Coords in config space and constraint surface for planar double pendulum : Exer 2.1

$X: (x_1, y_1, x_2, y_2)$

$Q: (\phi_1, \phi_2)$

0

Exercise: Bar in a hoop

(2.2)

A thin bar of length $l < 2r$ moves in a plane such its end points are always in contact of a hoop of radius r.

a) what are the constraints on the Cartesian coordinates of each end point $(x_1, y_1), (x_2, y_2)$?

b) what constraints are on the velocities?

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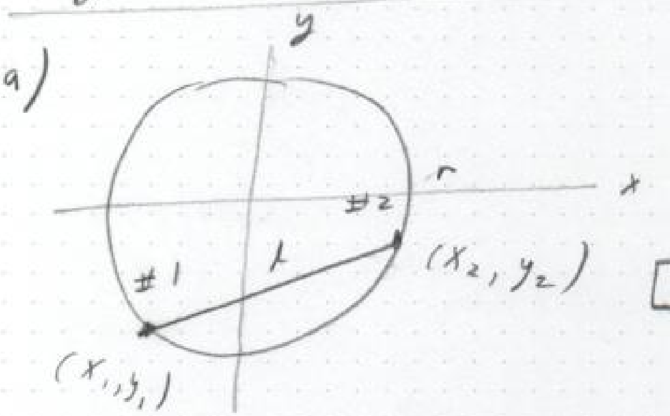
a)



i) $(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2$

ii) $x_2^2 + y_2^2 = r^2$

iii) $x_1^2 + y_1^2 = r^2$

[3 constraints on 4 DoF]

i), ii), iii) correspond to the bar having length l, and endpoints #1, #2 in contact with the hoop.

See if we can combine constraints:

$$y_2 = \pm\sqrt{r^2 - x_2^2}, \quad y_1 = \pm\sqrt{r^2 - x_1^2}$$

$$\rightarrow \quad l^2 = (x_2 - x_1)^2 + \left(\pm\sqrt{r^2 - x_2^2} - \pm\sqrt{r^2 - x_1^2}\right)^2$$

$$= x_2^2 + x_1^2 - 2x_1x_2 + (r^2 - x_2^2) + (r^2 - x_1^2) \mp 2\sqrt{(r^2 - x_2^2)(r^2 - x_1^2)}$$

$$= 2r^2 - 2x_1x_2 \mp 2\sqrt{(r^2 - x_2^2)(r^2 - x_1^2)}$$

[not very illuminating]

b) constraints on velocities by differentiating i), ii), iii)

i) $0 = 2(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + 2(y_2 - y_1)(\dot{y}_2 - \dot{y}_1)$

$= (x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + (y_2 - y_1)(\dot{y}_2 - \dot{y}_1)$

ii) $2x_2\dot{x}_2 + 2y_2\dot{y}_2 = 0$

$x_2\dot{x}_2 + y_2\dot{y}_2 = 0$

iii) $2x_1\dot{x}_1 + 2y_1\dot{y}_1 = 0$

$x_1\dot{x}_1 + y_1\dot{y}_1 = 0$

NOTE! i)

$$0 = (x_2-x_1)(\dot{x}_2-\dot{x}_1) + (y_2-y_1)(\dot{y}_2-\dot{y}_1)$$
$$= \underbrace{x_2\dot{x}_2}_{0} + \underbrace{x_1\dot{x}_1}_{0} - x_1\dot{x}_2 - x_2\dot{x}_1 + \underbrace{y_2\dot{y}_2}_{0} + \underbrace{y_1\dot{y}_1}_{0} - y_1\dot{y}_2 - y_2\dot{y}_1$$
$$= x_1\dot{x}_2 + x_2\dot{x}_1 + y_1\dot{y}_2 + y_2\dot{y}_1$$

Thus,

$$x_2\dot{x}_2 + y_2\dot{y}_2 = 0$$
$$x_1\dot{x}_1 + y_1\dot{y}_1 = 0$$
$$x_1\dot{x}_2 + x_2\dot{x}_1 + y_1\dot{y}_2 + y_2\dot{y}_1 = 0$$

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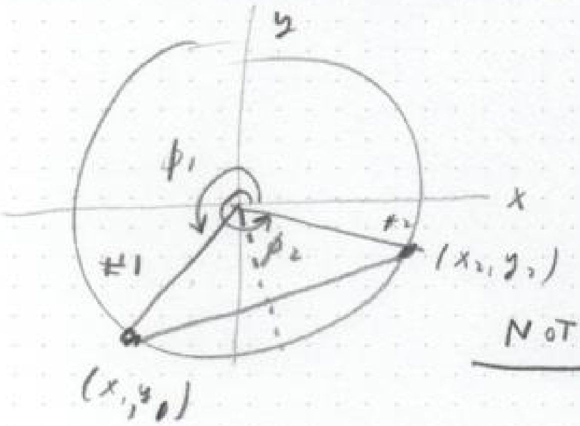
c) Generalized coordinate for the system



$x_1 = r\cos\phi_1$, $y_1 = r\sin\phi_1$

$x_2 = r\cos\phi_2$, $y_2 = r\sin\phi_2$

$\theta \equiv \frac{1}{2}(\phi_1+\phi_2)$ (bisector)

NOTE:

$$l^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$
$$= r^2(\cos\phi_2-\cos\phi_1)^2 + r^2(\sin\phi_2-\sin\phi_1)^2$$
$$= r^2[\cos^2\phi_2 + \cos^2\phi_1 - 2\cos\phi_1\cos\phi_2 + \sin^2\phi_2 + \sin^2\phi_1 - 2\sin\phi_1\sin\phi_2]$$
$$= r^2[2 - 2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)]$$
$$= 2r^2[1-\cos(\phi_1-\phi_2)]$$

Thus, $\frac{l^2}{2r^2} = 1 - \cos(\phi_1 - \phi_2)$

$l < 2r$

$1 - \frac{4r^2}{2r^2}$

$1 - 2 = -1$

$$\cos(\phi_1 - \phi_2) = 1 - \frac{l^2}{2r^2}$$

$$\rightarrow \underbrace{\phi_1 - \phi_2}_{\Delta\phi} = \left| \arccos\left[1 - \frac{l^2}{2r^2} \right] \right|$$

NOTE: $\phi_1 + \frac{1}{2}\Delta\phi = \phi_1 + \frac{1}{2}|\phi_1 - \phi_2|$
$= \phi_1 + \frac{1}{2}(\phi_2 - \phi_1)$
$= \frac{1}{2}(\phi_1 + \phi_2) = \theta$

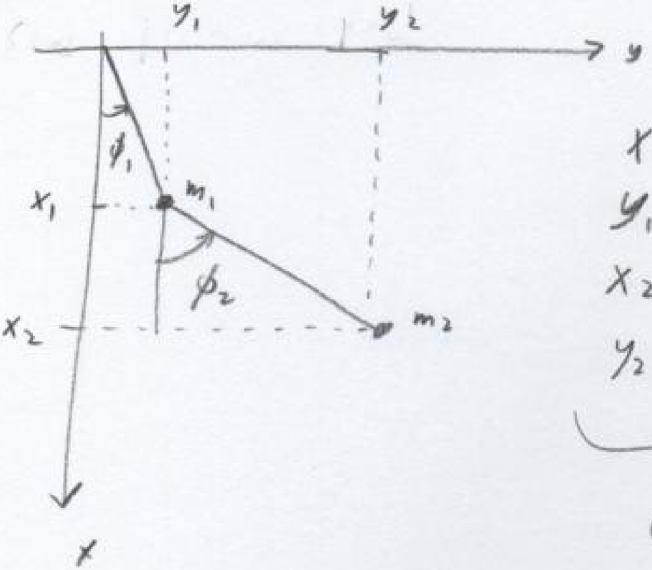
Conversely, given θ, I can calculate $\phi_1, \phi_2 \rightarrow (x_1, y_1), (x_2, y_2)$.

$$\rightarrow \boxed{\phi_1 = \theta - \Delta\phi = \theta - \left| \arccos\left[1 - \frac{l^2}{2r^2} \right] \right|}$$

$$\boxed{\phi_2 = \theta + \Delta\phi = \theta + \left| \arccos\left[1 - \frac{l^2}{2r^2} \right] \right|}$$

So θ = angle of $\perp$ bisector of the base of the triangle connecting $(0,0)$, (x_1, y_1), (x_2, y_2)

Constraint functions and embedding equations for the planar double pendulum Exer (2.3)



$$x_1 = l_1 \cos\phi_1$$
$$y_1 = l_1 \sin\phi_1$$
$$x_2 = l_1 \cos\phi_1 + l_2 \cos\phi_2$$
$$y_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2$$

embedding equations
$$\vec{r}_I = \vec{r}_I(q^a, t)$$

Constraint equations:
$$x_1^2 + y_1^2 = l_1^2$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$$

2.4

<u>Problem:</u> Check each constraint individually for holonomicity

a) $\underbrace{(y^2 - xy)\,dx + x^2\,dy}_{\alpha} = 0$

$$d\alpha = 2y\,dy\wedge dx - x\,dy\wedge dx + 2x\,dx\wedge dy$$
$$= (-2y + x + 2x)\,dx\wedge dy$$
$$= (3x - 2y)\,dx\wedge dy$$

$$d\alpha \wedge \alpha = (3x-2y)\,dx\wedge dy \wedge \left[(y^2 - xy)\,dx + x^2\,dy\right]$$
$$= 0 \quad (\text{since 3-form in 2-d})$$

so holonomic

b) $\underbrace{yz\,dx + zx\,dy + xy\,dz}_{\alpha} = 0$

$$d\alpha = \cancel{z\,dy\wedge dx} + \cancel{y\,dz\wedge dx} + \cancel{z\,dx\wedge dy} + \cancel{x\,dz\wedge dy} + \cancel{x\,dy\wedge dz} + \cancel{y\,dx\wedge dz}$$

$= 0$

so holonomic

c) $\underbrace{-y\,dx + x\,dy + dz}_{\alpha} = 0$

$$d\alpha = -dy\wedge dx + dx\wedge dy = 2\,dx\wedge dy$$

$$d\alpha\wedge\alpha = 2\,dx\wedge dy\,(-y\,dx + x\,dy + dz)$$
$$= 2\,dx\wedge dy\wedge dz \neq 0 \quad \rightarrow \text{so non-holonomic}$$

(2.5)

Problem: Non-holonomic individual constraints but holonomic system

$\alpha \equiv (x^2+y^2)\,dx + xz\,dz = 0$

$\beta \equiv (x^2+y^2)\,dy + yz\,dz = 0$

a)

$d\alpha = 2y\,dy\wedge dx + z\,dx\wedge dz$

$d\alpha\wedge\alpha = (2y\,dy\wedge dx + z\,dx\wedge dz)\wedge[\cancel{(x^2+y^2)\,dx} + xz\,dz]$

$= 2xyz\,dy\wedge dx\wedge dz$

$= -2xyz\,dx\wedge dy\wedge dz \neq 0$

$d\beta = 2x\,dx\wedge dy + z\,dy\wedge dz$

$d\beta\wedge\beta = (2x\,dx\wedge dy + z\,dy\wedge dz)\wedge[\cancel{(x^2+y^2)\,dy} + yz\,dz]$

$= 2xyz\,dx\wedge dy\wedge dz \neq 0$

so individual constraints are not holonomic

For the system of constraints, need to check

$\underbrace{d\alpha\wedge\alpha\wedge\beta}_{\text{4-form in 3-d}} = 0$, similarly $\underbrace{d\beta\wedge\alpha\wedge\beta}_{\text{4-form in 3-d}} = 0$

b)

$\left(\frac{x^2+y^2}{x}\right)dx = -z\,dz$

$\left(\frac{x^2+y^2}{y}\right)dy = -z\,dz$

Thus, $\left(\frac{\cancel{x^2+y^2}}{x}\right)dx = \left(\frac{\cancel{x^2+y^2}}{y}\right)dy$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\ln x = \ln y + \text{const}$$

$$\ln\left(\frac{y}{x}\right) = -\text{const} \rightarrow \frac{y}{x} = e^{-\text{const}} = k$$

$$\boxed{y = kx}$$

substitute

$$\alpha = (x^2 + k^2x^2)dx + x\,z\,dz = 0$$
$$\beta = k(x^2 + k^2x^2)dx + kx\,z\,dz = 0$$
equivalent

Thus, $(1+k^2)x^2dx + x\,z\,dz = 0$

$$\int (1+k^2)\,x\,dx = -\int z\,dz$$

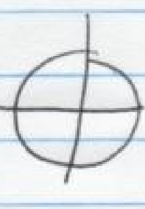
$$(1+k^2)\frac{x^2}{2} = -\frac{z^2}{2} + c$$

$$(1+k^2)x^2 + z^2 = 2c$$

$$(x^2+y^2+z^2) = R^2 \quad (= \text{const})$$

Thus, $\boxed{x^2+y^2+z^2 = R^2}$ $\boxed{y = kx}$

$$\boxed{\begin{aligned} \varphi^1(x,y,z) &= R^2 - (x^2+y^2+z^2) = 0 \\ \varphi^2(x,y,z) &= y - kx = 0 \end{aligned}}$$



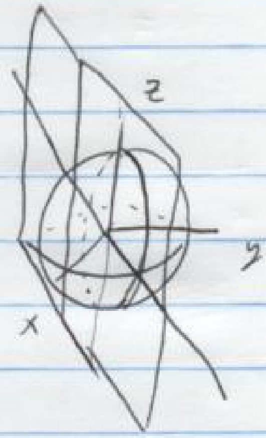
Integral curves:

intersection of $\varphi^1 = 0$, $\varphi^2 = 0$

$$R^2 - (x^2 + (kx)^2 + z^2) = 0$$

$$R^2 - x^2(1+k^2) - z^2 = 0$$

$$\boxed{x^2(1+k^2) + z^2 = R^2}$$



3 dim

2 constraints:

$\underbrace{3-2}_{n} = 1$ 1-dimensional surface

$$d\varphi^1 = -2x\,dx - 2y\,dy - 2z\,dz$$

$$d\varphi^2 = dy - k\,dx$$

$x^2+y^2+z^2=R$

surface of sphere

$y = kx$

Find μ_{AB} such that :

$$d\varphi^A = \sum_B \mu_{AB}\, c^B \quad \longleftrightarrow \quad \varphi^A_{,i} = \sum_B \mu_{AB}\, c^B_i$$

where $c^1 = (x^2+y^2)\,dx + xz\,dz$

$\quad\;\; c^2 = (x^2+y^2)\,dy + yz\,dz$

$$d\varphi^1 = -2x\,dx - 2y\,dy - 2z\,dz$$

$$d\varphi^2 = -k\,dx + dy$$

$$\begin{array}{c} \\ \begin{bmatrix} d\varphi^1 \\ d\varphi^2 \end{bmatrix} \end{array} = \begin{array}{c} \rightarrow i \\ {\scriptstyle A}\downarrow \begin{bmatrix} -2x & -2y & -2z \\ -k & 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} c^1 \\ c^2 \end{bmatrix} = \begin{bmatrix} x^2+y^2 & 0 & xz \\ 0 & x^2+y^2 & yz \end{bmatrix}$$

$$\begin{array}{c} 3 \rightarrow i \\ 2\,{\scriptstyle A}\downarrow \begin{bmatrix} -2x & -2y & -2z \\ -k & 1 & 0 \end{bmatrix} \end{array} = \begin{array}{c} 2 \rightarrow B \\ 2\,{\scriptstyle A}\downarrow \begin{bmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{bmatrix} \end{array} \begin{array}{c} \rightarrow i \\ {\scriptstyle B}\downarrow \begin{bmatrix} x^2+y^2 & 0 & xz \\ 0 & x^2+y^2 & yz \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} \mu_{11}(x^2+y^2) & \mu_{12}(x^2+y^2) & \mu_{11}xz + \mu_{12}yz \\ \mu_{21}(x^2+y^2) & \mu_{22}(x^2+y^2) & \mu_{21}xz + \mu_{22}yz \end{bmatrix}$$

$$\rightarrow \quad \mu_{11} = \frac{-2x}{x^2+y^2} \qquad \mu_{21} = \frac{-k}{x^2+y^2}$$

$$\mu_{12} = \frac{-2y}{x^2+y^2} \qquad \mu_{22} = \frac{1}{x^2+y^2}$$

check:

$$\mu_{11}xz + \mu_{12}yz = \frac{-2x^2z - 2y^2z}{(x^2+y^2)} = -2z$$

$$\mu_{21}\, xz + \mu_{22}\, yz = \frac{-kxz}{x^2+y^2} + \frac{yz}{x^2+y^2}$$

$$= 0 \quad (\text{since } y = kx)$$

Thus,

$$\cancel{A_{AB} = \frac{-2x}{x^2+y^2}}$$

$$\mu_{AB} = \begin{array}{c} \\ \downarrow \\ A \end{array} \begin{array}{c} \longrightarrow B \\ \boxed{\begin{array}{c|c} \dfrac{-2x}{x^2+y^2} & \dfrac{-2y}{x^2+y^2} \\ \hline \dfrac{-k}{x^2+y^2} & \dfrac{1}{x^2+y^2} \end{array}} \end{array}$$

where $k = \dfrac{y}{x}$

2.5

Exercise: $C_1 = (x^2+y^2)\,dx + xz\,dz = 0$
$C_2 = (x^2+y^2)\,dy + yz\,dz = 0$

Holonomic system of constraints

Individually:

$$dC_1 \wedge C_1 = \left(2y\,dy\wedge dx + z\,\cancel{dx}\wedge dz\right)\wedge\left(\cancel{(x^2+y^2)}dx + xz\,dz\right)$$
$$= 2xyz\,dy\wedge dx\wedge dz$$
$$\neq 0$$

$$dC_2 \wedge C_2 = \left(2x\,dx\wedge dy + z\,\cancel{dy}\wedge dz\right)\wedge\left(\cancel{(x^2+y^2)}dy + yz\,dz\right)$$
$$= 2xyz\,dx\wedge dy\wedge dz$$
$$\neq 0$$

System:

$$dC_1\wedge C_1\wedge C_2 = -2xyz\,dx\wedge dy\wedge dz \wedge \left((x^2+y^2)dy + yz\,dz\right)$$
$$= 0$$ (can't have a non-zero 4-form in 3-d)

$$dC_2\wedge C_1\wedge C_2 = -dC_2\wedge C_2\wedge C_1$$
$$= -2xyz\,dx\wedge dy\wedge dz\wedge\left((x^2+y^2)dx + xz\,dz\right)$$
$$= 0$$ (same reason)

NOTE: $0 = \dfrac{C_1}{x} - \dfrac{C_2}{y} = (x^2+y^2)\left(\dfrac{dx}{x} - \dfrac{dy}{y}\right)$

$$= \left(\frac{x^2+y^2}{xy}\right)(y\,dx - x\,dy)$$

so $y\,dx - x\,dy = 0 \rightarrow \dfrac{dx}{x} = \dfrac{dy}{y}$

$\ln x = \ln y + \text{const}$

$$\ln\left(\frac{y}{x}\right) = c$$

$$\frac{y}{x} = e^c = \text{const}$$

In terms of ϕ defined by $\tan\phi = \frac{y}{x}$, we see that $\phi = \text{const.}$ $\rightarrow$ $\boxed{F_1 = \phi - \phi_0 = 0}$

Work in terms of cylindrical coords:

$$\begin{aligned} \rho^2 &= x^2+y^2 \\ \tan\phi &= y/x \\ Z &= z \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ Z &= z \end{aligned}$$

Then:

$$C_1 = \rho^2(\cos\phi\, d\rho - \rho\sin\phi\, d\phi) + \rho z\cos\phi\, dz$$

$$C_2 = \rho^2(\sin\phi\, d\rho + \rho\cos\phi\, d\phi) + \rho z\sin\phi\, dz$$

use $d\phi = 0$

$$\rightarrow \quad C_1 = \rho^2\cos\phi\, d\rho + \rho z\cos\phi\, dz$$

$$= \rho\cos\phi\,[\rho\, d\rho + z\, dz]$$

$$C_2 = \rho^2\sin\phi\, d\rho + \rho z\sin\phi\, dz$$

$$= \rho\sin\phi\,[\rho\, d\rho + z\, dz]$$

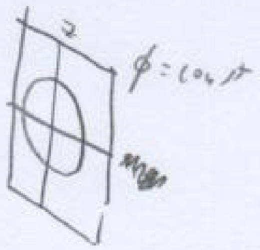
Thus,

$$\rho\, d\rho + z\, dz = 0$$

$$\frac{1}{2}\rho^2 + \frac{1}{2}z^2 = \text{const}$$

$$\rho^2 + z^2 = R^2 \quad \rightarrow \quad \boxed{F_2 = \rho^2 + z^2 - R^2 = 0}$$

So integral curves are circles in the $\phi = \text{const}$ plane.



$$dF_1 = d\phi$$
$$= \frac{1}{\rho^2}(x\,dy - y\,dx)$$
$$= \frac{1}{(x^2+y^2)}(x\,dy - y\,dz)$$

$$dF_2 = d(x^2+y^2) + dz^2$$
$$= 2\,x\,dx + y\,dy + 2z\,dz$$
$$= 2(x\,dx + y\,dy + z\,dz)$$

<u>NOTE:</u> $x\,dy - y\,dx$
$$= \rho\cos\phi\,[\cancel{\sin\phi\,d\rho} + \rho\cos\phi\,d\phi]$$
$$- \rho\sin\phi\,[\cancel{\cos\phi\,d\rho} - \rho\sin\phi\,d\phi]$$
$$= \rho^2(\cos^2\phi + \sin^2\phi)\,d\phi$$
$$= \rho^2\,d\phi$$

$$dF_1(x^2+y^2) = x\,dy - y\,dz$$
$$\frac{1}{2}\,dF_2 = x\,dx + y\,dy + z\,dz$$

$$\frac{1}{2}x\,dF_2 = x^2\,dx + xy\,dy + xz\,dz$$
$$\frac{1}{2}y\,dF_2 = xy\,dx + y^2\,dy + yz\,dz$$

Thus, $\frac{1}{2}x\,dF_2 + y^2\,dx - xy\,dy = (x^2+y^2)\,dx + xz\,dz$

$$\frac{1}{2}x\,dF_2 + y(y\,dx - x\,dy) = C_1$$

$$\boxed{\frac{1}{2}x\,dF_2 - y(x^2+y^2)\,dF_1 = C_1}$$

Similarly $\frac{1}{2}y\,dF_2 + x^2\,dy - xy\,dx = (x^2+y^2)\,dy + yz\,dz$

$$\frac{1}{2}y\,dF_2 + x(x\,dy - y\,dx) = C_2$$

$$\boxed{\frac{1}{2}y\,dF_2 + x(x^2+y^2)\,dF_1 = C_2}$$

Invert:

$$\frac{1}{2}x^2\,dF_2 \cancel{-xy(x^2+y^2)\,dF_1} = x\,C_1$$

$$\frac{1}{2}y^2\,dF_2 \cancel{+xy(x^2+y^2)\,dF_1} = y\,C_2$$

$$\frac{1}{2}(x^2+y^2)\,dF_2 = x\,C_1 + y\,C_2$$

$$\boxed{dF_2 = \frac{2}{(x^2+y^2)}\left[x\,C_1 + y\,C_2\right]}$$

$$\cancel{\frac{1}{2}xy\,dF_2} - y^2(x^2+y^2)\,dF_1 = y\,C_1$$

$$\cancel{-\frac{1}{2}xy\,dF_2} - x^2(x^2+y^2)\,dF_1 = -x\,C_2$$

$$-(x^2+y^2)^2\,dF_1 = y\,C_1 - x\,C_2$$

$$\boxed{dF_1 = \frac{-1}{(x^2+y^2)^2}\left[y\,C_1 - x\,C_2\right]}$$

Static equilibrium for rigid body (2.6)

Force eqn: $\vec{F}_I^{(a)} + \vec{F}_I^{(c)} = 0$

Principle of virtual work:

If $\delta\vec{r}_I$ is a virtual displacement, $\sum_I \delta\vec{r}_I \cdot \vec{F}_I^{(c)} = 0$

For a rigid body:

$$\delta\vec{r}_I = \delta\vec{c} \quad \text{(translation)}$$

$$\delta\vec{r}_I = \delta\vec{\omega} \times \vec{r}_I \quad \text{(rotation)}$$

Thus, $\vec{F}_I^{(a)} + \vec{F}_I^{(c)} = 0$

$$\rightarrow \quad 0 = \sum_I \delta\vec{r}_I \cdot \left(\vec{F}_I^{(a)} + \vec{F}_I^{(c)}\right)$$

$$= \sum_I \delta\vec{r}_I \cdot \vec{F}_I^{(a)} + \cancelto{0}{\sum_I \delta\vec{r}_I \cdot \vec{F}_I^{(c)}}$$

$$= \sum_I \delta\vec{c} \cdot \vec{F}_I^{(a)}$$

$$= \delta\vec{c} \cdot \left(\sum_I \vec{F}_I^{(a)}\right)$$

so $\boxed{\sum_I \vec{F}_I^{(a)} = 0}$

Similarly:

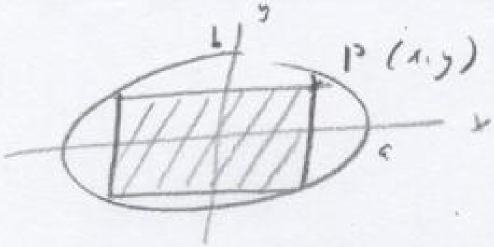
$$0 = \sum_I \left(\delta\vec{\omega} \times \vec{r}_I\right) \cdot \vec{F}_I^{(a)}$$

$$= \sum_I \left(\vec{r}_I \times \vec{F}_I^{(a)}\right) \cdot \delta\vec{\omega}$$

$$= \delta\vec{\omega} \cdot \sum_I \left(\vec{r}_I \times \vec{F}_I^{(a)}\right)$$

so $\boxed{\sum_I \left(\vec{r}_I \times \vec{F}_I^{(a)}\right) = 0}$

Problem (2.7)



Maximize area of inscribed rectangle in ellipse.

