac{\partial q^b}{\partial Q^a}\left(\frac{\partial L}{\partial q^b} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^b}\right)\right)$$

$$\cancel{+ \sum_b \frac{\partial L}{\partial \dot q^b}\left(\sum_c \frac{\partial^2 q^b}{\partial Q^a \partial Q^c}\dot Q^c + \frac{\partial^2 q^b}{\partial Q^a \partial t}\right)}$$

$$\cancel{- \sum_b \frac{\partial L}{\partial \dot q^b}\left(\frac{\partial^2 q^b}{\partial Q^c \partial Q^a}\dot Q^c + \frac{\partial^2 q^b}{\partial t \partial Q^a}\right)}$$

$$= \sum_b \underbrace{\frac{\partial q^b}{\partial Q^a}}_{\text{invertible}} \left(\frac{\partial L}{\partial q^b} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^b}\right)\right)$$

Thus, $\dfrac{\partial L'}{\partial Q^a} - \dfrac{d}{dt}\left(\dfrac{\partial L'}{\partial \dot{Q}^a}\right) = 0$ iff $\dfrac{\partial L}{\partial q^a} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}^a}\right) = 0$

Exer.: 3.4

EL: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a} = 0$, $a = 1, 2, \cdots, n$

I/II. If L indep. of q^a then $\frac{\partial L}{\partial q^a} = 0$

$\rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0 \iff p_a \equiv \frac{\partial L}{\partial \dot{q}^a} = \text{const}$

III. If L does not depend explicitly on time (i.e. $\frac{\partial L}{\partial t} = 0$)
then

$$\frac{dh}{dt} = \frac{d}{dt}\left(\sum_a p_a \dot{q}^a - L\right)$$

$$= \sum_a \left(\dot{p}_a \dot{q}^a + p_a \ddot{q}^a\right) - \frac{dL}{dt}$$

$$= \sum_a \left(\frac{\partial L}{\partial q^a}\dot{q}^a + \frac{\partial L}{\partial \dot{q}^a}\ddot{q}^a\right) - \sum_a \left(\frac{\partial L}{\partial q^a}\dot{q}^a + \frac{\partial L}{\partial \dot{q}^a}\ddot{q}^a\right)$$

$$= 0$$

Exer (3.5)

$$L = \frac{1}{2} m v^2 - U(\vec{r}) \quad , \quad \vec{F} = -\vec{\nabla} U$$

Now: $H = \left(\sum_a p_a \dot{q}^a - L \right) \Big|_{\dot{q} = \dot{q}(q,p,t)}$

where $p_a = \dfrac{\partial L}{\partial \dot{q}^a}$

Use Cartesian coords: $q^a = x^i = (x, y, z)$

Then $L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = m \dot{x}^i \quad \rightarrow \quad \dot{x}^i = \frac{p_i}{m}$$

$$\rightarrow \quad H = \sum_i p_i \left(\frac{p_i}{m} \right) - \frac{1}{2} m \left(\sum_i \left(\frac{p_i}{m} \right)^2 \right) + U(\vec{r})$$

$$= \sum_i \frac{p_i^2}{m} - \frac{1}{2} \sum_i \frac{p_i^2}{m} + U(r)$$

$$= \frac{1}{2m} \sum_i p_i^2 + U(r)$$

$$= \frac{p^2}{2m} + U(r)$$

Properties of Poisson brackets: 3.6

$$\{f,g\} \equiv \sum_a \left(\frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q^a} \right)$$

(i) Anti-symmetry: $\{f,g\} = -\{g,f\}$

Proof:
$$\{f,g\} = \sum_a \left(\frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q^a} \right)$$
$$= -\sum_a \left(\frac{\partial g}{\partial q^a} \frac{\partial f}{\partial p_a} - \frac{\partial g}{\partial p_a} \frac{\partial f}{\partial q^a} \right)$$
$$= -\{g,f\}$$

(ii) Linearity: $\{f, g+ah\} = \{f,g\} + a\{f,h\}$

Proof:
$$\{f,g+ah\} = \sum_a \left(\frac{\partial f}{\partial q^a} \frac{\partial}{\partial p_a}(g+ah) - \frac{\partial f}{\partial p_a} \frac{\partial (g+ah)}{\partial q^a} \right)$$
$$= \sum_a \left(\frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p_a} + a \frac{\partial f}{\partial q^a} \frac{\partial h}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q^a} - a \frac{\partial f}{\partial p_a} \frac{\partial h}{\partial q^a} \right)$$
$$= \sum_a \left(\frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q^a} \right) + a \sum_a \left(\frac{\partial f}{\partial q^a} \frac{\partial h}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial h}{\partial q^a} \right)$$
$$= \{f,g\} + a\{f,h\}$$

(iii) Jacobi: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

Proof: (drop the superscripts and subscripts on q^i, p_i)

$$\{f,\{g,h\}\} = \sum\left(\frac{\partial f}{\partial q}\frac{\partial}{\partial p}\{g,h\} - \frac{\partial f}{\partial p}\frac{\partial}{\partial q}\{g,h\}\right)$$

$$= \sum\left[\frac{\partial f}{\partial q}\frac{\partial}{\partial p}\left(\sum\left(\frac{\partial g}{\partial q}\frac{\partial h}{\partial p} - \frac{\partial g}{\partial p}\frac{\partial h}{\partial q}\right)\right) - \frac{\partial f}{\partial p}\frac{\partial}{\partial q}\left(\sum\left(\frac{\partial g}{\partial q}\frac{\partial h}{\partial p} - \frac{\partial g}{\partial p}\frac{\partial h}{\partial q}\right)\right)\right]$$

$$= \sum\sum\left[\cancel{\frac{\partial f}{\partial q}\frac{\partial^2 g}{\partial p\partial q}\frac{\partial h}{\partial p}} + \cancel{\frac{\partial f}{\partial q}\frac{\partial g}{\partial q}\frac{\partial^2 h}{\partial p^2}} - \cancel{\frac{\partial f}{\partial q}\frac{\partial^2 g}{\partial p^2}\frac{\partial h}{\partial q}} - \cancel{\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\frac{\partial^2 h}{\partial p\partial q}} - \cancel{\frac{\partial f}{\partial p}\frac{\partial^2 g}{\partial q^2}\frac{\partial h}{\partial p}} - \cancel{\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\frac{\partial^2 h}{\partial q\partial p}} + \cancel{\frac{\partial f}{\partial p}\frac{\partial^2 g}{\partial q\partial p}\frac{\partial h}{\partial q}} + \cancel{\frac{\partial f}{\partial p}\frac{\partial g}{\partial p}\frac{\partial^2 h}{\partial q^2}}\right]$$

similarly

$$\{g,\{h,f\}\} = \sum\sum\left[\cancel{\frac{\partial g}{\partial q}\frac{\partial^2 h}{\partial p\partial q}\frac{\partial f}{\partial p}} + \cancel{\frac{\partial g}{\partial q}\frac{\partial h}{\partial q}\frac{\partial^2 f}{\partial p^2}} - \cancel{\frac{\partial g}{\partial q}\frac{\partial^2 h}{\partial p^2}\frac{\partial f}{\partial q}} - \cancel{\frac{\partial g}{\partial q}\frac{\partial h}{\partial p}\frac{\partial^2 f}{\partial p\partial q}} - \cancel{\frac{\partial g}{\partial p}\frac{\partial^2 h}{\partial q^2}\frac{\partial f}{\partial p}} - \cancel{\frac{\partial g}{\partial p}\frac{\partial h}{\partial q}\frac{\partial^2 f}{\partial q\partial p}} + \cancel{\frac{\partial g}{\partial p}\frac{\partial^2 h}{\partial q\partial p}\frac{\partial f}{\partial q}} + \cancel{\frac{\partial g}{\partial p}\frac{\partial h}{\partial p}\frac{\partial^2 f}{\partial q^2}}\right]$$

$$\{h,\{f,g\}\} = \sum\sum \Big[\frac{\partial h}{\partial q}\frac{\partial^2 f}{\partial p\,\partial q}\frac{\partial g}{\partial p} + \cancel{\frac{\partial h}{\partial q}\frac{\partial f}{\partial q}\frac{\partial^2 g}{\partial p^2}}$$

$$- \frac{\partial h}{\partial q}\frac{\partial^2 f}{\partial p^2}\frac{\partial g}{\partial q} - \frac{\partial h}{\partial q}\frac{\partial f}{\partial p}\frac{\partial^2 g}{\partial p\,\partial q}$$

$$- \frac{\partial h}{\partial p}\frac{\partial^2 f}{\partial q^2}\frac{\partial g}{\partial p} - \cancel{\frac{\partial h}{\partial p}\frac{\partial f}{\partial q}\frac{\partial^2 g}{\partial q\,\partial p}}$$

$$+ \frac{\partial h}{\partial p}\frac{\partial^2 f}{\partial q\,\partial p}\frac{\partial g}{\partial q} + \frac{\partial h}{\partial p}\frac{\partial f}{\partial p}\frac{\partial^2 g}{\partial q^2} \Big]$$

Add:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}$$

= terms cancel as shown above

$= 0$

Problem: Product rule for PBs; chain rule for PBs

3.7

$$\{f, gh\} = \frac{\partial f}{\partial q}\frac{\partial (gh)}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial (gh)}{\partial q}$$

$$= \frac{\partial f}{\partial q}\left(\frac{\partial g}{\partial p} h + g\frac{\partial h}{\partial p}\right)$$

$$- \frac{\partial f}{\partial p}\left(\frac{\partial g}{\partial q} h + g\frac{\partial h}{\partial q}\right)$$

$$= \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right) h$$

$$+ \left(\frac{\partial f}{\partial q}\frac{\partial h}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial h}{\partial q}\right) g$$

$$= \{f, g\} h + g\{f, h\}$$

$$\{f, g \circ h\} = \frac{\partial f}{\partial q}\frac{\partial (g \circ h)}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial (g \circ h)}{\partial q}$$

$$= \frac{\partial f}{\partial q}\frac{dg}{dh}\frac{\partial h}{\partial p} - \frac{\partial f}{\partial p}\frac{dg}{dh}\frac{\partial h}{\partial q}$$

$$= \frac{dg}{dh}\left(\frac{\partial f}{\partial g}\frac{\partial h}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial h}{\partial q}\right)$$

$$= \frac{dg}{dh}\{f, h\}$$

Problem: 3.8 If f and g are conserved, so is $\{f,g\}$

Proof: $\{f,H\}=0$, $\{g,H\}=0$ since f,g conserved

Jacobi identity:

$$\{\{f,g\},H\} + \{\underbrace{\{g,H\}}_{\substack{0 \\ g \text{ conserved}}},f\} + \{\underbrace{\{H,f\}}_{\substack{0 \\ f \text{ conserved}}},g\} = 0$$