$A(x,y) = 4xy \qquad \rightarrow \quad 0 = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy$

constraint $\phi(x,y) \equiv \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1 = 0 \quad \rightarrow \quad 0 = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$

a) Use method of Lagrange multipliers

$$0 = \frac{\partial A}{\partial x} - \lambda \frac{\partial \phi}{\partial x}$$
$$0 = \frac{\partial A}{\partial y} - \lambda \frac{\partial \phi}{\partial y}$$

$$\rightarrow \quad 0 = 4y - \lambda 2 \frac{x}{a^2}$$
$$0 = 4x - \lambda 2 \frac{y}{b^2}$$

so
$$0 = 4xy - 2\lambda \left(\frac{x}{a}\right)^2$$
$$0 = 4xy - 2\lambda \left(\frac{y}{b}\right)^2$$

Add
$$0 = 8xy - 2\lambda \Big[\underbrace{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2}_{1} \Big]$$

$$\boxed{\lambda = 4xy = \text{Area}}$$

Thus
$$0 = 4y - 4xy \, 2 \frac{x}{a^2}$$
$$= 4y \left[1 - 2\left(\frac{x}{a}\right)^2 \right]$$

$A_{max} = 4xy = 2ab$

$\rightarrow y = 0 \rightarrow x = a$ [minimum area]

$1 - 2\left(\frac{x}{a}\right)^2 = 0 \rightarrow \frac{x}{a} = \frac{1}{\sqrt{2}} \rightarrow \boxed{x = \frac{a}{\sqrt{2}}}$

Using constraint $\left(\frac{y}{b}\right)^2 = 1 - \left(\frac{x}{a}\right)^2 = \frac{1}{2} \rightarrow \boxed{y = \frac{b}{\sqrt{2}}}$

b) Repeat calculation but eliminate constraint

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1 = 0$$

$$\rightarrow \left(\frac{y}{b}\right) = \sqrt{1-\left(\frac{x}{a}\right)^2}$$

$$\boxed{y = b\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

Thus, $$\bar{A}(x) = A(x,y)\Big|_{y = b\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

$$= 4xb\sqrt{1-\left(\frac{x}{a}\right)^2}$$

$$= 4b\, x\sqrt{1-\left(\frac{x}{a}\right)^2}$$

$$0 = \frac{d\bar{A}}{dx} = 4b\left[\sqrt{1-\left(\frac{x}{a}\right)^2} + \left(\frac{1}{2}\right)\frac{x}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\left(-\frac{2x}{a^2}\right)\right]$$

$$= \frac{4b}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\left[1-\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^2\right]$$

$$= \frac{4b}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\left[1-2\left(\frac{x}{a}\right)^2\right]$$

$$\rightarrow \left(\frac{x}{a}\right)^2 = \frac{1}{2}$$

$$\left(\frac{x}{a}\right) = \frac{1}{\sqrt{2}} \rightarrow \boxed{x = \frac{a}{\sqrt{2}}} \rightarrow \boxed{y = \frac{b}{\sqrt{2}}}$$

as before

Evaluating $\sum_I \frac{\partial \vec{r}_I}{\partial x^\alpha} \cdot \vec{p}_I$: (2.8)

Given: $\vec{r}_I = \vec{r}_I(x^1, x^2, \ldots, x^{3N})$

$\rightarrow \quad \vec{v}_I = \frac{d\vec{r}_I}{dt} = \sum_\alpha \frac{\partial \vec{r}_I}{\partial x^\alpha} \dot{x}^\alpha$

Thus, $\boxed{\frac{\partial \vec{v}_I}{\partial \dot{x}^\alpha} = \frac{\partial \vec{r}_I}{\partial x^\alpha}}$

Also, $$\frac{\partial \vec{v}_I}{\partial x^\alpha} = \sum_\beta \frac{\partial^2 \vec{r}_I}{\partial x^\alpha \partial x^\beta} \dot{x}^\beta = \frac{d}{dt}\left(\frac{\partial \vec{r}_I}{\partial x^\alpha}\right)$$

Using these relations:

$$\sum_I \frac{\partial \vec{r}_I}{\partial x^\alpha} \cdot \dot{\vec{p}}_I = \sum_I \left[\frac{d}{dt}\left(\frac{\partial \vec{r}_I}{\partial x^\alpha} \cdot \vec{p}_I\right) - \frac{d}{dt}\left(\frac{\partial \vec{r}_I}{\partial x^\alpha}\right) \cdot \vec{p}_I \right]$$

$$= \sum_I \left[\frac{d}{dt}\left(\frac{\partial \vec{v}_I}{\partial \dot{x}^\alpha} \cdot \vec{p}_I\right) - \frac{\partial \vec{v}_I}{\partial x^\alpha} \cdot \vec{p}_I \right]$$

$$= \sum_I \left[\frac{d}{dt}\left(\frac{\partial \vec{v}_I}{\partial \dot{x}^\alpha} \cdot m_I \vec{v}_I\right) - \frac{\partial \vec{v}_I}{\partial x^\alpha} \cdot m_I \vec{v}_I \right]$$

$$= \sum_I \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^\alpha}\left(\frac{1}{2} m_I \vec{v}_I \cdot \vec{v}_I\right)\right) - \frac{\partial}{\partial x^\alpha}\left(\frac{1}{2} m_I \vec{v}_I \cdot \vec{v}_I\right) \right]$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^\alpha}\right) - \frac{\partial T}{\partial x^\alpha}$$

where $T = \sum_I \frac{1}{2} m_I \vec{v}_I \cdot \vec{v}_I$

Total KE in terms of $\dot{x}^\alpha$: (2.9)

$$T = \sum_I \frac{1}{2} m_I \vec{v}_I \cdot \vec{v}_I$$

$$\vec{r}_I = \vec{r}_I(x^1, x^2, \ldots, x^{3N})$$

$$\vec{v}_I = \frac{d\vec{r}_I}{dt}$$

$$= \sum_\alpha \frac{\partial \vec{r}_I}{\partial x^\alpha} \dot{x}^\alpha$$

$$\vec{v}_I \cdot \vec{v}_I = \sum_\alpha \left(\frac{\partial \vec{r}_I}{\partial x^\alpha} \dot{x}^\alpha \right) \cdot \sum_\beta \left(\frac{\partial \vec{r}_I}{\partial x^\beta} \dot{x}^\beta \right)$$

$$= \sum_\alpha \sum_\beta \frac{\partial \vec{r}_I}{\partial x^\alpha} \cdot \frac{\partial \vec{r}_I}{\partial x^\beta} \dot{x}^\alpha \dot{x}^\beta$$

Thus, $T = \sum_I \frac{1}{2} m_I \vec{v}_I \cdot \vec{v}_I$

$$= \sum_I \frac{1}{2} m_I \sum_{\alpha,\beta} \frac{\partial \vec{r}_I}{\partial x^\alpha} \cdot \frac{\partial \vec{r}_I}{\partial x^\beta} \dot{x}^\alpha \dot{x}^\beta$$

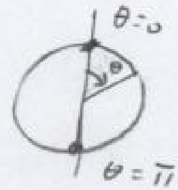
$$= \frac{1}{2} \sum_{\alpha,\beta} \left(\sum_I m_I \frac{\partial \vec{r}_I}{\partial x^\alpha} \cdot \frac{\partial \vec{r}_I}{\partial x^\beta} \right) \dot{x}^\alpha \dot{x}^\beta$$

$$= \frac{1}{2} \sum_{\alpha,\beta} T_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$$

where $T_{\alpha\beta} = \sum_I m_I \frac{\partial \vec{r}_I}{\partial x^\alpha} \cdot \frac{\partial \vec{r}_I}{\partial x^\beta}$

Stationary solution for bead on a rotating hoop: (2.10)

$$\ddot{\theta} = \sin\theta \left(\omega^2 \cos\theta + \frac{g}{R} \right)$$



$\ddot{\theta} = 0$ iff $\sin\theta = 0 \rightarrow \theta = 0, \pi$

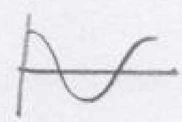
or $\omega^2 \cos\theta + \frac{g}{R} = 0$

$$\cos\theta = \frac{-g}{R\omega^2}$$

$$\boxed{\theta_0 = \cos^{-1}\left[\frac{-g}{R\omega^2}\right]}$$

a) For a given value of θ_0, need

$$\omega = \sqrt{\frac{-g}{R\cos\theta}}$$



└ need $\cos\theta < 0$
so $\theta > \pi/2$

b) Limiting value for $\omega \rightarrow \infty$

$\omega = \infty \iff \cos\theta = 0 \iff \boxed{\theta = \pi/2}$

$\theta_0 \in [\pi/2, \pi]$

c) For $\theta < \pi/2$, stationary solutions $(\theta = \theta_0)$ are not possible. Bead slide downward to values of $\theta \geq \pi/2$

Work done by constraint forces for bead on a rotating hoop.

(2.11)

Constraint Force

$$\vec{F}^{(c)} = m(g\cos\theta - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{r}$$
$$+ 2m(\dot{r}\dot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta)\hat{\phi}$$

Displacement

$$d\vec{s} = \overset{0\,(r=R)}{\cancel{dr}}\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$
$$= R\,d\theta\,\hat{\theta} + R\sin\theta\,d\phi\,\hat{\phi}$$

Thus,

$$W = \int \vec{F}^{(c)}\cdot d\vec{s} = \int 2mR(\underbrace{\dot{r}}_{0}\underbrace{\dot{\phi}}_{\omega}\sin^2\theta + \underbrace{r}_{R}\underbrace{\dot{\phi}}_{\omega}\dot{\theta}\sin\theta\cos\theta)\underbrace{d\phi}_{\omega\,dt}$$

$$= \int 2mR^2\,\omega^2\,\dot{\theta}\,\sin\theta\cos\theta\,dt$$

$$= 2mR^2\omega^2\int_{\theta_1}^{\theta_2}\sin\theta\,\underbrace{\cos\theta\,d\theta}_{d(\sin\theta)}$$

$$= \cancel{2}mR^2\omega^2\,\frac{1}{\cancel{2}}\sin^2\theta\Big|_{\theta_1}^{\theta_2}$$

$$= mR^2\omega^2(\sin^2\theta_2 - \sin^2\theta_1)$$

Total kinetic energy in terms of $\dot{q}^a$ (2.12)

$$T = \sum_I \frac{1}{2} m_I \vec{v}_I \cdot \vec{v}_I$$

$$\vec{r}_I = \vec{r}_I (q^1, q^2, \ldots, q^n, t)$$

$$\vec{v}_I = \frac{d\vec{r}_I}{dt}$$

$$= \sum_a \frac{\partial \vec{r}_I}{\partial q^a} \dot{q}^a + \frac{\partial \vec{r}_I}{\partial t}$$

Thus,
$$T = \sum_I \frac{1}{2} m_I \left(\sum_a \frac{\partial \vec{r}_I}{\partial q^a} \dot{q}^a + \frac{\partial \vec{r}_I}{\partial t} \right) \cdot \left(\sum_b \frac{\partial \vec{r}_I}{\partial q^b} \dot{q}^b + \frac{\partial \vec{r}_I}{\partial t} \right)$$

$$= \sum_I \frac{1}{2} m_I \sum_{a,b} \frac{\partial \vec{r}_I}{\partial q^a} \cdot \frac{\partial \vec{r}_I}{\partial q^b} \dot{q}^a \dot{q}^b$$

$$+ \sum_I \frac{1}{2} m_I \frac{\partial \vec{r}_I}{\partial t} \cdot \frac{\partial \vec{r}_I}{\partial t}$$

$$+ 2 \sum_I \frac{1}{2} m_I \left(\frac{\partial \vec{r}_I}{\partial q^a} \right) \cdot \frac{\partial \vec{r}_I}{\partial t} \dot{q}^a$$

$$= \frac{1}{2} \left(\sum_{a,b} T_{ab} \dot{q}^a \dot{q}^b + 2 \sum_a T_{a0} \dot{q}^a + T_{00} \right)$$

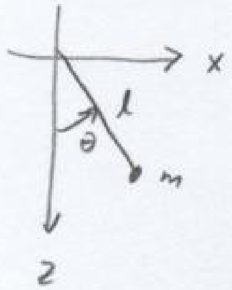
where
$$T_{00} = \sum_I m_I \frac{\partial \vec{r}_I}{\partial t} \cdot \frac{\partial \vec{r}_I}{\partial t}$$

$$T_{a0} = \sum_I m_I \left(\frac{\partial \vec{r}_I}{\partial q^a} \right) \cdot \frac{\partial \vec{r}_I}{\partial t}$$

$$T_{ab} = \sum_I m_I \left(\frac{\partial \vec{r}_I}{\partial q^a} \right) \cdot \left(\frac{\partial \vec{r}_I}{\partial q^b} \right)$$

Total KE examples: 2.13

a) simple planar pendulum



$x^2 + z^2 = l^2$

$\left.\begin{array}{l} x = l\sin\theta \\ z = l\cos\theta \end{array}\right\}$ embedding equations

No time dependence

$$\rightarrow T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

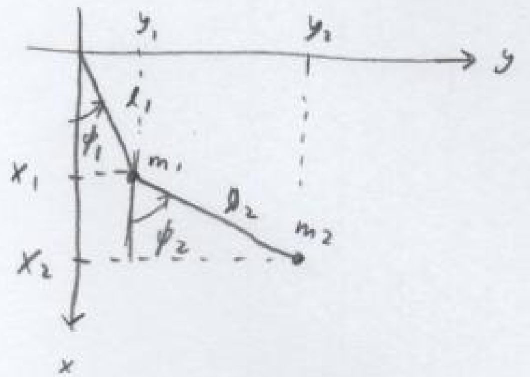
$$= \frac{1}{2}m\left[\left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial y}{\partial\theta}\right)^2\right]\dot{\theta}^2$$

$$= \frac{1}{2}m\left[(l\cos\theta)^2 + (-l\sin\theta)^2\right]\dot{\theta}^2$$

$$= \frac{1}{2}ml^2\left[\cos^2\theta + \sin^2\theta\right]\dot{\theta}^2$$

$$= \frac{1}{2}ml^2\dot{\theta}^2$$

b) planar double pendulum



$x_1 = l_1\cos\phi_1$

$y_1 = l_1\sin\phi_1$

$x_2 = l_2\cos\phi_2 + l_1\cos\phi_1$

$y_2 = l_2\sin\phi_2 + l_1\sin\phi_1$

(no time dependence)

$$T = \frac{1}{2}\left[m_1(\dot{x}_1^2 + \dot{y}_1^2) + m_2(\dot{x}_2^2 + \dot{y}_2^2)\right]$$

Generalized coords (ϕ_1, ϕ_2)

$$\frac{\partial x_1}{\partial\phi_1} = -l_1\sin\phi_1, \quad \frac{\partial x_1}{\partial\phi_2} = 0$$

$$\frac{\partial y_1}{\partial\phi_1} = l_1\cos\phi_1, \quad \frac{\partial y_1}{\partial\phi_2} = 0$$

$$\frac{\partial x_2}{\partial \phi_1} = -l_1 \sin\phi_1 \,, \quad \frac{\partial x_2}{\partial \phi_2} = -l_2 \sin\phi_2$$

$$\frac{\partial y_2}{\partial \phi_1} = +l_1 \cos\phi_1 \,, \quad \frac{\partial y_2}{\partial \phi_2} = l_2 \cos\phi_2$$