Thus, $\{\{f,g\},H\} = 0$

iff $\{f,g\}$ conserved

Problem: 3.9 $\vec{l}$ conserved for $U = U(r)$ (central force)

$$H = \frac{p^2}{2m} + U(r)$$

$$\vec{l} = \vec{r} \times \vec{p} \quad , \quad l_i = \epsilon_{ijk} x^j p_k$$

$$\{l_i, H\} = \epsilon_{ijk} \{x^j p_k , \frac{p^2}{2m} + U(r)\}$$

$$= \frac{\epsilon_{ijk}}{2m} \{x^j p_k, p^2\} + \epsilon_{ijk}\{x^j p_k, U(r)\}$$

Now: $\{x^j, U(r)\} = 0$

$$\{p_k, U(r)\} = \frac{dU}{dr}\{p_k, r\}$$

$$= \frac{dU}{dr}\{p_k, \sqrt{x_i^2}\}$$

$$= \frac{dU}{dr}\frac{1}{\cancel{2}r}\cancel{2}\underbrace{\{p_k, x_i\}}_{-\delta_{ik}}x_i$$

$$= -\frac{dU}{dr}\left(\frac{x_k}{r}\right)$$

$$\{x^j p_k, p^2\} = p_k\{x^j, p^2\}$$

$$= p_k \{x^j, p_i^2\}$$

$$= p_k 2p_i \underbrace{\{x^j, p_i\}}_{\delta_{ij}}$$

$$= 2p_j p_k$$

Thus,

$$\{l_i, H\} = \frac{1}{2m} \cancelto{0}{\epsilon_{ijk}\, 2 p_j p_k}$$

$$+ \cancelto{0}{\epsilon_{ijk}\, x^j \left(\frac{dU}{dr}\right)\left(\frac{x^k}{r}\right)}$$

$= 0$ since ~~p~~

$p_j p_k$ and $x_j x_k$ are both symmetric wrt jk

but ϵ_{ijk} is antisymmetric.

Prob: (3.16) Show $Q^a = p_a$, $P_a = -q^a$ is a canonical transformation with $F(q, Q) = \sum_a q^a Q^a$.

Proof: Time independent transform, so only need to show that $\sum_a p_a dq^a = \sum_a P_a dQ^a + dF$

$$\sum_a P_a dQ^a = \sum_a -q^a dp_a$$

$$= \sum_a \left[d(-q^a p_a) + p_a dq^a \right]$$

$$= -d\left(\sum_a q^a p_a\right) + \sum_a p_a dq^a$$

Thus, $F = \sum_a q^a p_a$

$$= \sum_a q^a Q^a$$

$$= F(q, Q) \qquad [\text{type I}]$$

$q\ q'$

$q\ p'$

$p\ q'$

$p\ p$

Problem: Linear momentum as inf. generator of translations

3.11

Consider:

$$G = \hat{n} \cdot \vec{p}$$
$$= \delta_{ij} n^i p_j$$

$$\frac{\partial G}{\partial x^i} = 0$$

$$\frac{\partial G}{\partial p_i} = n_i$$

Thus,

$$\Delta x^i = \Delta\lambda \frac{\partial G}{\partial p_i} = \Delta\lambda \, n_i$$

$$\Delta p_i = -\Delta\lambda \frac{\partial G}{\partial x^i} = 0$$

so

$$\boxed{\begin{aligned} \Delta \vec{r} &= \Delta\lambda \, \hat{n} \\ \Delta \vec{p} &= 0 \end{aligned}}$$

Problem: 3.17 Ang momentum as inf. generator of rotations

Consider:

$$G = \hat{n}\cdot\vec{l} \qquad \vec{l} = \vec{r}\times\vec{p}$$
$$= \epsilon_{ijk}\, n^i x^j p^k$$

$$\frac{\partial G}{\partial x^i} = \epsilon_{jik}\, n^j p^k = -\epsilon_{ijk}\, n^j p^k = -(\hat{n}\times\vec{p})_i$$

$$\frac{\partial G}{\partial p_i} = \epsilon_{jki}\, n^j x^k = \epsilon_{ijk}\, n^j x^k = (\hat{n}\times\vec{r})_i$$

Thus,

$$\Delta x^i = \Delta\theta \frac{\partial G}{\partial p_i} = \Delta\theta(\hat{n}\times\vec{r})_i$$
$$\Delta p_i = -\Delta\theta \frac{\partial G}{\partial x^i} = \Delta\theta(\hat{n}\times\vec{p})_i$$

$$\rightarrow \boxed{\begin{aligned}\Delta\vec{r} &= \Delta\theta(\hat{n}\times\vec{r})\\ \Delta\vec{p} &= \Delta\theta(\hat{n}\times\vec{p})\end{aligned}}$$

active rotation of a vector around $\hat{n}$ through $\Delta\theta$

Example: Take $\hat{n} = \hat{z}$

$$G = \hat{z}\cdot\vec{l}$$
$$= (\vec{r}\times\vec{p})_z$$
$$= xp_y - yp_x$$

$$\vec{l} = \vec{r}\times\vec{p}$$
$$= (yp_z - zp_y)\hat{i}$$
$$+ (zp_x - xp_z)\hat{j}$$
$$+ (xp_y - yp_x)\hat{k}$$

$$\hat{n} \times \vec{r} = \hat{z} \times \vec{r}$$

$$= \hat{x}\,(0 \cdot z - 1 \cdot y) + \hat{y}\,(1 \cdot x - 0 \cdot z) + \hat{z}\,(0 \cdot y - 0 x)$$

$$= -y\hat{x} + x\hat{y}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \downarrow \begin{bmatrix} 1 & -\Delta\theta & 0 \\ \Delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \Delta\theta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Delta x = -y\,\Delta\theta$$
$$\Delta y = x\,\Delta\theta$$
$$\Delta z = 0$$

$$\rightarrow \boxed{\begin{aligned} x &= x_0 + \Delta x = x_0 - y\,\Delta\theta \\ y &= y_0 + \Delta y = y_0 + x\,\Delta\theta \\ z &= z_0 + \Delta z = z_0 \end{aligned}}$$

Problem: (3.12) Harmonic oscillator orbit

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -kq$$

Differentiate again: $\ddot{q} = \frac{1}{m}\dot{p} = -\frac{k}{m}q$

$$\ddot{p} = -k\dot{q} = -\frac{k}{m}p$$

solve for $q(t), p(t)$ with $q(0)=0$, $\dot{q}(0) = \frac{p(0)}{m} = 0$

$$q(t) = A\cos(\omega t) + B\sin(\omega t) \quad , \quad \omega = \sqrt{\frac{k}{m}}$$

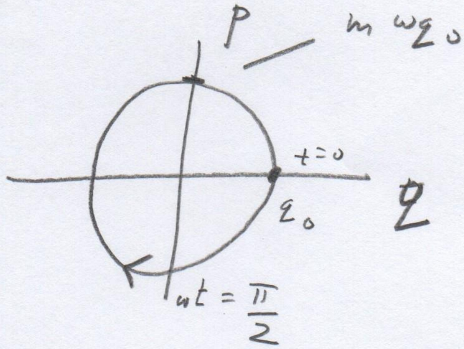
$$p(t) = m\dot{q}$$

$$= m\omega[-A\sin(\omega t) + B\cos(\omega t)]$$

$q(0) = q_0 \rightarrow q_0 = A$

$p(0) = 0 \rightarrow 0 = m\omega[-A \cdot 0 + B] \rightarrow B = 0$

Thus, $$\boxed{\begin{aligned} q(t) &= q_0\cos(\omega t) \\ p(t) &= -m\omega q_0\sin(\omega t) \end{aligned}}$$



(3.13) Cons. laws:

I/II: if L indep of q^a then p_a conserved

III: if $\frac{\partial L}{\partial t} = 0$ then $\sum_a p_a \dot{q}^a - L$ is conserved

---

Now $G(q, p)$ is conserved iff $\{H, G\} = 0$

From Exercise 3.12:

$G = \hat{n} \cdot \vec{p}$ generates inf. spatial translations

$G = \hat{n} \cdot \vec{J}$ " spatial rotations

$G = H$ " time translations

Thus $\vec{p}$ is conserved iff $\{H, \hat{n} \cdot \vec{p}\} = 0$

$\vec{l}$ " $\{H, \hat{n} \cdot \vec{l}\} = 0$

$E = H$ " $\cancelto{0}{\{H, H\}} + \frac{\partial H}{\partial t} = 0$

iff $\frac{\partial H}{\partial t} = 0$

$\cancel{\{H, G\}} =$

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

Problem: Action along actual path for SHO

Exer 3.14

$$S[q] = \int_{t_1}^{t_2} dt \; L(q,\dot{q},t)$$

SHO:

$$L = T - U$$
$$= \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$$
$$= \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2$$
$$= \frac{1}{2}m(\dot{q}^2 - \omega^2 q^2)$$

$\omega = \sqrt{\frac{k}{m}}$

$k = m\omega^2$

General solution:

$$q(t) = A\cos(\omega t + \alpha)$$
$$\rightarrow \dot{q}(t) = -A\omega\sin(\omega t + \alpha)$$