Thus,

$$T_{ab} = \sum_I m_I \frac{\partial \vec{r}_I}{\partial q^a} \cdot \frac{\partial \vec{r}_I}{\partial q^b}$$

$$T_{11} = m_1\left[\left(\frac{\partial x_1}{\partial \phi_1}\right)^2 + \left(\frac{\partial y_1}{\partial \phi_1}\right)^2\right] + m_2\left[\left(\frac{\partial x_2}{\partial \phi_1}\right)^2 + \left(\frac{\partial y_2}{\partial \phi_1}\right)^2\right]$$

$$= m_1\left[l_1^2 \sin^2\phi_1 + l_1^2 \cos^2\phi_1\right] + m_2\left[l_1^2 \sin^2\phi_1 + l_1^2\cos^2\phi_1\right]$$

$$= (m_1 + m_2)\, l_1^2$$

$$T_{22} = m_1\left[\cancel{\left(\frac{\partial x_1}{\partial \phi_2}\right)^2}^{0} + \cancel{\left(\frac{\partial y_1}{\partial \phi_2}\right)^2}^{0}\right] + m_2\left[\left(\frac{\partial x_2}{\partial \phi_2}\right)^2 + \left(\frac{\partial y_2}{\partial \phi_2}\right)^2\right]$$

$$= m_2\, l_2^2$$

$$T_{12} = m_1\left[\left(\frac{\partial x_1}{\partial \phi_1}\right)\cancel{\left(\frac{\partial x_1}{\partial \phi_2}\right)}^{0} + \left(\frac{\partial y_1}{\partial \phi_1}\right)\cancel{\left(\frac{\partial y_1}{\partial \phi_2}\right)}^{0}\right]$$

$$+ m_2\left[\left(\frac{\partial x_2}{\partial \phi_1}\right)\left(\frac{\partial x_2}{\partial \phi_2}\right) + \left(\frac{\partial y_2}{\partial \phi_1}\right)\left(\frac{\partial y_2}{\partial \phi_2}\right)\right]$$

$$= m_2\left[l_1 l_2 \sin\phi_1 \sin\phi_2 + l_1 l_2 \cos\phi_1 \cos\phi_2\right]$$

$$= m_2\, l_1 l_2 \left[\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2\right]$$

$$= m_2\, l_1 l_2 \cos(\phi_1 - \phi_2)$$

$$T_{ab} = T_{\theta\theta}$$

$$= m\left[\left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial y}{\partial\theta}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2\right]$$

$$= m\left[l^2\cos^2\theta\cos^2\omega t + l^2\cos^2\theta\sin^2\omega t + l^2\sin^2\theta\right]$$

$$= ml^2\left[\cos^2\theta\underbrace{\left(\cos^2\omega t + \sin^2\omega t\right)}_{1} + \sin^2\theta\right]$$

$$= ml^2$$

$$T_{a0} = T_{\theta 0}$$

$$= m\left[\left(\frac{\partial x}{\partial\theta}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial y}{\partial\theta}\right)\left(\frac{\partial y}{\partial t}\right) + \left(\frac{\partial z}{\partial\theta}\right)\cancelto{0}{\left(\frac{\partial z}{\partial t}\right)}\right]$$

$$= m\left[-l^2\omega\sin\theta\cos\theta\sin\omega t\cos\omega t + l^2\omega\sin\theta\cos\theta\sin\omega t\cos\omega t\right]$$

$$= 0$$

$$T_{00} = m\left[\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \cancelto{0}{\left(\frac{\partial z}{\partial t}\right)^2}\right]$$

$$= m\left[l^2\omega^2\sin^2\theta\sin^2\omega t + l^2\omega^2\sin^2\theta\cos^2\omega t\right]$$

$$= ml^2\omega^2\sin^2\theta$$

Thus,

$$T = \frac{1}{2}\left(T_{\theta\theta}\dot\theta^2 + 2\cancelto{0}{T_{\theta 0}}\dot\theta + T_{00}\right)$$

$$= \frac{1}{2}\left(ml^2\dot\theta^2 + ml^2\omega^2\sin^2\theta\right)$$

$$= \frac{1}{2}ml^2\left(\dot\theta^2 + \omega^2\sin^2\theta\right)$$

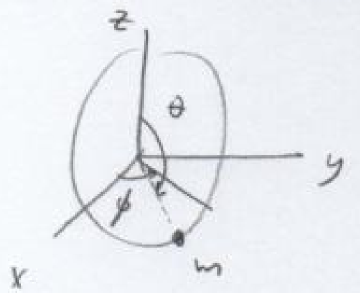
Thus,

$$T = \frac{1}{2} \sum_{a,b} T_{ab} \dot{q}^a \dot{q}^b$$

$$= \frac{1}{2} \left[T_{11} \dot{\phi}_1^2 + 2 T_{12} \dot{\phi}_1 \dot{\phi}_2 + T_{22} \dot{\phi}_2^2 \right]$$

$$= \frac{1}{2} \left[(m_1 + m_2) l_1^2 \dot{\phi}_1^2 + 2 m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 + m_2 l_2^2 \dot{\phi}_2^2 \right]$$

c) Bead on a rotating hoop:



$\phi = \omega t$

Cartesian: (x, y, z)

Sph. polar: (r, θ, ϕ)

$$\left.\begin{aligned} x &= r \sin\theta \cos\phi = l \sin\theta \cos\omega t \\ y &= r \sin\theta \sin\phi = l \sin\theta \sin\omega t \\ z &= r \cos\theta = l \cos\theta \end{aligned}\right\} \text{time dependence}$$

Generalized coord: θ

$$\frac{\partial x}{\partial \theta} = l \cos\theta \cos\omega t, \quad \frac{\partial x}{\partial t} = -l\omega \sin\theta \sin\omega t$$

$$\frac{\partial y}{\partial \theta} = l \cos\theta \sin\omega t, \quad \frac{\partial y}{\partial t} = l\omega \sin\theta \cos\omega t$$

$$\frac{\partial z}{\partial \theta} = -l \sin\theta, \quad \frac{\partial z}{\partial t} = 0$$

Lorentz force law from generalized potential (2.14)

$$U(\vec{r},\dot{\vec{r}},t) = q\left[\Phi(\vec{r},t) - \vec{A}(\vec{r},t)\cdot\dot{\vec{r}}\right]$$

$$\rightarrow \vec{F} = -\vec{\nabla}U + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\vec{r}}}\right)$$

$$= -q\,\partial_i\Phi + q(\partial_i A_j)\dot{r}_j + \frac{d}{dt}(-q A_i)$$

$$= -q\,\partial_i\Phi + q(\partial_i A_j)\dot{r}_j - q\left[(\partial_j A_i)\dot{r}_j + \frac{\partial A_i}{\partial t}\right]$$

$$= -q\left[\partial_i\Phi + \frac{\partial A_i}{\partial t}\right] + q(\partial_i A_j - \partial_j A_i)\dot{r}_j$$

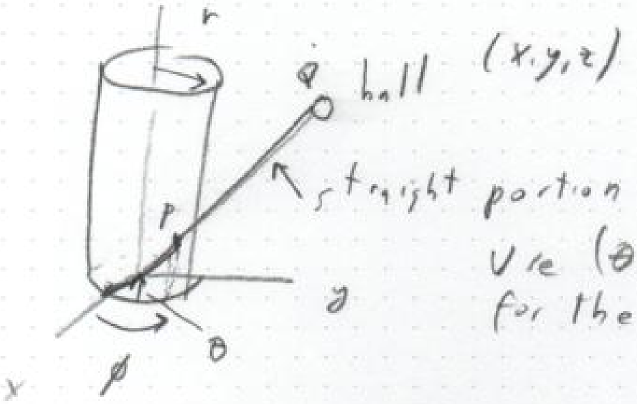
$$= q\left[-\partial_i\Phi - \frac{\partial A_i}{\partial t}\right] + q\,\epsilon_{ijk}(\vec{\nabla}\times\vec{A})_k v_j$$

$$= q E_i + q(\vec{v}\times\vec{B})_i$$

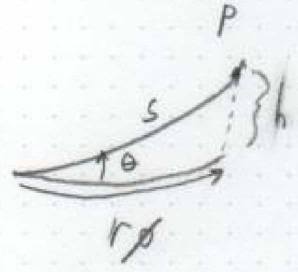
Thus, $\vec{F} = q(\vec{E} + (\vec{v}\times\vec{B}))$

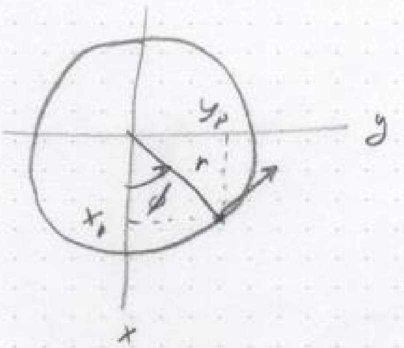
Problem: Embedding equations for a tether ball

(2,1)



Use (θ, ϕ) as generalized coords for the tether-ball





Coords of P:

$$x_p = r\cos\phi$$
$$y_p = r\sin\phi$$
$$z_p = r\phi\tan\theta$$

(unwrapped)

$$\cos\theta = \frac{r\phi}{s}$$

$$\sin\theta = \frac{h}{s}$$

Thus, $\tan\theta = \dfrac{h}{r\phi}$

$$\rightarrow h = r\phi\tan\theta$$

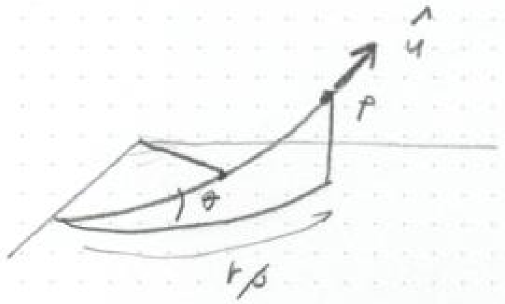
Length of string in contact with pole:

$$s = \frac{r\phi}{\cos\theta}$$

Alternatively

$$s^2 = (r\phi)^2 + h^2$$
$$= (r\phi)^2\left[1 + \tan^2\theta\right]$$
$$= (r\phi)^2\left(1 + \frac{\sin^2\theta}{\cos^2\theta}\right)$$
$$= \frac{(r\phi)^2}{\cos^2\theta}$$

$$\rightarrow s = \frac{r\phi}{\cos\theta}$$



$x_p = r\cos\phi$

$y_p = r\sin\phi$

$z_p = r\phi\tan\theta$ ← fixed

tgt vector to curve:

$$\frac{dx_p}{d\phi} = -r\sin\phi$$

$$\frac{dy_p}{d\phi} = r\cos\phi$$

$$\frac{dz_p}{d\phi} = r\tan\theta$$

$$\rightarrow \vec{u} = \frac{dx_p}{d\phi}\hat{x} + \frac{dy_p}{d\phi}\hat{y} + \frac{dz_p}{d\phi}\hat{z}$$

$$= -r\sin\phi\,\hat{x} + r\cos\phi\,\hat{y} + r\tan\theta\,\hat{z}$$

$$|\vec{u}|^2 = r^2\sin^2\phi + r^2\cos^2\phi + r^2\tan^2\theta$$

$$= r^2(1+\tan^2\theta)$$

$$= \frac{r^2}{\cos^2\theta}$$

Thus, $|\vec{u}| = \dfrac{r}{\cos\theta}$

so $$\hat{u} = \frac{\cos\theta}{r}\left[-r\sin\phi\,\hat{x} + r\cos\phi\,\hat{y} + r\tan\theta\,\hat{z}\right]$$

$$= -\cos\theta\sin\phi\,\hat{x} + \cos\theta\cos\phi\,\hat{y} + \sin\theta\,\hat{z}$$

$$\rightarrow Q: \vec{r} = \vec{r}_p + \overline{PQ}\,\hat{u}$$

$$= (r\cos\phi,\ r\sin\phi,\ r\phi\tan\theta)$$

$$+ \left(l - \frac{r\phi}{\cos\theta}\right)(-\cos\theta\sin\phi,\ \cos\theta\cos\phi,\ \sin\theta)$$

③

So:

$$x = r\cos\phi + \left(l - \frac{r\phi}{\cos\theta}\right)(-\cos\theta\sin\phi)$$

$$y = r\sin\phi + \left(l - \frac{r\phi}{\cos\theta}\right)\cos\theta\cos\phi$$

$$z = r\phi\tan\theta + \left(l - \frac{r\phi}{\cos\theta}\right)\sin\theta = l\sin\theta$$

---

simplify:

$$x = r\cos\phi - l\cos\theta\sin\phi + r\phi\sin\phi$$

$$= r(\cos\phi + \phi\sin\phi) - l\cos\theta\sin\phi$$

$$y = r\sin\phi + l\cos\theta\cos\phi - r\phi\cos\phi$$

$$= r(\sin\phi - \phi\cos\phi) + l\cos\theta\cos\phi$$

$$z = l\sin\theta$$

Find expression for kinetic energy in terms of $\dot{\theta}, \dot{\phi}$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\dot{x} = r(\cancel{-\sin\phi\,\dot{\phi}} + \cancel{\dot{\phi}\sin\phi} + \phi\cos\phi\,\dot{\phi})$$

$$+ l\sin\theta\sin\phi\,\dot{\theta} - l\cos\theta\cos\phi\,\dot{\phi}$$

$$= l\sin\theta\sin\phi\,\dot{\theta} + (r\phi\cos\phi - l\cos\theta\cos\phi)\dot{\phi}$$

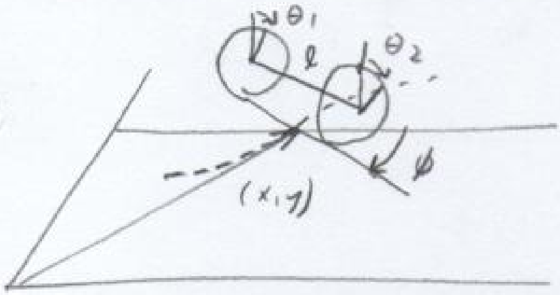
$$= l\sin\theta\sin\phi\,\dot{\theta} + \cos\phi(r\phi - l\cos\theta)\dot{\phi}$$

$$\dot{y} = r(\cancel{\cos\phi\,\dot{\phi}} - \cancel{\dot{\phi}\cos\phi} + \phi\sin\phi\,\dot{\phi})$$

$$- l\sin\theta\,\dot{\theta}\cos\phi - l\cos\theta\sin\phi\,\dot{\phi}$$

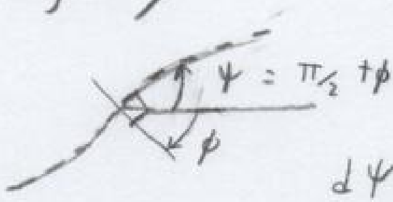
$$= -l\sin\theta\cos\phi\,\dot{\theta} + \sin\phi(r\phi - l\cos\theta)\dot{\phi}$$

Rolling wheel constraints (2.2)

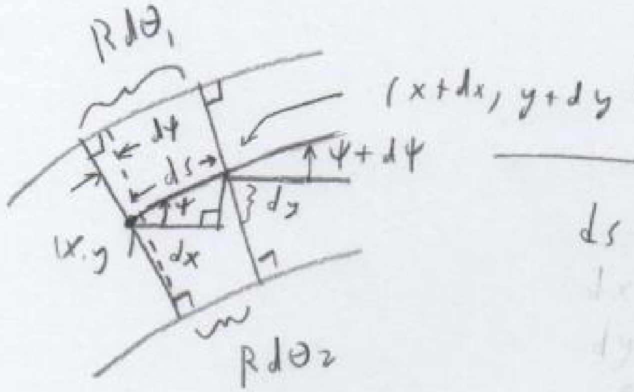


(x,y), ϕ, θ_1, θ_2

trajectory



$d\psi = d\phi$



note: $d\psi < 0$

$ds = \sqrt{dx^2 + dy^2}$

1.) $R d\theta_1 - R d\theta_2 = -l\, d\psi = -l\, d\phi$

$$\boxed{R(d\theta_1 - d\theta_2) = -l\, d\phi}$$

2.) $\dfrac{dy}{dx} = \tan\psi = \dfrac{\sin\psi}{\cos\psi} = \dfrac{\sin(\pi/2+\phi)}{\cos(\pi/2+\phi)} = \dfrac{\cos\phi}{-\sin\phi}$

$\rightarrow \; -\sin\phi\, dy = \cos\phi\, dx$

$$\boxed{0 = \cos\phi\, dx + \sin\phi\, dy}$$

3.) $ds = \frac{1}{2}(R d\theta_1 + R d\theta_2) = \frac{1}{2} R (d\theta_1 + d\theta_2)$

<u>NOTE</u>: $dx = ds\cos\psi = -\sin\phi\, ds$
$dy = ds \sin\psi = \cos\phi\, ds$

Thus, $ds = -\sin\phi\, dx + \cos\phi\, dy$

$$\rightarrow \boxed{-\sin\phi\, dx + \cos\phi\, dy = \frac{1}{2} R (d\theta_1 + d\theta_2)}$$

Integrable constraints:

$\sum_{\alpha=1}^{n} \delta x^\alpha C_\alpha = 0$ is integrable iff $\exists\, \varphi$ such that $C_\alpha = \frac{\partial \varphi}{\partial x^\alpha}$ iff

$$\epsilon_{\alpha_1 \alpha_2 \cdots \alpha_n} \partial_{\alpha_{n-1}} C_{\alpha_n} = 0$$

(a) $\underline{\underline{n=2}}$: $A(x,y)\,dx + B(x,y)\,dy = 0$

is integrable iff $A = \frac{\partial \varphi}{\partial x}$, $B = \frac{\partial \varphi}{\partial y}$

iff $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$

so $\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} = 0$

$\underline{\underline{n=3}}$: $A(x,y,z)\,dx + B(x,y,z)\,dy + C(x,y,z)\,dz = 0$

is integrable iff $A = \frac{\partial \varphi}{\partial x}$, $B = \frac{\partial \varphi}{\partial y}$, $C = \frac{\partial \varphi}{\partial z}$

$$\text{iff } \left(\frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz\right) \wedge dx + \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial z} dz\right) \wedge dy + \left(\frac{\partial C}{\partial x} dx + \frac{\partial C}{\partial y} dy\right) \wedge dz = 0$$

$$\text{iff } 0 = -\frac{\partial A}{\partial y} dz + \frac{\partial A}{\partial z} dy + \frac{\partial B}{\partial x} dz - \frac{\partial B}{\partial z} dx - \frac{\partial C}{\partial x} dy + \frac{\partial C}{\partial y} dx$$

$$= (\partial_y C - \partial_z B)\,dx + (\partial_z A - \partial_x C)\,dy + (\partial_x B - \partial_y A)\,dz$$

$\partial_y C - \partial_z B = 0$, $\partial_z A - \partial_x C = 0$, $\partial_x B - \partial_y A = 0$

(b) Check the following

(i) $(2x+y)\,dx + x\,dy = 0$

$A(x,y) = 2x+y$
$B(x,y) = x$

$\frac{\partial A}{\partial y} = 1$, $\frac{\partial B}{\partial x} = 1$

so $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$ $\rightarrow$ integrable

(ii) $(y^2 - xy)\,dx + x^2\,dy = 0$

$A = y^2 - xy$
$B = x^2$

$\frac{\partial A}{\partial y} = 2y - x$, $\frac{\partial B}{\partial x} = 2x$

so $\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$ $\rightarrow$ not integrable

(iii) $yz\,dx + zx\,dy + xy\,dz = 0$

$A(x,y,z) = yz$
$B(x,y,z) = zx$
$C(x,y,z) = xy$

$$\left.\begin{aligned} \partial_y C - \partial_z B &= x - x = 0 \\ \partial_z A - \partial_x C &= y - y = 0 \\ \partial_x B - \partial_y A &= z - z = 0 \end{aligned}\right\} \Rightarrow \text{integrable}$$

(c)

$$\cos\phi \, dx + \sin\phi \, dy = 0$$
$$-\sin\phi \, dx + \cos\phi \, dy = \frac{1}{2} R (d\theta_1 + d\theta_2)$$
$$-\ell \, d\phi = R(d\theta_1 - d\theta_2)$$

3 constraints on 5 variables $dx, dy, d\phi, d\theta_1, d\theta_2$

~~2nd constraint~~

~~multiply 3rd constraint by $\frac{1}{2}$ and add to 2nd constraint~~

~~$\frac{1}{2}\ell \, d\phi - \sin\phi \, dx + \cos\phi \, dy = R \, d\theta_1$~~

(1) $\rightarrow$ $dy = -\dfrac{\cos\phi}{\sin\phi} dx$ (solve)

(2) $-\sin\phi \, dx + \cos\phi \left(\dfrac{-\cos\phi}{\sin\phi} dx \right) = \frac{1}{2} R (d\theta_1 + d\theta_2)$

$$(-\sin^2\phi - \cos^2\phi) \, dx = \frac{1}{2} R \sin\phi \, (d\theta_1 + d\theta_2)$$

$$-dx = \frac{1}{2} R \sin\phi \, (d\theta_1 + d\theta_2)$$

(3) $\rightarrow$ $R \, d\theta_2 = R \, d\theta_1 + \ell \, d\phi$ (solve)

Thus, $-dx = \frac{1}{2} \sin\phi \, (R \, d\theta_1 + R \, d\theta_1 + \ell \, d\phi)$

$$= \frac{1}{2} \sin\phi \, (2 R \, d\theta_1 + \ell \, d\phi)$$

$$= R \sin\phi \, d\theta_1 + \frac{1}{2} \ell \sin\phi \, d\phi$$

so $\boxed{0 = dx + \frac{1}{2} \ell \sin\phi \, d\phi + R \sin\phi \, d\theta_1}$

$$0 = dx + \frac{1}{2} l \sin\phi \, d\phi + R \sin\phi \, d\theta,$$

$$\rightarrow \quad 0 = dx + \frac{1}{2} l \sin y \, dy + R \sin y \, dz$$

$$A(x,y,z) = 1$$
$$B(x,y,z) = \frac{1}{2} l \sin y$$
$$C(x,y,z) = R \sin y$$

<u>check:</u>

$$\partial_y C - \partial_z B = R\cos y - 0 \; \boxed{\neq 0}$$

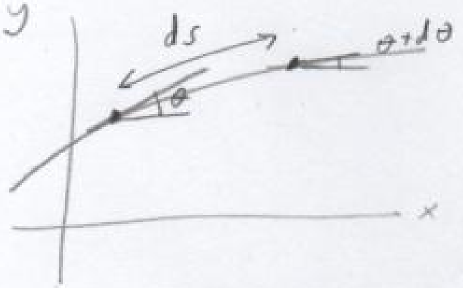
so not integrable.

$$\partial_z A - \partial_x C = 0 - 0 = 0 \; \checkmark$$

$$\partial_x B - \partial_y A = 0 - 0 = 0 \; \checkmark$$

Curvature:

$\kappa = \left|\frac{d\hat{T}}{ds}\right|$, $\hat{T}$: unit tgt vector to a curve



$$\vec{r} = x\hat{x} + y\hat{y}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y}$$

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$

$$\kappa = \left|\frac{d\hat{T}}{ds}\right| = \left|\frac{d\lambda}{ds}\frac{d\hat{T}}{d\lambda}\right|$$

Take $\lambda = x$: Then $ds = \sqrt{dx^2 + dy^2}$

$= dx\sqrt{1+y'^2} \rightarrow \frac{dx}{ds} = \frac{1}{\sqrt{1+y'^2}}$

$$\kappa = \left|\frac{dx}{ds}\frac{d\hat{T}}{dx}\right|$$

$$= \frac{1}{\sqrt{1+y'^2}}\left|\frac{d\hat{T}}{dx}\right|$$

$$= \frac{1}{\sqrt{1+y'^2}}\left|\frac{d}{dx}\left(\frac{\hat{x} + y'\hat{y}}{\sqrt{1+y'^2}}\right)\right|$$

$$= \frac{1}{\sqrt{1+y'^2}}\left|\left\{\frac{-\frac{1}{2}(2y')y''}{(1+y'^2)^{3/2}}(\hat{x}+y'\hat{y}) + \frac{1}{\sqrt{1+y'^2}}y''\hat{y}\right\}\right|$$

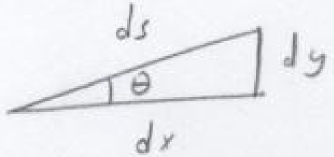
$$= \frac{1}{(1+y'^2)^2}\left|\left\{-y'y''(\hat{x}+\cancel{y'\hat{y}}) + (1+\cancel{y'^2})y''\hat{y}\right\}\right|$$

$$= \frac{1}{(1+y'^2)^2}\left|y''\left[-y'\hat{x} + \hat{y}\right]\right|$$

$$= \frac{|y''|}{(1+y'^2)^2}\sqrt{1+y'^2} = \boxed{\frac{y''}{(1+y'^2)^{3/2}}}$$

Another derivation:

$$\kappa = \left|\frac{d\theta}{ds}\right|$$



$$\tan\theta = \frac{dy}{dx} = y' \quad , \quad \cos\theta = \frac{dx}{ds} = \frac{1}{\sqrt{1+y'^2}}$$

$$\rightarrow \quad \frac{1}{\cos^2\theta}\frac{d\theta}{ds} = \frac{dy'}{ds}$$

Thus, $\kappa = \left|\frac{d\theta}{ds}\right|$

$$= \cos^2\theta \left|\frac{dy'}{ds}\right|$$

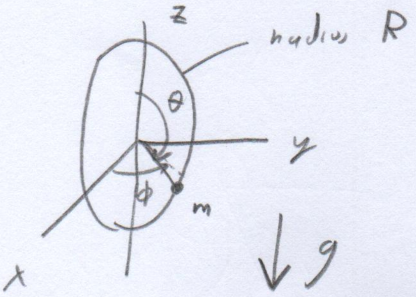
$$= \cos^2\theta \left|\frac{dx}{ds}\frac{dy'}{dx}\right|$$

$$= \left|\frac{dx}{ds}\right|^3 |y''|$$

$$= \frac{1}{(1+y'^2)^{3/2}}|y''|$$

2.3

Problem: Rotating hoop with $\alpha = \text{const}$



$$\alpha = \text{const}$$

$$\phi = \phi_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

Also, $r = R$

Constraint functions:

$$\varphi^1 \equiv r - R = 0$$

$$\varphi^2 \equiv \phi - \left(\phi_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2\right)$$