$\sqrt{\frac{2E}{k}} = A$

$\frac{1}{2}A^2 m\omega^2 = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}E}{\cancel{k}}\right)\cancel{m\omega^2}$

$= E$

Thus,
$$L = \frac{1}{2}m\left[A^2\omega^2\sin^2(\omega t+\alpha) - \omega^2 A^2\cos^2(\omega t+\alpha)\right]$$
$$= -\frac{1}{2}m\omega^2 A^2\left(\cos^2(\omega t+\alpha) - \sin^2(\omega t+\alpha)\right)$$
$$= -\frac{1}{2}m\omega^2 A^2\cos[2(\omega t+\alpha)]$$

$$S = \int_{t_1}^{t_2} dt \; L(q,\dot{q},t)$$
$$= -\frac{1}{2}m\omega^2 A^2\int_{t_1}^{t_2} dt\cos[2(\omega t+\alpha)]$$
$$= -\frac{1}{2}m\omega^2 A^2\left(\frac{1}{2\omega}\right)\sin[2(\omega t+\alpha)]\Big|_{t_1}^{t_2}$$

$$= -\frac{1}{4} m\omega A^2 \left(\sin\left[2(\omega t_2+\alpha)\right] - \sin\left[2(\omega t_1+\alpha)\right] \right)$$

$$= \frac{1}{2} m\omega A^2 \Big(\sin(\omega t_1+\alpha)\cos(\omega t_1+\alpha) - \sin(\omega t_2+\alpha)\cos(\omega t_2+\alpha) \Big)$$

<u>Now:</u>

$$\cancel{A\cos(\omega t_1+\alpha) = q_1}, \qquad \cancel{A\cos(\omega t_2+\alpha)} = q_2$$

$$\cancel{A\sin(\omega t_1+\alpha) = A\sin\left(\omega(t_1-t_2)+(\omega t_2+\alpha)\right)}$$

$$\cancel{= A\sin\left(-\omega\Delta t+(\omega t_2+\alpha)\right)}$$

$$\cancel{= A\left[\sin(-\omega\Delta t)\cos(\omega t_2+\alpha) + \cos(-\omega\Delta t)\sin(\omega t_2+\alpha)\right]}$$

$$q_1 = A\cos(\omega t_1+\alpha)$$

$$\begin{aligned}
q_2 &= A\cos(\omega t_2+\alpha) \\
&= A\cos\left(\omega(t_2-t_1) + \omega t_1+\alpha\right) \\
&= A\cos\left(\omega\Delta t + (\omega t_1+\alpha)\right) \\
&= A\left[\cos(\omega\Delta t)\cos(\omega t_1+\alpha) - \sin(\omega\Delta t)\sin(\omega t_1+\alpha)\right] \\
&= \cos(\omega\Delta t)\, q_1 - \sin(\omega\Delta t)\, A\sin(\omega t_1+\alpha)
\end{aligned}$$

Thus,

$$\boxed{A\sin(\omega t_1+\alpha) = \frac{\cos(\omega\Delta t)\, q_1 - q_2}{\sin(\omega\Delta t)}}$$

Similarly,

$$A\sin(\omega t_2+\alpha) = \frac{\cos(\omega\Delta t)q_2 - q_1}{-\sin(\omega\Delta t)}$$

$$= \frac{-\cos(\omega\Delta t)q_2 + q_1}{\sin(\omega\Delta t)}$$

Thus,

$$S = \frac{1}{2}m\omega\Big[A\sin(\omega t_1+\alpha)\underbrace{A\cos(\omega t_1+\alpha)}_{q_1}$$

$$- A\sin(\omega t_2+\alpha)\underbrace{A\cos(\omega t_2+\alpha)}_{q_2}\Big]$$

$$= \frac{1}{2}m\omega\Bigg[\left(\frac{\cos(\omega\Delta t)q_1 - q_2}{\sin(\omega\Delta t)}\right)q_1$$

$$- \left(\frac{-\cos(\omega\Delta t)q_2 + q_1}{\sin(\omega\Delta t)}\right)q_2\Bigg]$$

$$= \boxed{\frac{1}{2}\frac{m\omega}{\sin(\omega\Delta t)}\left[\cos(\omega\Delta t)(q_1^2+q_2^2) - 2q_1q_2\right]}$$

Proof of quantum time evolution expression:

To show: $\frac{d}{dt}\langle\hat{A}\rangle = \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle + \langle\frac{\partial\hat{A}}{\partial t}\rangle$

when $\langle\hat{A}\rangle \equiv \int_V dV\, \Psi^*\hat{A}\,\Psi$

proof: $\frac{d}{dt}\langle\hat{A}\rangle = \int_V dV\left[\frac{\partial\Psi^*}{\partial t}\hat{A}\Psi + \Psi^*\frac{\partial\hat{A}}{\partial t}\Psi + \Psi^*\hat{A}\frac{\partial\Psi}{\partial t}\right]$

$= \left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle + \int_V dV\left[\frac{\partial\Psi^*}{\partial t}\hat{A}\Psi + \Psi^*\hat{A}\frac{\partial\Psi}{\partial t}\right]$

Now: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$

$\rightarrow \frac{\partial\Psi}{\partial t} = \frac{1}{i\hbar}\hat{H}\Psi$

and $\frac{\partial\Psi^*}{\partial t} = \frac{1}{-i\hbar}(\hat{H}\Psi)^*$

Thus, $\int_V dV\left[\frac{\partial\Psi^*}{\partial t}\hat{A}\Psi + \Psi^*\hat{A}\frac{\partial\Psi}{\partial t}\right]$

$= \frac{1}{-i\hbar}\int_V dV\left[(\hat{H}\Psi)^*\hat{A}\Psi - \Psi^*\hat{A}\hat{H}\Psi\right]$

$= \frac{1}{-i\hbar}\int_V dV\left[\Psi^*\hat{H}\hat{A}\Psi - \Psi^*\hat{A}\hat{H}\Psi\right]$

$= \frac{1}{-i\hbar}\int_V dV\, \Psi^*[\hat{H},\hat{A}]\Psi$

$= \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle$

$\rightarrow \boxed{\frac{d}{dt}\langle\hat{A}\rangle = \left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle + \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle}$

Poisson bracket - commutator relation (3.15)

$x, \ p_x$: $\{x, p_x\} = 1$, $\{x, x\} = 0$, $\{p_x, p_x\} = 0$

$\hat{x} = x$, $\hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$

$[\hat{x}, \hat{x}] = 0$

$[\hat{p}_x, \hat{p}_x] = 0$

$$[\hat{x}, \hat{p}_x] f = x \frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}(x f)$$

$$= \cancel{x \frac{\hbar}{i}\frac{\partial f}{\partial x}} - \frac{\hbar}{i} f - \cancel{\frac{\hbar}{i} x \frac{\partial f}{\partial x}}$$

$$= i\hbar f$$

Thus, $\boxed{[\hat{x}, \hat{p}_x] = i\hbar = i\hbar \{x, p\}}$

Ehrenfest's theorem: 3.16

Particle moving in a potential $U(\vec{r},t)$

Show: $$m \frac{d}{dt}\langle \hat{x} \rangle = \langle \hat{p}_x \rangle$$

$$\frac{d}{dt}\langle \hat{p}_x \rangle = -\left\langle \frac{\partial \hat{U}}{\partial x} \right\rangle$$

Proof: $$\langle \hat{A} \rangle \equiv \int dV \, \Psi^* \hat{A} \, \Psi$$

Then $$\frac{d}{dt}\langle \hat{A} \rangle = \frac{1}{i\hbar}\langle [\hat{A},\hat{H}] \rangle + \frac{\partial \hat{A}}{\partial t}$$

Using this general result, taking $\hat{H} = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + \underbrace{V(\vec{r},t)}_{\hat{U}}$, we have

$$m \frac{d}{dt}\langle \hat{x} \rangle = \frac{m}{i\hbar}\langle [\hat{x},\hat{H}] \rangle + \cancel{m\frac{\partial \hat{x}}{\partial t}}^{\,0}$$

$$= \frac{m}{i\hbar}\left\langle \left[\hat{x}, \frac{\hat{\vec{p}}\cdot\hat{\vec{p}}}{2m} + \hat{U}(\vec{r},t)\right] \right\rangle$$

$$= \frac{\cancel{m}}{i\hbar}\frac{1}{2\cancel{m}}\left\langle [\hat{x}, \vec{p}\cdot\vec{p}] \right\rangle$$

$$= \frac{1}{\cancel{i\hbar}}\frac{1}{\cancel{2}}\cancel{2}\,\cancel{i\hbar}\langle p_x \rangle$$

$$= \langle p_x \rangle$$

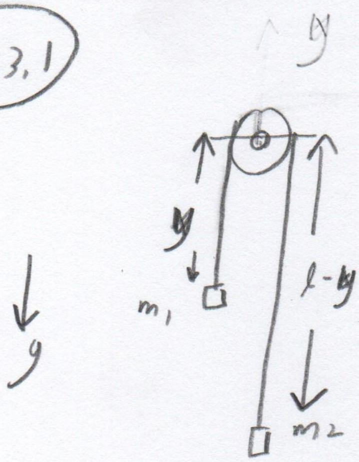
$$\frac{d}{dt}\langle \hat{p}_x \rangle = \frac{1}{i\hbar}\langle [\hat{p}_x, \hat{H}] \rangle + \cancel{\frac{\partial \hat{p}_x}{\partial t}}^{\,0}$$

$$= \frac{1}{i\hbar}\left\langle \left[\hat{p}_x, \frac{\hat{\vec{p}}\cdot\hat{\vec{p}}}{2m} + \hat{U}(\vec{r},t)\right] \right\rangle$$

$$= \frac{1}{i\hbar}\left(\frac{\hbar}{i}\right)\left\langle \frac{\partial \hat{U}}{\partial x} \right\rangle$$

$$= -\left\langle \frac{\partial \hat{U}}{\partial x} \right\rangle$$

Prob 3.1



$$U = -m_1 g y - m_2 g (l-y)$$

$$T = \frac{1}{2} m_1 \dot{y}^2 + \frac{1}{2} m_2 \dot{y}^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{y}^2$$

a) $$L = T - U$$

$$= \frac{1}{2}(m_1 + m_2)\dot{y}^2 + m_1 g y + m_2 g (l-y)$$

b) $$0 = \frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)$$

$$= m_1 g - m_2 g - \frac{d}{dt}\left((m_1 + m_2)\dot{y}\right)$$

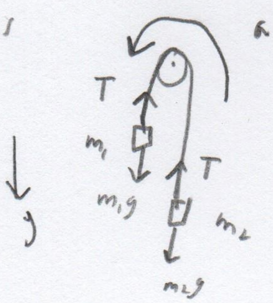
$$= (m_1 - m_2) g - (m_1 + m_2)\ddot{y}$$

$$\boxed{\ddot{y} = \frac{(m_1 - m_2)}{m_1 + m_2} g} = \text{const} \equiv a$$

c) $$y = y_0 + \underbrace{v_0}_{=0 \text{ (assume start from rest)}} t + \frac{1}{2} a t^2$$

$$= y_0 + \frac{1}{2}\left(\frac{m_1 - m_2}{m_1 + m_2}\right) g t^2$$

d) Freshman physics



$$m_1 a = m_1 g - T$$

$$m_2 a = T - m_2 g$$

Add: $(m_1 + m_2) a = (m_1 - m_2) g$

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) g$$ which agrees with $\ddot{y}$