Take 2nd time derivative:

$$\ddot{\varphi}^1 \equiv \ddot{r} = 0$$

$$\ddot{\varphi}^2 = \ddot{\phi} - \alpha = 0$$

Use
$$\ddot{\vec{F}} = (\ddot{r} - r\dot{\phi}^2 \sin^2\theta - r\dot{\theta}^2)\,\hat{r}$$
$$+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)\,\hat{\phi}$$
$$+ (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\,\hat{\theta}$$

$$\rightarrow \dot{\vec{p}} = m\ddot{\vec{r}}$$

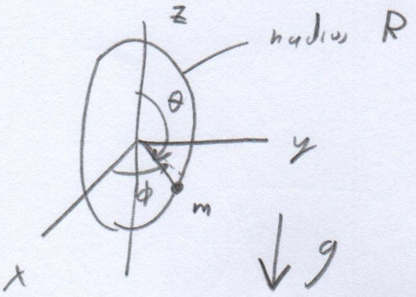
$$\vec{F}^{(a)} = -mg\,\hat{z} = -mg\cos\theta\,\hat{r} + mg\sin\theta\,\hat{\theta}$$

$$\vec{\nabla}\varphi^1 = \frac{\partial}{\partial r}(r - R)\,\hat{r} = \hat{r}$$

$$\vec{\nabla}\varphi^2 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\phi - \tfrac{1}{2}\alpha t^2\right) = \frac{1}{r\sin\theta}\,\hat{\phi}$$

2.3

<u>Problem:</u> Rotating hoop with $\alpha = \text{const}$



$\alpha = \text{const}$

$\phi = \phi_0 + \omega_0 t + \frac{1}{2}\alpha t^2$

Also, $r = R$

<u>Constraint functions:</u>

$$\varphi^1 \equiv r - R = 0$$

$$\varphi^2 \equiv \phi - \left(\phi_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2\right)$$

Take 2nd time derivative:

$$\ddot{\varphi}^1 \equiv \ddot{r} = 0$$

$$\ddot{\varphi}^2 = \ddot{\phi} - \alpha = 0$$

Use

$$\ddot{\vec{F}} = (\ddot{r} - r\dot{\phi}^2 \sin^2\theta - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)\,\hat{\phi} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\,\hat{\theta}$$

$\rightarrow \dot{\vec{p}} = m\ddot{\vec{r}}$

$$\vec{F}^{(a)} = -mg\,\hat{z} = -mg\cos\theta\,\hat{r} + mg\sin\theta\,\hat{\theta}$$

$$\vec{\nabla}\varphi^1 = \frac{\partial}{\partial r}(r - R)\,\hat{r} = \hat{r}$$

$$\vec{\nabla}\varphi^2 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\phi - \tfrac{1}{2}\alpha t^2\right) = \frac{1}{r\sin\theta}\,\hat{\phi}$$

$$0 = \dot{\vec{p}} - \vec{F}^{(a)} - \sum_A \lambda_A \nabla \varphi^A$$

$$= m\left[\left(\quad\right)\hat{r} + \left(\quad\right)\hat{\theta} + \left(\quad\right)\hat{\phi}\right]$$

$$+ mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{\theta}$$

$$- \lambda_1 \hat{r} - \lambda_2 \frac{1}{r\sin\theta}\hat{\phi}$$

$\hat{r}$: $\quad 0 = m\ddot{r} - mr\dot{\phi}^2\sin^2\theta - mr\dot{\theta}^2 + mg\cos\theta - \lambda_1$

$\hat{\phi}$: $\quad 0 = mr\ddot{\phi}\sin\theta + 2m\dot{r}\dot{\phi}\sin\theta + 2mr\dot{\phi}\dot{\theta}\cos\theta - \frac{\lambda_2}{r\sin\theta}$

$\hat{\theta}$: $\quad 0 = mr\ddot{\theta} + 2m\dot{r}\dot{\theta} - mr\dot{\phi}^2\sin\theta\cos\theta - mg\sin\theta$

---

solve for $\ddot{r}, \ddot{\phi}$:

$$\ddot{r} = -r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2 + g\cos\theta - \frac{\lambda_1}{m} \qquad = 0$$

$$\ddot{\phi} = -2\frac{\dot{r}}{r}\dot{\phi} - 2\dot{\phi}\dot{\theta}\frac{\cos\theta}{\sin\theta} + \frac{\lambda_2}{mr^2\sin^2\theta} \qquad = \alpha$$

substitute into $\ddot{r}=0$, $\ddot{\phi}=\alpha$

$$\rightarrow \quad -r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2 + g\cos\theta - \frac{\lambda_1}{m} = 0$$

$$\boxed{\lambda_1 = mg\cos\theta - mr\dot{\theta}^2 - mr\dot{\phi}^2\sin^2\theta}$$

$$\boxed{\begin{aligned}\lambda_2 = \; & mr^2\sin^2\theta\,\alpha + 2mr\dot{r}\dot{\phi}\sin^2\theta \\ & + 2mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta\end{aligned}}$$

Constraint forces:

$$\vec{F}^{(c)} = \lambda_1 \vec{\nabla} \varphi^1 + \lambda_2 \vec{\nabla} \varphi^2$$

$$= \left(mg\cos\theta - mr\dot{\theta}^2 - mr\dot{\phi}^2 \sin^2\theta \right) \hat{r}$$

$$+ \left(mr^2 \sin^2\theta \, \alpha + 2mr\dot{r}\dot{\phi}\sin^2\theta + 2mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta \right) \frac{1}{r\sin\theta} \hat{\phi}$$

$$= m\left(g\cos\theta - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta \right) \hat{r}$$

$$+ m\left(r\sin\theta \, \alpha + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta \right) \hat{\phi}$$

EOMs:

$$\cancel{0 = \dot{\vec{p}} - \vec{F}^{(a)} - \vec{F}^{(c)}}$$

$$0 = m\ddot{r} - mr\dot{\phi}^2 \sin^2\theta - mr\dot{\theta}^2 \cancel{+ mg\cos\theta}$$
$$\cancel{- mg\cos\theta} + mr\dot{\theta}^2 + mr\dot{\phi}^2 \sin^2\theta$$

$$\rightarrow \boxed{\ddot{r} = 0}$$

$$0 = mr\ddot{\phi}\sin\theta \cancel{+ 2m\dot{r}\dot{\phi}\sin\theta} \cancel{+ 2mr\dot{\phi}\dot{\theta}\cos\theta}$$
$$- mr\sin\theta \, \alpha \cancel{- 2m\dot{r}\dot{\phi}\sin\theta}$$
$$\cancel{- 2mr\dot{\phi}\dot{\theta}\cos\theta}$$
$$= mr\sin\theta \left(\ddot{\phi} - \alpha \right)$$

$$\rightarrow \boxed{\ddot{\phi} - \alpha = 0}$$

$\ddot{r} = 0 \rightarrow \dot{r} = A$

But $\dot{\varphi}' = \dot{r} = 0$ implies $\dot{r} = 0$ so $\boxed{r = R}$

$\ddot{\phi} - \alpha = 0 \rightarrow \ddot{\phi} = \alpha \rightarrow \dot{\phi} = \alpha t + \omega_0$

$\rightarrow \boxed{\phi = \phi_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2}$

(most general expression)

$\underline{\theta}$:

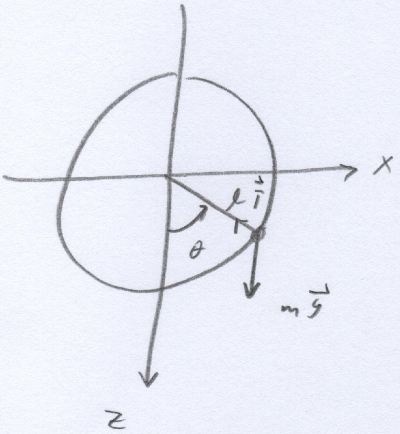
$$0 = \cancel{m}\underset{\overset{\parallel}{R}}{r}\ddot{\theta} + 2\cancel{m}\underset{\overset{\downarrow}{0}}{\dot{r}}\dot{\theta} - \cancel{m}\underset{\overset{\parallel}{R}}{r}(\alpha t + \omega_0)^2 \sin\theta\cos\theta - \cancel{m}g\sin\theta$$

$$\boxed{\begin{aligned}\ddot{\theta} &= \frac{g}{R}\sin\theta + (\alpha t + \omega_0)^2\sin\theta\cos\theta \\ &= \sin\theta\left(\frac{g}{R} + (\omega_0 + \alpha t)^2\cos\theta\right)\end{aligned}}$$

For $\alpha = 0$ case
this was just ω^2

2.4

Problem: Planar pendulum using Lagrange's equations of the 1st kind



$$\dot{\vec{p}}_I - \vec{F}_I^{(a)} - \sum_A \lambda_A \vec{\nabla}_I \varphi^A = 0$$

One particle, 2-d (x,y)

$$\vec{F}^{(a)} = mg\hat{z}$$

Constraint:

$$\varphi \equiv x^2 + z^2 - R^2 = 0$$

$$\vec{\nabla}\varphi = 2x\hat{x} + 2z\hat{z}$$

$$\vec{p} = m\vec{v} = m(\dot{x}\hat{x} + \dot{z}\hat{z})$$

$$\dot{\vec{p}} = m(\ddot{x}\hat{x} + \ddot{z}\hat{z})$$

$$\boxed{\begin{aligned} 0 &= m(\ddot{x}\hat{x} + \ddot{z}\hat{z}) - mg\hat{z} - \lambda(2x\hat{x} + 2z\hat{z}) \\ &= (m\ddot{x} - 2\lambda x)\hat{x} + (m\ddot{z} - mg - 2\lambda z)\hat{z} \end{aligned}}$$

Lagrange's eqs of the 1st kind

(i) $0 = \dot{\varphi} = 2x\dot{x} + 2z\dot{z}$

$0 = \ddot{\varphi} = 2(\dot{x}^2 + \dot{z}^2) + 2(x\ddot{x} + z\ddot{z}) = \dot{x}^2 + \dot{y}^2 + x\ddot{x} + z\ddot{z}$

(ii) substitute into $\ddot{\varphi} = 0$ equation using ~~Eq~~ Lagrange's equations of the 1st kind

$$\boxed{\ddot{x} = \frac{2\lambda}{m}x}, \quad \boxed{\ddot{z} = g + \frac{2\lambda}{m}z}$$

(iii) solve for λ:

Thus,

$$\begin{aligned} 0 &= \dot{x}^2 + \dot{y}^2 + \frac{2\lambda x^2}{m} + \frac{2\lambda z^2}{m} + gz \\ &= \frac{2\lambda(x^2 + z^2)}{m} + (\dot{x}^2 + \dot{y}^2) + gz \\ &= \frac{2\lambda}{m}R^2 + v^2 + gz \end{aligned}$$

$$\frac{2\lambda R^2}{m} = -v^2 - gz$$

$$\boxed{\lambda} = \frac{m}{2R^2}\left(-v^2 - gz\right)$$

$$= -\frac{mv^2}{2R^2} - \frac{mgz}{2R^2}$$

$$= \boxed{-\frac{mv^2}{2R^2} - \frac{mg\cos\theta}{2R}} \qquad \left(\text{using } \frac{z}{R} = \cos\theta\right)$$

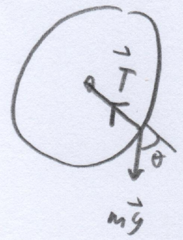
(iv) constraint force:

$$\boxed{\vec{F}^{(c)}} = \lambda \vec{\nabla}\phi$$

$$= \left(-\frac{mv^2}{2R^2} - \frac{mg\cos\theta}{2R}\right)\Big[\underbrace{2x\hat{x} + 2y\hat{y}}_{2\vec{r} = 2R\hat{r}}\Big]$$

$$= \boxed{-\left(\frac{mv^2}{R} + mg\cos\theta\right)\hat{r}}$$

This agrees with:



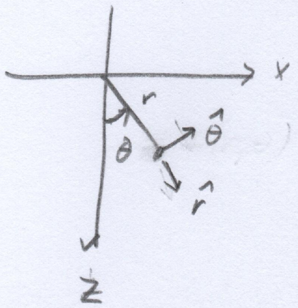
$$T - mg\cos\theta = \frac{mv^2}{R}$$

$$T = \frac{mv^2}{R} + mg\cos\theta$$

$$mg\sin\theta = -mR\ddot{\theta} \qquad (-m\cdot(\text{ang acceleration}))$$

$$\boxed{\ddot{\theta} = -\frac{g}{R}\sin\theta}$$

Suppose we do the analysis in polar coords (r, θ):



$$\vec{r} = z\hat{z} + x\hat{x}$$
$$= r\cos\theta\,\hat{z} + r\sin\theta\,\hat{x}$$

$$\vec{v} = \dot{\vec{r}}$$
$$= (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})\hat{z} + (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})\hat{x}$$

$$\hat{r} = \cos\theta\,\hat{z} + \sin\theta\,\hat{x}$$
$$\hat{\theta} = -\sin\theta\,\hat{z} + \cos\theta\,\hat{x}$$