Semiholonomic constraints: (Prob 3.2) ①

$$\varphi_A(x, \dot{x}, t) = 0$$

Virtual displacement $\delta x^\alpha(t)$ (Fixed time)

$$\delta\varphi_A = \sum_\alpha \left(\frac{\partial \varphi_A}{\partial x^\alpha} \delta x^\alpha + \frac{\partial \varphi_A}{\partial \dot{x}^\alpha} \delta \dot{x}^\alpha \right)$$

$$= \sum_\alpha \left(\frac{\partial \varphi_A}{\partial x^\alpha} \delta x^\alpha + \frac{\partial \varphi_A}{\partial \dot{x}^\alpha} \frac{d}{dt}(\delta x^\alpha) \right)$$

$$= \sum_\alpha \left[\frac{\partial \varphi_A}{\partial x^\alpha} \delta x^\alpha + \frac{d}{dt}\left(\frac{\partial \varphi_A}{\partial \dot{x}^\alpha} \delta x^\alpha \right) - \frac{d}{dt}\left(\frac{\partial \varphi_A}{\partial \dot{x}^\alpha} \right) \delta x^\alpha \right]$$

$$= \sum_\alpha \left[\frac{\partial \varphi_A}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial \varphi_A}{\partial \dot{x}^\alpha} \right) \right] \delta x^\alpha$$

$$+ \frac{d}{dt}\left(\sum_\alpha \frac{\partial \varphi_A}{\partial \dot{x}^\alpha} \delta x^\alpha \right)$$

Variational problem:

$$0 = \int dt \sum_\alpha \delta x^\alpha E_\alpha$$

$$= \int dt \sum_\alpha \delta x^\alpha \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) - \frac{\partial L}{\partial x^\alpha} \right]$$

subject to auxiliary conditions

$$0 = \sum_\alpha \delta x^\alpha \left[\frac{\partial \varphi_A}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial \varphi_A}{\partial \dot{x}^\alpha} \right) \right] + \frac{d}{dt}\left(\sum_\alpha \frac{\partial \varphi_A}{\partial \dot{x}^\alpha} \delta x^\alpha \right)$$

$$\rightarrow \quad 0 = \int dt \sum_A \lambda^A(t) \left(\sum_\alpha \delta x^\alpha \left[\frac{\partial \varphi_A}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial \varphi_A}{\partial \dot{x}^\alpha} \right) \right] + \frac{d}{dt}\left(\sum_\alpha \frac{\partial \varphi_A}{\partial \dot{x}^\alpha} \delta x^\alpha \right) \right)$$

So consider

$$0 = \int dt \Big\{ \sum_\alpha \delta x^\alpha \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha}\right]$$

$$+ \sum_A \lambda^A(t) \sum_\alpha \delta x^\alpha \left[\frac{\partial \varphi_A}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial\varphi_A}{\partial \dot{x}^\alpha}\right)\right]$$

$$+ \sum_A \lambda^A(t) \frac{d}{dt}\left(\sum_\alpha \frac{\partial\varphi_A}{\partial \dot{x}^\alpha}\delta x^\alpha\right)\Big\}$$

$$= \int dt \sum_\alpha \delta x^\alpha \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} - \sum_A \lambda^A \left(\frac{\partial \varphi_A}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial\varphi_A}{\partial \dot{x}^\alpha}\right)\right)\right]$$

$$- \int dt \left[\frac{d}{dt}\left(\sum_A \lambda^A \sum_\alpha \frac{\partial\varphi_A}{\partial \dot{x}^\alpha}\delta x^\alpha\right) - \sum_A \frac{d\lambda^A}{dt} \sum_\alpha \frac{\partial\varphi_A}{\partial \dot{x}^\alpha}\delta x^\alpha \right]$$

$$= \int dt \sum_\alpha \delta x^\alpha \left\{\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} - \sum_A \left[\lambda^A \left(\frac{\partial \varphi_A}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial\varphi_A}{\partial \dot{x}^\alpha}\right)\right) - \frac{d\lambda^A}{dt}\frac{\partial\varphi_A}{\partial \dot{x}^\alpha}\right]\right]\right\}$$

$$- \cancel{\sum_A \lambda^A \left(\sum_\alpha \frac{\partial\varphi_A}{\partial \dot{x}^\alpha}\delta x^\alpha\right)\Bigg|_{t_1}^{t_2}}$$

0 for $\delta x^\alpha\big|_{t_1} = 0$, $\delta x^\alpha\big|_{t_2} = 0$

$$\rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} - \sum_A \left[\lambda^A \left(\frac{\partial \varphi_A}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial\varphi_A}{\partial \dot{x}^\alpha}\right)\right) - \frac{d\lambda^A}{dt}\frac{\partial\varphi_A}{\partial \dot{x}^\alpha}\right] = 0}$$

Action:

$$\bar{S}[x,\lambda] = \int dt \left[L(x,\dot{x},t) + \sum_A \lambda^A(t)\varphi_A(x,\dot{x},t) \right]$$

Vary λ^A:

$$\delta\bar{S} = \int dt\, \delta\lambda^A \varphi_A$$

$$\delta\bar{S} = 0 \rightarrow \varphi_A = 0$$

Vary x^α:

$$\delta\bar{S} = \int dt \sum_\alpha \Bigg[\frac{\partial L}{\partial x^\alpha}\delta x^\alpha + \frac{\partial L}{\partial \dot{x}^\alpha}\underbrace{\delta\dot{x}^\alpha}_{\frac{d}{dt}\delta x^\alpha} + \sum_A \lambda^A\left(\frac{\partial\varphi_A}{\partial x^\alpha}\delta x^\alpha + \frac{\partial\varphi_A}{\partial\dot{x}^\alpha}\delta\dot{x}^\alpha\right)\Bigg]$$

$$= \int dt \sum_\alpha \Bigg[\frac{\partial L}{\partial x^\alpha}\delta x^\alpha + \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}^\alpha}\delta x^\alpha\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}^\alpha}\right)\delta x^\alpha + \sum_A \lambda^A\left(\frac{\partial\varphi_A}{\partial x^\alpha}\delta x^\alpha + \frac{d}{dt}\left(\frac{\partial\varphi_A}{\partial\dot{x}^\alpha}\delta x^\alpha\right) - \frac{d}{dt}\left(\frac{\partial\varphi_A}{\partial\dot{x}^\alpha}\right)\delta x^\alpha\right)\Bigg]$$

$$= \int dt \sum_\alpha \delta x^\alpha \Bigg[\frac{\partial L}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}^\alpha}\right) + \sum_A \lambda^A\left(\frac{\partial\varphi_A}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial\varphi_A}{\partial\dot{x}^\alpha}\right)\right)\Bigg]$$

$$+ \int dt \left[\frac{d}{dt}\left(\sum_A \lambda^A \frac{\partial\varphi^A}{\partial\dot{x}^\alpha}\delta x^\alpha\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}^\alpha}\delta x^\alpha\right)\right]$$

boundary term $= 0$ since $\delta x^\alpha\big|_{t_1,t_2} = 0$

$$- \int dt \sum_A \left(\frac{d\lambda^A}{dt}\right)\frac{\partial\varphi^A}{\partial\dot{x}^\alpha}\delta x^\alpha$$

Thus,

$$\delta \bar{S} = 0 \quad \text{iff} \quad \frac{\partial L}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) + \sum_A \left(\lambda^A \left(\frac{\partial \varphi_A}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial \varphi_A}{\partial \dot{x}^\alpha}\right) \right) - \frac{d\lambda^A}{dt} \frac{\partial \varphi}{\partial \dot{x}^\alpha} \right) = 0$$

$$\text{iff} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} - \sum_A \left[\lambda^A \left(\frac{\partial \varphi_A}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial \varphi_A}{\partial \dot{x}^\alpha}\right)\right) - \frac{d\lambda^A}{dt} \frac{\partial \varphi_A}{\partial \dot{x}^\alpha} \right] = 0$$

Prob 3.3

Holonomic constraints

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^\alpha}\right) - \frac{\partial T}{\partial x^\alpha} - F_\alpha - \sum_A \lambda_A \frac{\partial \varphi^A}{\partial x^\alpha} = 0$$

multiply by δx^α, $\sum_\alpha$ and integrate

$$\int_{t_1}^{t_2} dt \left\{ \sum_\alpha \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^\alpha}\right) - \frac{\partial T}{\partial x^\alpha} \right) \delta x^\alpha - \sum_\alpha F_\alpha \delta x^\alpha - \sum_A \lambda_A \underbrace{\sum_\alpha \frac{\partial \varphi^A}{\partial x^\alpha} \delta x^\alpha}_{\delta \varphi^A} \right\} = 0$$

$$\delta \int_{t_1}^{t_2} dt \left[-T - \sum_A \lambda_A \varphi^A \right] - \int_{t_1}^{t_2} dt \sum_\alpha F_\alpha \delta x^\alpha = 0$$

$$-\delta \left(\int_{t_1}^{t_2} dt \left[T + \sum_A \lambda_A \varphi^A \right] \right) - \int_{t_1}^{t_2} dt \sum_\alpha F_\alpha \delta x^\alpha = 0$$

$$\boxed{\delta \left(\int_{t_1}^{t_2} dt \left[T + \sum_A \lambda_A \varphi^A \right] \right) + \int_{t_1}^{t_2} dt \sum_\alpha F_\alpha \delta x^\alpha = 0}$$

not the variation of anything

Hamiltonian in sph. polar coords: Prob 3.4

$$U = U(\vec{r}, t) = U(r, \theta, \phi, t)$$

$$H = \left[\sum_a p_a \dot{q}^a - L(q, \dot{q}, t) \right] \Bigg|_{\dot{q}^a = \dot{q}^a(q, p, t)}$$

$$L = T - U$$
$$= \frac{1}{2} m \vec{v} \cdot \vec{v} - U(\vec{r}, t)$$
$$= \frac{1}{2} m \left[\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right] - U(r, \theta, \phi, t)$$

$$p_r \equiv \frac{\partial L}{\partial \dot{r}} = m\dot{r} \rightarrow \dot{r} = p_r / m$$

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \rightarrow \dot{\theta} = \frac{p_\theta}{m r^2}$$

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\theta \, \dot{\phi} \rightarrow \dot{\phi} = \frac{p_\phi}{m r^2 \sin^2\theta}$$

$$\rightarrow H(r, \theta, \phi, p_r, p_\theta, p_\phi) = \Big[p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi}$$
$$- \frac{1}{2} m \left[\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right] + U(r, \theta, \phi, t) \Big] \Bigg|_{\dot{q}^a = \dot{q}^a(q, p, t)}$$

$$= \frac{p_r^2}{m} + \frac{p_\theta^2}{m r^2} + \frac{p_\phi^2}{m r^2 \sin^2\theta} - \frac{1}{2} m \left[\frac{p_r^2}{m^2} + \frac{r^2 p_\theta^2}{m^2 r^4} + \frac{r^2 \sin^2\theta \, p_\phi^2}{m^2 r^4 \sin^4\theta} \right]$$
$$+ U(r, \theta, \phi, t)$$

$$= \frac{1}{2m} \left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta} \right] + U(r, \theta, \phi, t)$$

Hamiltonian for pt charge in an EM field: Prob 3.5

$$U(\vec{r},\dot{\vec{r}},t) = q\left[\Phi(\vec{r},t) - \vec{A}(\vec{r},t)\cdot\dot{\vec{r}}\right]$$

$$T = \frac{1}{2} m \dot{\vec{r}}\cdot\dot{\vec{r}}$$

$$L = T - U$$

Hamiltonian:

$$H(\vec{r},\vec{p},t) = \left[\sum_i p_i \dot{x}_i - L(\vec{r},\dot{\vec{r}},t)\right]\Big|_{\dot{\vec{r}}=\dot{\vec{r}}(\vec{r},\vec{p},t)}$$

Now:

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}_i + qA_i$$

Thus, $\vec{p} = m\dot{\vec{r}} + q\vec{A}$

or $\dot{\vec{r}} = \frac{1}{m}\left(\vec{p} - q\vec{A}\right)$

$$\rightarrow \quad H(\vec{r},\vec{p},t) = \sum_i p_i\left(\frac{1}{m}\right)(p_i - qA_i) - \frac{1}{2}m\left(\frac{1}{m}\right)^2(\vec{p}-q\vec{A})\cdot(\vec{p}-q\vec{A})$$

$$+ q\left[\Phi - \vec{A}\cdot\frac{1}{m}(\vec{p}-q\vec{A})\right]$$

$$= \frac{1}{m}\left(\vec{p}\cdot\vec{p} - q\,\vec{p}\cdot\vec{A}\right) - \frac{|\vec{p}-q\vec{A}|^2}{2m} + q\Phi$$

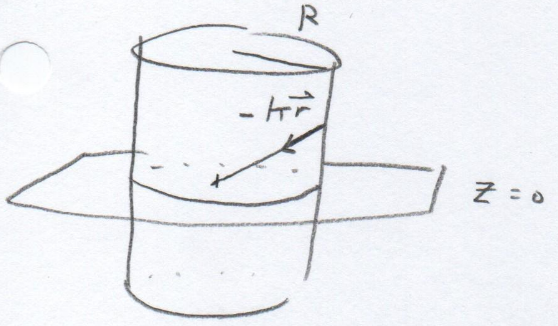
$$- \frac{q}{m}\vec{A}\cdot(\vec{p}-q\vec{A})$$

$$= \frac{1}{m}\vec{p}\cdot\left(\vec{p}-q\vec{A}\right) - \frac{q}{m}\vec{A}\cdot(\vec{p}-q\vec{A}) - \frac{|\vec{p}-q\vec{A}|^2}{2m} + q\Phi$$

$$= \frac{1}{m}\left(\vec{p}-q\vec{A}\right)\cdot\left(\vec{p}-q\vec{A}\right) - \frac{1}{2m}|\vec{p}-q\vec{A}|^2 + q\Phi$$

$$= \frac{|\vec{p}-q\vec{A}|^2}{2m} + q\Phi$$

Hamiltonian, etc. for particle moving on cylinder (Prob 3.6)



$$\vec{F} = -k\vec{r}$$
$$= -k(R\hat{\rho} + z\hat{k})$$
$$= \underbrace{-kR\hat{\rho}}_{\perp \text{ to cylinder}} - kz\hat{k}$$

(balanced by constraint force)

$(\vec{r} = R\hat{\rho} + z\hat{k})$

Thus, $\vec{F}^{(a)} = -kz\hat{k} = -\vec{\nabla}U$

where $U = \frac{1}{2}kz^2$

$$T = \frac{1}{2}m\,\vec{v}\cdot\vec{v}$$
$$= \frac{1}{2}m\left[R^2\dot{\phi}^2 + \dot{z}^2\right] \quad (\text{since } \dot{\rho} = 0)$$

$$\rightarrow L = T - U$$
$$= \frac{1}{2}m\left[R^2\dot{\phi}^2 + \dot{z}^2\right] - \frac{1}{2}kz^2$$

<u>Now</u>: $p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi} \rightarrow \dot{\phi} = \frac{p_\phi}{mR^2}$

$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = \frac{p_z}{m}$

<u>Hamiltonian</u>:

$$H = \left[p_\phi\dot{\phi} + p_z\dot{z} - L(\phi, z, \dot{\phi}, \dot{z}, t)\right]\Big|_{\dot{\phi} = \frac{p_\phi}{mR^2},\ \dot{z} = \frac{p_z}{m}}$$

$$= \frac{p_\phi^2}{mR^2} + \frac{p_z^2}{m} - \frac{1}{2}m\left[R^2\frac{p_\phi^2}{m^2R^4} + \frac{p_z^2}{m^2}\right] + \frac{1}{2}kz^2$$

$$= \frac{1}{2m}\left(\frac{p_\phi^2}{R^2} + p_z^2\right) + \frac{1}{2}kz^2$$

Hamilton's eqs: (1st order)

$$\dot{q}^a = \frac{\partial H}{\partial p_a} \quad , \quad \dot{p}^a = -\frac{\partial H}{\partial q_a}$$

$$\rightarrow \quad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mR^2} \quad \rightarrow \quad p_\phi = mR^2\dot{\phi}$$

$$\dot{z} = \frac{\partial H}{\partial z} = \frac{p_z}{m} \quad \rightarrow \quad p_z = m\dot{z}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \rightarrow \quad \boxed{p_\phi = \text{const}}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz$$

Convert to second-order equations:

$$p_\phi = \text{const} \quad \rightarrow \quad mR^2\dot{\phi} = \text{const} \quad \rightarrow \quad \dot{\phi} = \Omega \ (\text{const})$$

$$\rightarrow \quad \boxed{\phi = \Omega t + \phi_0}$$

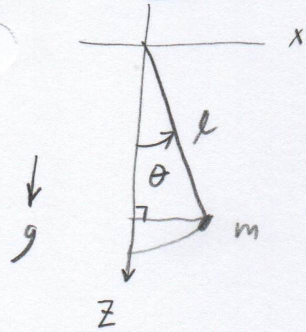
$$\dot{p}_z = -kz \quad \longrightarrow \quad m\ddot{z} = -kz$$

$$\ddot{z} = -\frac{k}{m}z = -\omega^2 z \quad \left(\omega \equiv \sqrt{\frac{k}{m}}\right)$$

General solution: $\boxed{z = A\sin(\omega t) + B\cos(\omega t)}$

Hamiltonian and Hamilton's equations for a simple pendulum

Prob 3.7



$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = -mgl\cos\theta + mgl$$

$$= mgl\left[1-\cos\theta\right]$$

$$\left(U(0) = 0, \quad U(\theta = \tfrac{\pi}{2}) = mgl \right)$$

$$L = T - U$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\left[1-\cos\theta\right]$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \rightarrow \dot{\theta} = \frac{p_\theta}{m l^2}$$

Thus,

$$\boxed{H = \left(p_\theta \dot{\theta} - \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\left[1-\cos\theta\right] \right)_{\dot{\theta} = \frac{p_\theta}{m l^2}}}$$

$$= \frac{p_\theta^2}{m l^2} - \frac{1}{2} m l^2 \left(\frac{p_\theta^2}{m^2 l^4} \right) + mgl\left[1-\cos\theta\right]$$

$$= \frac{1}{2} \frac{p_\theta^2}{m l^2} + mgl\left[1-\cos\theta\right]$$

Hamilton's eqs:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m l^2} \quad \left(\rightarrow \quad p_\theta = m l^2 \dot{\theta} \right)$$