$\Leftarrow\Rightarrow$

$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$
$$\hat{x} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$$

$$\vec{a} = \ddot{\vec{r}}$$
$$= (\ddot{r}\cos\theta - \dot{r}\sin\theta\,\dot{\theta} - \dot{r}\sin\theta\,\dot{\theta} - r\cos\theta\,\dot{\theta}^2 - r\sin\theta\,\ddot{\theta})\hat{z}$$
$$+ (\ddot{r}\sin\theta + \dot{r}\cos\theta\,\dot{\theta} + \dot{r}\cos\theta\,\dot{\theta} - r\sin\theta\,\dot{\theta}^2 + r\cos\theta\,\ddot{\theta})\hat{x}$$

$$= (\ddot{r}\cos\theta - 2\dot{r}\sin\theta\,\dot{\theta} - r\cos\theta\,\dot{\theta}^2 - r\sin\theta\,\ddot{\theta})\hat{z}$$
$$+ (\ddot{r}\sin\theta + 2\dot{r}\cos\theta\,\dot{\theta} - r\sin\theta\,\dot{\theta}^2 + r\cos\theta\,\ddot{\theta})\hat{x}$$

$$= (\ddot{r}\cos\theta - 2\dot{r}\sin\theta\,\dot{\theta} - r\cos\theta\,\dot{\theta}^2 - r\sin\theta\,\ddot{\theta})(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$
$$+ (\ddot{r}\sin\theta + 2\dot{r}\cos\theta\,\dot{\theta} - r\sin\theta\,\dot{\theta}^2 + r\cos\theta\,\ddot{\theta})(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta})$$

$$= \hat{r}(\ddot{r} - r\dot{\theta}^2) + \hat{\theta}(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

$$\varphi \equiv r - R = 0 \qquad , \qquad \vec{\nabla}\varphi = \frac{\partial}{\partial r}(r-R)\hat{r} = \hat{r}$$

$$\dot{\varphi} \equiv \dot{r} = 0 \qquad \vec{F}^{(a)} = mg\hat{z} = mg(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

$$\ddot{\varphi} = \ddot{r} = 0$$

$$0 = \dot{\vec{p}} - \vec{F}^{(a)} - \lambda\vec{\nabla}\varphi$$

$$= m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} - mg(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) - \lambda\hat{r}$$

$$= (m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta - \lambda)\hat{r} + (2m\dot{r}\dot{\theta} + mr\ddot{\theta} + mg\sin\theta)\hat{\theta}$$

so

$$\boxed{\begin{aligned} &m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta - \lambda = 0 \\ &2m\dot{r}\dot{\theta} + mr\ddot{\theta} + mg\sin\theta = 0 \end{aligned}}$$

$$0 = \ddot{r} \qquad (\ddot{\varphi} = 0)$$

$$= r\dot{\theta}^2 + g\cos\theta + \frac{\lambda}{m}$$

Thus,

$$\lambda = -(mr\dot{\theta}^2 + mg\cos\theta)$$

$$= -\left(m\frac{r^2\dot{\theta}^2}{r} + mg\cos\theta\right)$$

$$\boxed{\lambda = -\left(m\frac{v^2}{R} + mg\cos\theta\right)}$$

Thus, ~~ct~~ constraint force

$$\vec{F}^{(c)} = \lambda \nabla \varphi$$

$$= -\left(\frac{m v^2}{R} + m g \cos\theta \right) \hat{r}$$

EOMs:

$$0 = m \ddot{r} - m r \dot{\theta}^2 - \cancel{m g \cos\theta} + \left(\frac{m v^2}{R} + \cancel{m g \cos\theta} \right)$$

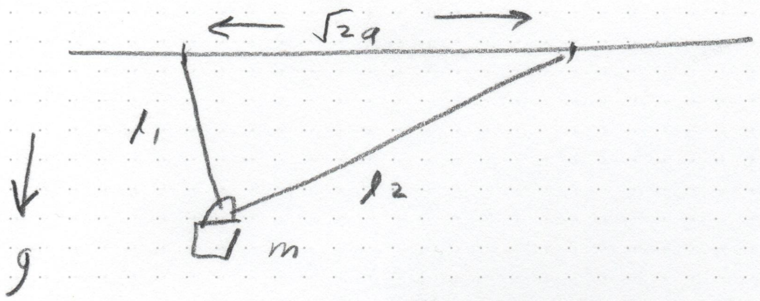
$$= 0 \quad \boxed{\checkmark} \quad \text{automatically satisfied}$$

$$0 = 2 m \dot{r} \dot{\theta} + m r \ddot{\theta} + m g \sin\theta$$

$$\rightarrow \boxed{\ddot{\theta} = -\frac{g}{R} \sin\theta}$$

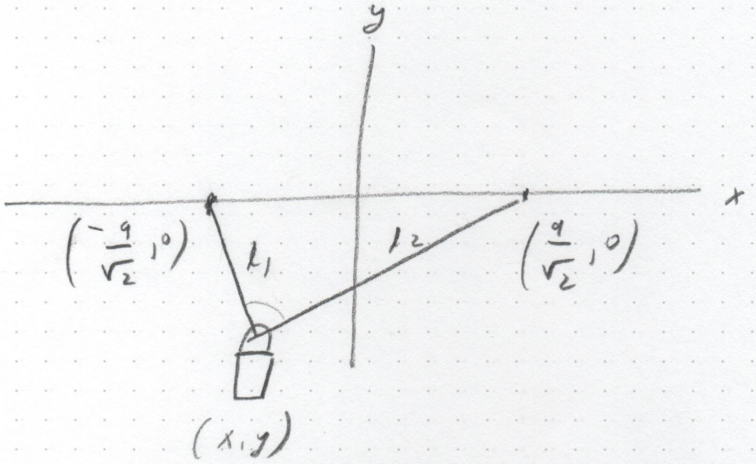
Problem: sliding bucket

2.5 2.6



$l_1 + l_2 = 2a$

$l_1 + l_2 - 2a = 0$



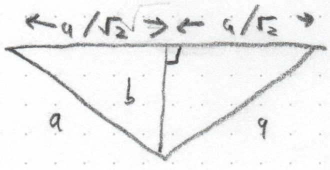
$$l_1 = \sqrt{\left(x + \frac{a}{\sqrt{2}}\right)^2 + y^2}, \quad l_2 = \sqrt{\left(x - \frac{a}{\sqrt{2}}\right)^2 + y^2}$$

Thus

$$\underbrace{\sqrt{\left(x + \frac{a}{\sqrt{2}}\right)^2 + y^2} + \sqrt{\left(x - \frac{a}{\sqrt{2}}\right)^2 + y^2} - 2a}_{\Phi(x,y)} = 0$$

$$\underset{\text{(constraint)}}{\Phi(x,y)} = 0$$

Simplify: Noting that the bucket moves along an ellipse with focal points at $(\frac{a}{\sqrt{2}}, 0)$, $(-\frac{a}{\sqrt{2}}, 0)$ with $d_1 + d_2 = 2a$ (a = semi-major axis)



$$b^2 + \left(\frac{a}{\sqrt{2}}\right)^2 = a^2$$

$$b^2 = a^2\left(1 - \frac{1}{2}\right) = \frac{a^2}{2}$$

$$\rightarrow \quad b = \frac{a}{\sqrt{2}}$$

To show: $d_1 + d_2 = 2a \rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

Proof: $2a = d_1 + d_2$

$$= \sqrt{(x-ae)^2+y^2} + \sqrt{(x+ae)^2+y^2}$$

where $b^2 = a^2(1-e^2)$ (see below)

$$2a - \sqrt{(x-ae)^2+y^2} = \sqrt{(x+ae)^2+y^2}$$

Square both sides:

$$4a^2 + (x-ae)^2+y^2 - 4a\sqrt{(x-ae)^2+y^2} = (x+ae)^2+y^2$$

$$4a^2 + \cancel{x^2} + a^2e^2 - 2aex + \cancel{y^2} - 4a\sqrt{(x-ae)^2+y^2} = \cancel{x^2} + a^2e^2 + 2aex + \cancel{y^2}$$

$$-4a\sqrt{(x-ae)^2+y^2} = 4aex - 4a^2$$

$$\sqrt{(x-ae)^2+y^2} = -ex + a$$

Square both sides:

$$(x-ae)^2+y^2 = e^2x^2 + a^2 - 2aex$$

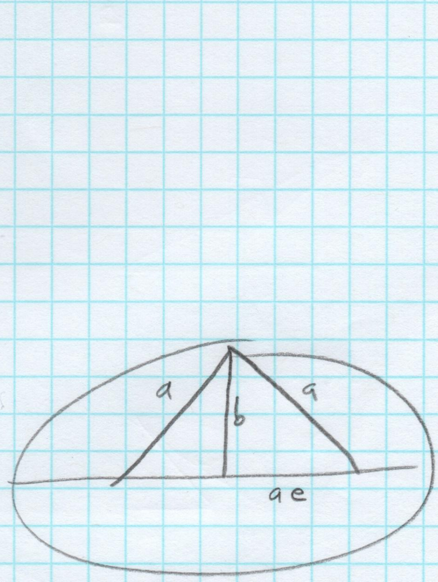
$$x^2+a^2e^2 \cancel{-2aex} + y^2 = e^2x^2 + a^2 \cancel{-2aex}$$

$$x^2(1-e^2) + y^2 = a^2(1-e^2)$$

$$x^2(1-e^2) + y^2 = b^2$$

$$x^2 \frac{(1-e^2)}{b^2} \left(\frac{y}{b}\right)^2 = 1$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$



$$a^2 = b^2 + a^2e^2$$

$$a^2(1-e^2) = b^2$$

Thus, constraint is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a/\sqrt{2}}\right)^2 = 1$$

$$\frac{x^2}{a^2} + \frac{2y^2}{a^2} = 1$$

So $\boxed{x^2 + 2y^2 - a^2 = 0} \equiv \Phi(x,y)$

Lagrange's equation of 1st kind:

$\vec{F} = -mg\,\vec{y}$ (impressed force)

$$\left.\begin{aligned} &(i) \quad m\ddot{x} - \lambda \frac{\partial \phi}{\partial x} = 0 \\ &(ii) \quad m\ddot{y} + mg - \lambda \frac{\partial \phi}{\partial y} = 0 \end{aligned}\right\}$$

λ: Lagrange multiplier

$\lambda \vec{\nabla} \Phi$: constraint force

Thus,

$$0 = m\ddot{x} - \lambda 2x$$
$$0 = m\ddot{y} + mg - \lambda 4y$$

~~Can eliminate λ by multiplying 1st equation~~ by $2y$,
~~the 2nd by x~~, ~~and then subtracting~~:

$$\cancel{0 = 2m\ddot{x}y - m\ddot{y}x} - mgx$$

multiply 1st equation by $\dot{x}$, 2nd by $\dot{y}$

$$0 = m\ddot{x}\dot{x} - 2\lambda x\dot{x}$$
$$0 = m\ddot{y}\dot{y} + mg\dot{y} - \lambda 4y\dot{y}$$

Then add

$$0 = m(\ddot{x}\dot{x} + \ddot{y}\dot{y}) + mg\dot{y} - 2\lambda(x\dot{x} + 2y\dot{y})$$

But note that

$$0 = \dot{\Phi} = 2x\dot{x} + 4y\dot{y} \rightarrow x\dot{x} + 2y\dot{y} = 0$$

Thus,

$$0 = \frac{d}{dt}\left[\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy\right]$$

$$\text{so} \quad \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy \equiv E$$
$$= \text{const.}$$

Use the constraint equation to eliminate y in the above equation.

$$x^2 + 2y^2 - a^2 = 0 \rightarrow 2y^2 = a^2 - x^2$$

$$y = -\sqrt{\frac{a^2 - x^2}{2}} \quad \left(\frac{\text{NOTE:}}{y<0}\right)$$

Also:

$$\dot{y} = -\frac{1}{2}\frac{(-x\dot{x})}{\sqrt{\frac{a^2-x^2}{2}}} = \frac{x\dot{x}}{\sqrt{2}\sqrt{a^2-x^2}}$$

$$\text{so} \quad \dot{y}^2 = \frac{\cancel{2}x^2\dot{x}^2}{2(a^2-x^2)}$$

Thus,

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$$
$$= \frac{1}{2}m\dot{x}^2\left(1 + \frac{\cancel{2}x^2}{2(a^2-x^2)}\right) - mg\sqrt{\frac{a^2-x^2}{2}}$$

$$\text{so} \quad \pm\sqrt{\frac{\frac{2}{m}\left(E + mg\sqrt{\frac{a^2-x^2}{2}}\right)}{1 + \frac{x^2}{2(a^2-x^2)}}} = \dot{x} = \frac{dx}{dt}$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{\frac{2}{m}\left(E + mg\sqrt{\frac{a^2-x^2}{2}}\right)}{\frac{2(a^2-x^2)+x^2}{2(a^2-x^2)}}}$$

$$= \pm 2\sqrt{\frac{(a^2-x^2)\left(E + mg\sqrt{\frac{a^2-x^2}{2}}\right)}{m\,(2a^2-x^2)}}$$

Thus

$$t - t_0 = \int_{x_0}^{x} \frac{dx}{\pm 2\sqrt{\frac{(a^2-x^2)\left(E + mg\sqrt{\frac{a^2-x^2}{2}}\right)}{m\,(2a^2-x^2)}}}$$

Finding λ:

$$\Phi(x,y) = x^2 + 2y^2 - a^2 = 0$$

$$\rightarrow \dot{\Phi} = 2x\dot{x} + 4y\dot{y} = 0 \quad \rightarrow \quad x\dot{x} + 2y\dot{y} = 0$$

$$\ddot{\Phi} = \dot{x}^2 + x\ddot{x} + 2\dot{y}^2 + 2y\ddot{y} = 0$$

substitute into these equations using

$$0 = m\ddot{x} - \lambda 2x$$
$$0 = m\ddot{y} + mg - \lambda 4y$$

$$\rightarrow \quad \ddot{x} = +\frac{2\lambda}{m}x$$
$$\ddot{y} = -g + \frac{4\lambda y}{m}$$

so

$$0 = \dot{x}^2 + x\left(\frac{2\lambda}{m}x\right) + 2\dot{y}^2 + 2y\left(-g + \frac{4\lambda y}{m}\right)$$

$$= \dot{x}^2 + \lambda\frac{2x^2}{m} + 2\dot{y}^2 - 2yg + 8\lambda\frac{y^2}{m}$$

$$= \dot{x}^2 + 2\dot{y}^2 - 2yg + \lambda\left(\frac{2x^2}{m} + \frac{8y^2}{m}\right)$$

$$\rightarrow \lambda = \frac{2yg - \dot{x}^2 - 2\dot{y}^2}{2\frac{x^2}{m} + \frac{8y^2}{m}}$$

$$= \frac{2mgy - m\dot{x}^2 - 2m\dot{y}^2}{2x^2 + 8y^2}$$

$$= \frac{2mgy - m(\dot{x}^2 + \dot{y}^2) - m\dot{y}^2}{2x^2 + 8y^2}$$

$$= \frac{2mgy + 2mgy - 2E - m\dot{y}^2}{2x^2 + 8y^2}$$

$$= \frac{4mgy - 2E - m\dot{y}^2}{2x^2 + 8y^2}$$

Recall: $$\dot{y}^2 = \frac{x^2\dot{x}^2}{2(a^2-x^2)} = \frac{x^2}{\cancel{2}\,\cancel{(a^2-x^2)}}\;\frac{\overset{2}{\cancel{4}}\,\cancel{(a^2-x^2)}\left(E + mg\sqrt{\frac{a^2-x^2}{2}}\right)}{m(2a^2-x^2)}$$

$$= \frac{2x^2}{m}\;\frac{\left(E + mg\sqrt{\frac{a^2-x^2}{2}}\right)}{(2a^2-x^2)}$$

and $$y = -\sqrt{\frac{a^2-x^2}{2}}$$

Thus,

$$\lambda = \frac{-4\,mg\sqrt{\frac{a^2-x^2}{2}} - 2E - \frac{2x^2\left(E + mg\sqrt{\frac{a^2-x^2}{2}}\right)}{(2a^2-x^2)}}{\underbrace{2x^2 + 8\left(\frac{a^2-x^2}{2}\right)}}$$

$$2x^2 + 4a^2 - 4x^2 = 4a^2 - 2x^2$$
$$= 2(2a^2-x^2)$$
$$= -2(x^2-2a^2)$$

$$= \frac{2E + 4mg\sqrt{\frac{a^2-x^2}{2}} + 2x^2\left(\frac{E + mg\sqrt{\frac{a^2-x^2}{2}}}{2a^2-x^2}\right)}{2(x^2-2a^2)}$$