Converting the two 1st-order equations to a single 2nd-order equation:

$$-mgl\sin\theta = \dot{p}_\theta = m l^2 \ddot{\theta}$$

$$\rightarrow \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$

Problem: $\sum p\,dq = \sum P\,dQ$ for Cartesian & sph. polar coords

3.8

$$L = T - U$$

where $U = U(\vec{r})$ time-independent potential

a) $T = \frac{1}{2} m v^2$

$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$\left.\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ p_y &= m\dot{y} \\ p_z &= m\dot{z} \end{aligned}\right\} \text{Cartesian}$$

b) $T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2)$

$$\left.\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \\ p_\phi &= m r^2 \sin^2\theta\, \dot{\phi} \end{aligned}\right\} \text{sph. polar}$$

c) $p_x\,dx + p_y\,dy + p_z\,dz = (p_x \dot{x} + p_y \dot{y} + p_z \dot{z})\,dt$

$= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\,dt$

$= \frac{1}{2} T\,dt$

$p_r\,dr + p_\theta\,d\theta + p_\phi\,d\phi = (p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi})\,dt$

$= m(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2)\,dt$

$= \frac{1}{2} T\,dt$

d) $\sum p_a\,dq^a = \frac{1}{2} T\,dt$ so $\sum_a p_a \dot{q}^a = \frac{1}{2}$(Kinetic energy)

Prob 3.10

$$dF = \sum_a \left(p_a dq^a - P_a dQ^a\right) + (H' - H)dt$$

<u>Type I:</u>
$F(q, Q, t)$

$$p_a = \frac{\partial F}{\partial q^a}, \quad P_a = \frac{-\partial F}{\partial Q^a}, \quad H' = H + \frac{\partial F}{\partial t}$$

<u>Type II:</u>
$F(q, P, t)$

$$dF = \sum_a \left(p_a dq^a - d(P_a Q^a) + Q^a dP_a\right) + (H' - H)dt$$

$$d\underbrace{\left[F + \sum_a P_a Q^a\right]}_{F_2} = \sum_a \left(p_a dq^a + Q^a dP_a\right) + (H' - H)dt$$

Thus, $p_a = \frac{\partial F_2}{\partial q_a}$, $Q^a = \frac{\partial F_2}{\partial P_a}$, $H' = H + \frac{\partial F_2}{\partial t}$

<u>Type III:</u>
$F(p, Q, t)$

$$dF = \sum_a \left(d(p_a q^a) - q^a dp_a - P_a dQ^a\right) + (H' - H)dt$$

$$d\underbrace{\left[F - \sum_a p_a q^a\right]}_{F_3} = \sum_a \left(-q^a dp_a - P_a dQ^a\right) + (H' - H)dt$$

Thus, $-q^a = \frac{\partial F_3}{\partial p_a}$, $-P_a = \frac{\partial F_3}{\partial Q^a}$, $H' = H + \frac{\partial F_3}{\partial t}$

<u>Type IV:</u>
$F(p, P, t)$

$$d\left[F - \sum_a p_a q^a\right] = \sum_a \left(-q^a dp_a - d(P_a Q^a) + Q^a dP_a\right) + (H' - H)dt$$

$$d\underbrace{\left[F - \sum_a p_a q^a + \sum_a P_a Q^a\right]}_{F_4} = \sum_a \left(-q^a dp_a + Q^a dP_a\right) + (H' - H)dt$$

Thus, $-q^a = \frac{\partial F_4}{\partial p_a}$, $Q^a = \frac{\partial F_4}{\partial P_a}$, $H' = H + \frac{\partial F_4}{\partial t}$

Problem: 3.11 $\{f,g\}_{(q,p)} = \{f,g\}_{(Q,P)}$

Let y^α any set of coords on Γ

Define

$$\Omega_{\alpha\beta} = \left(\sum_a dp_a \wedge dq^a\right)_{\alpha\beta}$$

$$= \sum_a \left(\frac{\partial p_a}{\partial y^\alpha}\frac{\partial q^a}{\partial y^\beta} - \alpha \leftrightarrow \beta\right)$$

a) Take $y^\alpha = (q^a, p_a)$

Then

$$\Omega_{cd}\Big|_{(q,q)} = \sum_a \left(\cancelto{}{\frac{\partial p_a}{\partial q^c}}\frac{\partial q^a}{\partial q^d} - \alpha \leftrightarrow \beta\right)$$

$$= 0$$

$$\Omega_{cd}\Big|_{(p,p)} = \sum_a \left(\frac{\partial p_a}{\partial p_c}\cancelto{0}{\frac{\partial q^a}{\partial p_d}} - \alpha \leftrightarrow \beta\right)$$

$$= 0$$

$$\Omega_{cd}\Big|_{(q,p)} = \sum_a \left(\cancelto{0}{\frac{\partial p_a}{\partial q^c}}\cancelto{0}{\frac{\partial q^a}{\partial p_d}} - \frac{\partial p_a}{\partial p_d}\frac{\partial q^a}{\partial q^c}\right)$$

$$= -\sum_a \delta_a^d \delta_c^a$$

$$= -\delta_c^d$$

$$= -\mathbb{1}_{n\times n}$$

$$\Omega_{cd}\Big|_{(p,q)} = \sum_a \left(\frac{\partial p_a}{\partial p_c}\,\frac{\partial q^a}{\partial q^d} - \cancel{\frac{\partial p_a}{\partial q^d}}^{\,0}\,\cancel{\frac{\partial q^a}{\partial p_c}}^{\,0} \right)$$

$$= \sum_a \delta^c_a\,\delta^a_d$$

$$= \delta^c_d$$

$$= \mathbb{1}_{n\times n}$$

Thus $\Omega_{\alpha\beta} = \begin{array}{c c} & \begin{array}{cc} q & p \end{array} \\ \begin{array}{c} q \\ p \end{array} & \begin{bmatrix} 0_{n\times n} & -\mathbb{1}_{n\times n} \\ \mathbb{1}_{n\times n} & 0_{n\times n} \end{bmatrix} \end{array}$

Inverse of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Thus, $(\Omega^{-1})^{\alpha\beta} = \begin{array}{c c} & \begin{array}{cc} q & p \end{array} \\ \begin{array}{c} q \\ p \end{array} & \begin{bmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{bmatrix} \end{array}$

---

b) $$\{f,g\}_{(q,p)} = \sum_a \left(\frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a} \right)$$

Compare: $$\sum_{\alpha,\beta} (\Omega^{-1})^{\alpha\beta}\,\frac{\partial f}{\partial y^\alpha}\,\frac{\partial g}{\partial \beta^2} = \sum_{\alpha,\beta} \frac{\partial f}{\partial y^\alpha}\,(\Omega^{-1})^{\alpha\beta}\,\frac{\partial g}{\partial y^\beta}$$

$$= \sum_{a,b} \left(\frac{\partial f}{\partial q^a}\,\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\,\frac{\partial g}{\partial q^a} \right)$$

$$= \{f,g\}_{(q,p)}$$

c) Type I:

$$\sum_a p_a \dot{q}^a - H = \sum_a P_a \dot{Q}^a - H' + \frac{dF}{dt}$$

For time independent CT, $H = H'$

$$\rightarrow \quad \sum_a p_a \dot{q}^a = \sum_a P_a \dot{Q}^a + \frac{dF}{dt}$$

$$\sum_a \left(p_a \, dq^a - P_a \, dQ^a \right) = dF$$

$$\sum_a \sum_\beta \left(p_a \left(\frac{\partial q^a}{\partial y^\beta} \right) - P_a \left(\frac{\partial Q^a}{\partial y^\beta} \right) \right) dy^\beta = dF$$

d) Integrability condition

$$dF = \sum_\beta \frac{\partial F}{\partial y^\beta} \, dy^\beta$$

$$0 = \sum_a \left\{ \left(\frac{\partial p_a}{\partial y^\alpha} \right)\left(\frac{\partial q^a}{\partial y^\beta} \right) - \left(\frac{\partial P_a}{\partial y^\alpha} \right)\left(\frac{\partial Q^a}{\partial y^\beta} \right) - \left(\frac{\partial p_a}{\partial y^\beta} \right)\left(\frac{\partial q^a}{\partial y^\alpha} \right) + \left(\frac{\partial P_a}{\partial y^\beta} \right)\left(\frac{\partial Q^a}{\partial y^\alpha} \right) \right\}$$

$$\Leftrightarrow \quad \sum_a \left(\frac{\partial P_a}{\partial y^\alpha} \frac{\partial Q^a}{\partial y^\beta} - \frac{\partial P_a}{\partial y^\beta} \frac{\partial Q^a}{\partial y^\alpha} \right) = \sum_a \left(\frac{\partial p_a}{\partial y^\alpha} \frac{\partial q^a}{\partial y^\beta} - \frac{\partial p_a}{\partial y^\beta} \frac{\partial q^a}{\partial y^\alpha} \right)$$

Prob 3.12

$$\Omega = \sum_a dp_a \wedge dq^a \qquad (2\text{-form})$$

$$\{f, g\}_{(q,p)} = (\Omega^{-1})^{\alpha\beta} \frac{\partial f}{\partial y^\alpha} \frac{\partial g}{\partial y^\beta}$$

Type I time-indep canonical transformation

$$dF = \sum_a (p_a dq^a - P_a dQ^a)$$

$$LHS = d(dF) = 0 \qquad (\text{all exact forms are closed})$$

$$RHS = \sum_a \Big(dp_a \wedge dq^a + p_a \cancel{ddq^a}^{\,0}$$

$$- dP_a \wedge dQ^a - P_a \cancel{ddQ^a}^{\,0}\Big)$$

$$= \sum_a dp_a \wedge dq^a - \sum_a dP_a \wedge dQ^a$$

LHS = RHS iff

$$\boxed{\sum_a dp_a \wedge dq^a = \sum_a dP_a \wedge dQ^a}$$

Poisson brackets for angular momentum: Prob 3.13

$\vec{L} = \vec{r} \times \vec{p}$

$L_i = \epsilon_{ijk} \, x_j \, p_k$

$$\{L_i, L_j\} = \sum_k \left(\frac{\partial L_i}{\partial x_k} \frac{\partial L_j}{\partial p_k} - i \leftrightarrow j \right)$$

$$\frac{\partial L_i}{\partial x_k} = \frac{\partial}{\partial x^k}[\epsilon_{i\ell m} \, x_\ell \, p_m]$$

$$= \epsilon_{ikm} \, p_m$$

$$\frac{\partial L_j}{\partial p_k} = \frac{\partial}{\partial p_k}[\epsilon_{j\ell m} \, x_\ell \, p_m]$$

$$= \epsilon_{j\ell k} \, x_\ell$$

Thus, $$\{L_i, L_j\} = \sum_k \left(\epsilon_{ikm} p_m \, \epsilon_{j\ell k} x_\ell - i \leftrightarrow j \right)$$

$$= \epsilon_{mik} \, \epsilon_{j\ell k} \, p_m x_\ell - i \leftrightarrow j$$

$$= (\delta_{mj} \delta_{i\ell} - \delta_{m\ell} \cancel{\delta_{ij}}) \, p_m x_\ell - i \leftrightarrow j$$

$$= \delta_{mj} \delta_{i\ell} \, p_m x_\ell - \delta_{mi} \delta_{j\ell} \, p_m x_\ell$$

$$= p_j x_i - p_i x_j$$

$$= \boxed{\epsilon_{ijk} \, L_k}$$

check: $\epsilon_{ijk} L_k = \epsilon_{ijk} \, \epsilon_{k\ell m} \, x_\ell p_m$

$$= (\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}) \, x_\ell p_m = x_i p_j - x_j p_i$$

Problem: Hamilton principle function for SHO

3.14

a) $$H = \frac{p^2}{2m} + \frac{1}{2} k q^2$$

$$= \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \qquad \text{where } \omega = \sqrt{\frac{k}{m}}$$

$$= \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$$

$$0 = H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t}$$

$$= \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2} k q^2 + \frac{\partial S}{\partial t}$$

$$S(q, E, t, t_0) = W(q, E) - E(t - t_0)$$

$$\frac{\partial S}{\partial t} = -E \quad , \quad p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q}$$

Thus,

$$0 = \frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2} k q^2 - E$$

$$\boxed{\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2} k q^2 = E}$$

$$\rightarrow \quad \frac{\partial W}{\partial q} = \sqrt{2m\left(E - \frac{1}{2} k q^2\right)}$$

so $$\boxed{W = \int dq \sqrt{2m\left(E - \frac{1}{2} k q^2\right)}}$$ — indefinite integral

Define:

$$\tau = \frac{\partial S}{\partial E}$$

$$= \frac{\partial W}{\partial E} - (t - t_0)$$

$$\text{so } (t - t_0) + \tau = \frac{\partial W}{\partial E}$$

$$= \int dq \, \frac{1}{\cancel{2}\sqrt{2m(\ )}} \, \cancel{2}m$$

$$= \sqrt{\frac{m}{2}} \int dq \, \frac{1}{\sqrt{E - \frac{1}{2} k q^2}}$$

$$= \sqrt{\frac{m}{2E}} \int dq \, \frac{1}{\sqrt{1 - \frac{k}{2E} q^2}}$$

$$\text{Let } \sin\theta = q\sqrt{\frac{k}{2E}}$$

$$- \cos\theta \, d\theta = \sqrt{\frac{k}{2E}} \, dq$$

$$\rightarrow \boxed{(t - t_0) + \tau} = \sqrt{\frac{m}{\cancel{2E}}} \sqrt{\frac{\cancel{2E}}{k}} \int \cancel{\cos\theta} \, d\theta \, \frac{1}{\cancel{\sqrt{1 - \sin^2\theta}}}$$

$$= \sqrt{\frac{m}{k}} \, \theta$$

$$= \boxed{\frac{1}{\omega} \sin^{-1}\left(q \sqrt{\frac{k}{2E}} \right)}$$

Invert:

$$\boxed{q = \sqrt{\frac{2E}{k}} \sin\left(\omega(t-t_0) + \omega T\right)}$$

$$\alpha = E$$

$$\beta = \frac{\partial S}{\partial \alpha}$$

$$= A \sin\left(\omega(t-t_0) + \beta\right), \quad \beta \equiv \omega T, \quad A = \sqrt{\frac{2E}{k}}$$

Also

$$p = \frac{\partial S}{\partial q}$$

$$= \frac{\partial W}{\partial q}$$

$$= \sqrt{2m\left(E - \frac{1}{2}kq^2\right)}$$

$$= \sqrt{2m\left(E - \cancel{\frac{1}{2}}\left(\frac{\cancel{2}E}{\cancel{k}}\right)\cancel{k}\sin^2\left(\omega(t-t_0)+\beta\right)\right)}$$

$$= \sqrt{2mE}\cos\left(\omega(t-t_0)+\beta\right) = A m\omega \cos\left(\omega(t-t_0)+\beta\right)$$

NOTE: $\sqrt{2mE} = \sqrt{\frac{2E}{k}}\sqrt{km}$

$$= \sqrt{\frac{2E}{k}}\, m\omega \qquad k = m\omega^2$$

$$= A m\omega$$

$$q_0 = q(t_0) = \sqrt{\frac{2E}{k}}\cos\beta$$

$$p_0 = p(t_0) = \sqrt{2mE}\sin\beta$$

check: $\frac{1}{2}kq_0^2 + \frac{p_0^2}{2m} = \frac{1}{\cancel{2}}\cancel{k}\left(\frac{\cancel{2}E}{\cancel{k}}\right)\sin^2\beta + \frac{1}{\cancel{2m}}(\cancel{2m}E)\cos^2\beta$

$$= E\left(\sin^2\beta + \cos^2\beta\right)$$

$$= E$$

Return to W:

$$W(q, E) = \int dq \sqrt{2m\left(E - \tfrac{1}{2} k q^2\right)}$$

$$S(q, E, t) = \int dq \sqrt{2m\left(E - \tfrac{1}{2} k q^2\right)} - \underline{E(t-t_0)}$$

substitute for q

$$= \int dq \sqrt{2mE}\left(1 - \frac{k}{2E} q^2\right)^{\frac{1}{2}} - \underline{E(t-t_0)}$$

$$= \int_{t_0}^{t} \sqrt{\frac{2E}{\cancel{k}}} \cos(\omega(\Delta t) + \beta)\, \cancel{\omega}\, dt \sqrt{2\cancel{m}E} \cos(\omega\Delta t + \beta) - E(t-t_0)$$

$$= 2E \int_{t_0}^{t} dt \left(\cos^2(\omega\Delta t + \beta) - \frac{1}{2}\right)$$

$$= 2E \int_{t_0}^{t} dt \left(\frac{1}{2}\left(\cancel{1} + \cos\left(2(\omega\Delta t + \beta)\right)\right) - \cancel{\frac{1}{2}}\right)$$

$$= E \int_{t_0}^{t} dt \cos\left(2(\omega\Delta t + \beta)\right) \qquad \Delta t = t - t_0$$

$$= \frac{E}{2\omega} \sin\left(2(\omega\Delta t + \beta)\right)\Big|_{t_0}^{t}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1 \rightarrow \cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

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$$= \frac{E}{2w}\left[\sin(2(w\Delta t+\beta)) - \sin(2\beta)\right]$$

$$= \frac{E}{w}\left[\underbrace{\sin(w\Delta t+\beta)}_{\frac{q}{\sqrt{\frac{2E}{k}}}}\cos(w\Delta t+\beta) - \underbrace{\sin\beta}_{\frac{q_0}{\sqrt{\frac{2E}{k}}}}\cos\beta\right]$$

NOW:

$$q = \sqrt{\frac{2E}{k}}\sin(w\Delta t+\beta)$$

$$= \sqrt{\frac{2E}{k}}\left(\sin(w\Delta t)\cos\beta + \cos(w\Delta t)\overbrace{\sin\beta}^{\frac{q_0}{\sqrt{\frac{2E}{k}}}}\right)$$

$$= \sqrt{\frac{2E}{k}}\sin(w\Delta t)\cos\beta + q_0\cos(w\Delta t)$$

thus $\boxed{\sqrt{\frac{2E}{k}}\cos\beta = \frac{q - q_0\cos(wt)}{\sin(w\Delta t)}}$

similarly,

$$\sqrt{\frac{2E}{k}}\cos(w\Delta t+\beta) = \sqrt{\frac{2E}{k}}\left(\cos(w\Delta t)\cos\beta - \sin(w\Delta t)\sin\beta\right)$$

$$= \cos(w\Delta t)\left(\frac{q-q_0\cos(wt)}{\sin(w\Delta t)}\right) - \sin(wt)\,q_0$$

$$= \frac{-q_0 + q\cos(w\Delta t)}{\sin(w\Delta t)}$$

Thus,

$$S = \frac{E}{\omega}\left[\frac{q}{\sqrt{\frac{2E}{K}}}\frac{\left(-q_0 + q\cos(\omega \Delta t)\right)}{\sqrt{\frac{2E}{K}}\sin(\omega \Delta t)} + \frac{q_0}{\sqrt{\frac{2E}{K}}}\frac{q - q_0\cos(\omega \Delta t)}{\sqrt{\frac{2E}{K}}\sin(\omega \Delta t)}\right]$$

$$= \frac{\cancel{E}}{\omega}\frac{1}{\left(\frac{2\cancel{E}}{K}\right)\sin(\omega \Delta t)}\left[-qq_0 + q^2\cos(\omega \Delta t) - q_0 q + q_0^2\cos(\omega \Delta t)\right]$$

$$= \frac{\overset{m\omega^2}{K}}{2\omega}\frac{1}{\sin(\omega \Delta t)}\left[\left(q_0^2 + q^2\right)\cos(\omega \Delta t) - 2qq_0\right]$$

$$= \frac{m\omega}{2\sin(\omega \Delta t)}\left[\left(q_0^2 + q^2\right)\cos(\omega \Delta t) - 2qq_0\right]$$

$$\cancel{S(q, E, t_0) = W(q, E) - E(t-t_0)}$$

$$\cancel{= \frac{E}{\omega}\left[\sin^{-1}\left(q\sqrt{\frac{k}{2E}}\right) + q\sqrt{\frac{k}{2E}}\sqrt{1-\frac{k}{2E}q^2}\right]}$$

$$\cancel{- E(t-t_0)}$$

$$\cancel{\omega \Delta t} + \beta$$

EXTRA

---

convert E to ~~q~~ $\quad E = E(q, q_0)$

$$E = \frac{1}{2}kq^2 + \frac{p^2}{2m}$$

Need p as a function of (q_0, q):

$$q = A\sin(\omega(t-t_0)+\beta)$$

$$= A\sin(\omega(t-t_0))\cos\beta + A\cos(\omega(t-t_0))\sin\beta$$

$$= \underbrace{A\sin\beta}_{q_0}\cos(\omega(t-t_0)) + A\cos\beta\,\sin(\omega(t-t_0))$$

$$p = Am\omega\cos(\omega(t-t_0)+\beta)$$

$$= Am\omega\left(\cos(\omega(t-t_0))\cos\beta - \sin(\omega(t-t_0))\sin\beta\right)$$

$$= \underbrace{Am\omega\cos\beta}_{p_0}\cos(\omega(t-t_0)) - Am\omega\sin\beta\,\sin(\omega(t-t_0))$$

NOTE: $A\cos\beta = \dfrac{p_0}{m\omega}$, $\quad Am\omega\sin\beta = m\omega q_0$

$$q = q_0 \cos(\omega(t-t_0)) + \frac{p_0}{m\omega}\sin(\omega(t-t_0))$$

$$p = p_0 \cos(\omega(t-t_0)) - m\omega q_0 \sin(\omega(t-t_0))$$

Thus,

$$\begin{pmatrix} q \\ p \end{pmatrix} = \underbrace{\begin{pmatrix} \cos & \frac{1}{m\omega}\sin \\ -m\omega\sin & \cos \end{pmatrix}}_{\det = 1} \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$$

$$\boxed{\underline{\text{Note:}} \quad \left(\frac{p_0}{m\omega}\right) = \frac{q - q_0\cos(\omega\Delta t)}{\sin(\omega\Delta t)}}$$

Thus,

$$E = \frac{1}{2}k q_0^2 + \frac{p_0^2}{2m} = \frac{1}{2}m\omega^2 q_0^2 + \frac{p_0^2}{2m}$$

$$= \frac{1}{2m}\left(p_0^2 + m^2\omega^2 q_0^2\right)$$

$$= \frac{1}{2m}\left(m^2\omega^2\left(\frac{q - q_0\cos(\omega\Delta t)}{\sin(\omega\Delta t)}\right)^2 + m^2\omega^2 q_0^2\right)$$

$$= \frac{m\omega^2}{2}\left(\frac{q^2 + q_0^2\cos^2(\omega\Delta t) - 2qq_0\cos(\omega\Delta t)}{\sin^2(\omega\Delta t)} + \frac{q_0^2\sin^2(\omega\Delta t)}{\sin^2(\omega\Delta t)}\right)$$

$$\boxed{= \frac{m\omega^2}{2}\frac{1}{\sin^2(\omega\Delta t)}\left[q^2 + q_0^2 - 2qq_0\cos(\omega\Delta t)\right]}$$

Problem: 3.15 HJ from Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) + U(\vec{r},t)\,\Psi(\vec{r},t) = i\hbar\frac{\partial\Psi}{\partial t}$$

Assume: $\Psi(\vec{r},t) = A(\vec{r},t)\, e^{\frac{iS(\vec{r},t)}{\hbar}}$

RHS: $$i\hbar\frac{\partial\Psi}{\partial t} = i\hbar\left(\frac{\partial A}{\partial t}e^{iS/\hbar} + \frac{i}{\hbar}Ae^{iS/\hbar}\frac{\partial S}{\partial t}\right)$$

$$= i\hbar\left(\frac{\partial A}{\partial t} + \frac{i}{\hbar}A\frac{\partial S}{\partial t}\right)e^{iS/\hbar}$$

$$= e^{iS/\hbar}\left[i\hbar\frac{\partial A}{\partial t} - A\frac{\partial S}{\partial t}\right]$$

LHS: $$-\frac{\hbar^2}{2m}\vec{\nabla}\cdot\vec{\nabla}\Psi = -\frac{\hbar^2}{2m}\vec{\nabla}\cdot\left[e^{\frac{iS}{\hbar}}\vec{\nabla}A + Ae^{\frac{iS}{\hbar}}\frac{i}{\hbar}\vec{\nabla}S\right]$$

$$= -\frac{\hbar^2}{2m}\left[\frac{i}{\hbar}e^{iS/\hbar}\vec{\nabla}S\cdot\vec{\nabla}A + e^{\frac{iS}{\hbar}}\nabla^2A + \frac{i}{\hbar}e^{\frac{iS}{\hbar}}\vec{\nabla}A\cdot\vec{\nabla}S - \frac{1}{\hbar^2}e^{\frac{iS}{\hbar}}A\vec{\nabla}S\cdot\vec{\nabla}S + Ae^{iS/\hbar}\frac{i}{\hbar}\nabla^2S\right]$$

$$= -\frac{\hbar^2}{2m}e^{\frac{iS}{\hbar}}\left[2\frac{i}{\hbar}\vec{\nabla}A\cdot\vec{\nabla}S + \nabla^2A + \frac{i}{\hbar}A\nabla^2S - \frac{1}{\hbar^2}A|\vec{\nabla}S|^2\right]$$

$$= e^{\frac{iS}{\hbar}}\left[-\frac{i\hbar}{m}\vec{\nabla}A\cdot\vec{\nabla}S - \frac{\hbar^2}{2m}\nabla^2A + \frac{A}{2m}|\vec{\nabla}S|^2 - \frac{i\hbar}{2m}A\nabla^2S\right]$$

$$= e^{\frac{iS}{\hbar}}\left[-\frac{\hbar^2}{2m}\nabla^2A + \frac{A}{2m}|\vec{\nabla}S|^2 - \frac{i\hbar}{2m}\left(A\nabla^2S + 2\vec{\nabla}A\cdot\vec{\nabla}S\right)\right]$$

Torque law from Poisson brackets: (Prob 3.9)

$$L_i = \sum_{j,k} \epsilon_{ijk}\, x^j p^k$$

$$\frac{dL_i}{dt} = \cancel{\frac{\partial L_i}{\partial t}}^{\,0} + \{L_i, H\}$$

where $$H = \sum_\ell \frac{p_\ell^2}{2m} + U(\vec{r})$$

Thus, $$\{L_i, H\} = \{\sum_{j,k} \epsilon_{ijk}\, x^j p^k, \sum_\ell \frac{p_\ell^2}{2m} + U(\vec{r})\}$$

$$= \sum_{j,k,\ell} \epsilon_{ijk} \frac{1}{2m} \{x^j p^k, p_\ell^2\}$$

$$+ \sum_{j,k} \epsilon_{ijk} \{x^j p^k, U(\vec{r})\}$$

$$= \sum_{j,k,\ell} \frac{1}{\cancel{2}m} \epsilon_{ijk}\, p^k \underbrace{\{x^j, p_\ell^2\}}_{\cancel{2} p_\ell \delta^j_\ell}$$

$$+ \sum_{j,k} \epsilon_{ijk}\, x^j \underbrace{\{p^k, U(\vec{r})\}}_{-\frac{\partial U}{\partial x^k}}$$

$$= \cancel{\sum_{j,k} \frac{1}{m} \epsilon_{ijk}\, p^k p_j}^{\,0} - \sum_{j,k} \epsilon_{ijk}\, x^j \partial_k U$$

$$= \sum_{j,k} \epsilon_{ijk}\, x_j F_k$$

$$= (\vec{r} \times \vec{F})_i$$

$$\rightarrow \frac{dL_i}{dt} = (\vec{r} \times \vec{F})_i \quad \text{or} \quad \boxed{\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}}$$

Thus,

LHS = RHS iff

$$\boxed{-\frac{\hbar^2}{2m}\nabla^2 A + \frac{A}{2m}|\vec{\nabla} S|^2 - \frac{i\hbar}{2m}\left(A\nabla^2 S + 2\vec{\nabla}A\cdot\vec{\nabla}S\right) + UA = i\hbar\frac{\partial A}{\partial t} - A\frac{\partial S}{\partial t}}$$

Real part: $-\frac{\hbar^2}{2m}\nabla^2 A + \frac{A}{2m}|\vec{\nabla} S|^2 + UA = -A\frac{\partial S}{\partial t}$

$$\boxed{\frac{|\vec{\nabla} S|^2}{2m} + U = -\frac{\partial S}{\partial t} + \frac{\hbar^2}{2m}\frac{\nabla^2 A}{A}}$$

$\xrightarrow[\hbar\to 0]{}$ $\boxed{\frac{|\vec{\nabla} S|^2}{2m} + U = -\frac{\partial S}{\partial t}}$ HJ equation

Imag part: $\boxed{-\frac{\hbar}{2m}\left(A\nabla^2 S + 2\vec{\nabla}A\cdot\vec{\nabla}S\right) = \hbar\frac{\partial A}{\partial t}}$

$\lim_{\hbar\to 0}$: $\boxed{0 = 0}$

So $\hbar\to 0$ limit of SE becomes $\frac{|\vec{\nabla} S|^2}{2m} + U = -\frac{\partial S}{\partial t}$

$\frac{|\vec{p}|^2}{2m} + U = H$

with $\vec{p} = \frac{\partial S}{\partial \vec{r}}$, $H = -\frac{\partial S}{\partial t}$