Rewrite in terms of y:

use $\Phi(x,y) = 0 = x^2 + 2y^2 - a^2$

$\rightarrow \quad x^2 = a^2 - 2y^2 \quad$ or $\quad a^2 - x^2 = 2y^2$

Thus, $\sqrt{\frac{a^2-x^2}{2}} = \sqrt{\frac{2y^2}{2}} = |y| = -y \quad$ (since $y<0$)

$$E + mg\sqrt{\frac{a^2-x^2}{2}} = E - mgy$$

$$2a^2 - x^2 = 2a^2 - a^2 + 2y^2 = a^2 + 2y^2$$

$$x^2 - 2a^2 = -(a^2+2y^2)$$

$$\lambda = \frac{2E - 4mgy + 2(a^2-2y^2)\dfrac{(E-mgy)}{(a^2+2y^2)}}{-2(a^2+2y^2)}$$

$$= \frac{-E(a^2+2y^2) + 2mgy(a^2+2y^2) - (a^2-2y^2)(E-mgy)}{(a^2+2y^2)^2}$$

$$= \frac{-Ea^2 \cancel{-2Ey^2} + 2mgya^2 \boxed{+4mgy^3} - Ea^2 \boxed{-2mgy^3} + mgya^2 \cancel{+2y^2E}}{(a^2+2y^2)^2}$$

$$= \frac{-2Ea^2 + 2mgy^3 + 3mgya^2}{(a^2+2y^2)^2}$$

$$= \frac{m\left(-\frac{2E}{m}a^2 + 2gy^3 + 3gya^2\right)}{(a^2+2y^2)^2}$$

Summary:

$$\lambda = \frac{2E + 4mg\sqrt{\dfrac{a^2-x^2}{2}} + 2x^2\left(\dfrac{E+mg\sqrt{\dfrac{a^2-x^2}{2}}}{2a^2-x^2}\right)}{2(x^2-2a^2)}$$

or

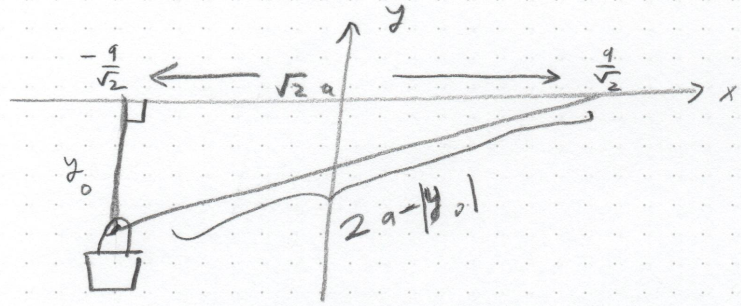
$$\boxed{\lambda = \frac{m\left(2gy^3 + 3gya^2 - \frac{2E}{m}a^2\right)}{(a^2+2y^2)^2}}$$

← for arbitrary initial conditions

(simpler than the one in terms of x)

Initial conditions

$(y_0 < 0)$



$\dot{x}_0 = 0 \qquad x_0 = -\frac{a}{\sqrt{2}}$

$\dot{y}_0 = 0 \qquad y_0 = -\frac{a}{2}$ (see below)

$$y_0^2 + 2a^2 = (2a-|y_0|)^2$$

$$\cancel{y_0^2} + 2a^2 = 4a^2 + \cancel{y_0^2} - 4a|y_0|$$

$$4a|y_0| = 2a^2$$

$$|y_0| = \frac{a}{2} \quad \rightarrow \quad \boxed{y_0 = -\frac{a}{2}}$$

$$E = \left[\tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) + mgy\right]\Big|_0$$

$$= mgy_0$$

$$= -\frac{mga}{2} \quad \rightarrow \quad -\frac{2E}{m} = ga$$

$$\rightarrow \boxed{\lambda} = \frac{m\left(2gy^3 + 3gya^2 + ga\,a^2\right)}{(a^2+2y^2)^2}$$

$$\boxed{= \frac{mg\left(2y^3 + 3ya^2 + a^3\right)}{(a^2+2y^2)^2}}$$

and a similar (but more complicated) expression for λ in terms of x.

Components of constraint force:

$$C_x = \lambda \Phi_{,x} \quad , \quad C_y = \lambda \Phi_{,y}$$

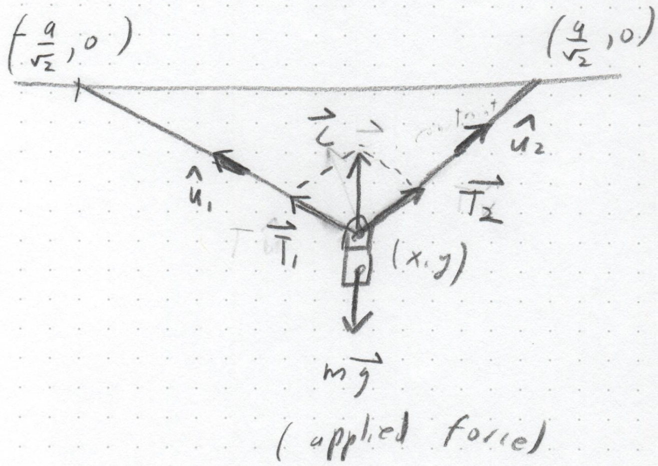
$$\Phi(x,y) = x^2 + 2y^2 - a^2$$

$$\Phi_{,x} = 2x$$

$$\Phi_{,y} = 4y$$

$$\rightarrow \boxed{C_x = 2x\lambda = \frac{2x\, mg\,(2y^3 + 3ya^2 + a^3)}{(a^2+2y^2)^2}}$$

$$\boxed{C_y = 4y\lambda = \frac{4y\, mg\,(2y^3 + 3ya^2 + a^3)}{(a^2+2y^2)^2}}$$



$$\vec{T}_1 = T\hat{u}_1$$

$$\vec{T}_2 = T\hat{u}_2$$

where

$$\vec{C} = \vec{T}_1 + \vec{T}_2 = T(\hat{u}_1 + \hat{u}_2)$$

Dot into $\vec{C}$

$$\vec{C}\cdot\vec{C} = T(\vec{C}\cdot\hat{u}_1 + \vec{C}\cdot\hat{u}_2)$$

$$C^2 = T(\vec{C}\cdot\hat{u}_1 + \vec{C}\cdot\hat{u}_2)$$

~~Thus, T =~~

Need to calculate

$\vec{C}\cdot\hat{u}_1$, $\vec{C}\cdot\hat{u}_2$

$$\hat{u}_1 = \frac{\left(-\frac{a}{\sqrt{2}}-x,\ -y\right)}{\sqrt{\left(-\frac{a}{\sqrt{2}}-x\right)^2+y^2}}, \quad \hat{u}_2 = \frac{\left(\frac{a}{\sqrt{2}}-x,\ -y\right)}{\sqrt{\left(\frac{a}{\sqrt{2}}-x\right)^2+y^2}}$$

$$\vec{C} = (2x\lambda,\ 4y\lambda) = 2\lambda(x, 2y)$$

Thus,
$$\vec{C}\cdot\hat{u}_1 = \frac{2\lambda}{\sqrt{}}\left[x\left(-\frac{a}{\sqrt{2}}-x\right)-2y^2\right]$$

Now,
$x^2+2y^2-a^2=0$
$2y^2 = a^2-x^2$

$$= \frac{2\lambda\left[-\frac{ax}{\sqrt{2}}-x^2-2y^2\right]}{\left[\frac{a^2}{2}+x^2+\sqrt{2}ax+y^2\right]^{\frac{1}{2}}}$$

$$= \frac{2\lambda\left[-\frac{ax}{\sqrt{2}}-\cancel{x^2}-a^2+\cancel{x^2}\right]}{\left[\frac{a^2}{2}+x^2+\sqrt{2}ax+\frac{a^2}{2}-\frac{x^2}{2}\right]^{\frac{1}{2}}}$$

$$= \frac{2\lambda\left[-\frac{ax}{\sqrt{2}}-a^2\right]}{\left[a^2+\frac{1}{2}x^2+\sqrt{2}ax\right]^{\frac{1}{2}}}$$

$$\sqrt{\frac{1}{2}\left(x^2+2\sqrt{2}ax+2a^2\right)}$$
$$=\sqrt{\frac{1}{2}\left(x+\sqrt{2}a\right)^2}$$
$$=\frac{1}{\sqrt{2}}\left(x+\sqrt{2}a\right)$$

$$= \frac{2\lambda\left[-\frac{a}{\cancel{\sqrt{2}}}\cancel{\left(x+\sqrt{2}a\right)}\right]}{\cancel{\sqrt{\tfrac{1}{2}}}\cancel{\left(x+\sqrt{2}a\right)}}$$

$$= \boxed{-2\lambda a}$$

Similarly

$$\vec{C}\cdot\hat{u}_2 = \frac{2\lambda}{\sqrt{}}\left[x\left(\frac{a}{\sqrt{2}}-x\right)-2y^2\right]$$

$$= \frac{2\lambda\left[\frac{ax}{\sqrt{2}}-x^2-2y^2\right]}{\left[\frac{a^2}{2}+x^2-\sqrt{2}ax+y^2\right]^{1/2}}$$

Use: $2y^2 = a^2 - x^2$

$$= \frac{2\lambda\left[\frac{ax}{\sqrt{2}}-\cancel{x^2}-a^2+\cancel{x^2}\right]}{\left[\frac{a^2}{2}+x^2-\sqrt{2}ax+\left(\frac{a^2-x^2}{2}\right)\right]^{\frac{1}{2}}}$$

$$= \frac{2\lambda\frac{a}{\sqrt{2}}\left(x-\sqrt{2}a\right)}{\left(\frac{1}{2}x^2-\sqrt{2}ax+a^2\right)^{\frac{1}{2}}} \leftarrow \sqrt{\frac{1}{2}\left(x-\sqrt{2}a\right)^2}$$

$$= \frac{2\lambda a\cancel{\frac{1}{\sqrt{2}}}\left(x-\sqrt{2}a\right)}{\cancel{\sqrt{\frac{1}{2}}}\,\underbrace{\left|x-\sqrt{2}a\right|}_{=-(x-\sqrt{2}a)}}$$

$$= \boxed{-2\lambda a}$$

$-\frac{a}{\sqrt{2}} < x < \frac{a}{\sqrt{2}}$

so $x-\sqrt{2}a < 0$

$\frac{a}{\sqrt{2}}-\sqrt{2}a = \left(\frac{1}{\sqrt{2}}-\sqrt{2}\right)a$

$= (.707-1.414)a$

Thus, $\vec{C}\cdot\hat{u}_1 = -2\lambda a = \vec{C}\cdot\vec{u}_2$

so

$$C^2 = T\left(-2\lambda a - 2\lambda a\right)$$

$$\rightarrow T = \frac{C^2}{-4\lambda a} = \frac{4\lambda^2\left(x^2+4y^2\right)}{-4\lambda a}$$

$$= -\frac{\lambda}{a}\left(x^2+4y^2\right) = \frac{-\lambda}{a}\left(a^2+2y^2\right)$$

$x^2 = a^2 - 2y^2$

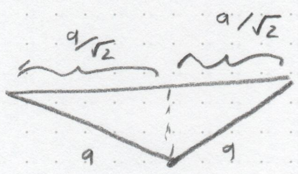
$$T = -\frac{\lambda}{a}\left(a^2+2y^2\right)$$

$$= -\frac{mg}{a}\,\frac{\left(2y^3+3y\,a^2+a^3\right)}{\left(a^2+2y^2\right)^{\cancel{2}}}\ \cancel{\left(a^2+2y^2\right)}$$

$$= \boxed{-\frac{mg}{a}\,\frac{\left(2y^3+3y\,a^2+a^3\right)}{\left(a^2+2y^2\right)}}$$

At lowest point:

$x = 0\,,\quad y = -\frac{a}{\sqrt{2}}$



$y = -\frac{a}{\sqrt{2}}$

$$T\Big|_{\text{lowest}} = -\frac{mg}{a}\,\frac{\left(\cancel{2}\left(\frac{-a^3}{\cancel{2}\sqrt{2}}\right) - \frac{3}{\sqrt{2}}a^3 + a^3\right)}{\left(a^2 + \cancel{2}\left(\frac{a^2}{\cancel{2}}\right)\right)}$$

$$= -mg\,\frac{\left(-\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}} + 1\right)}{2}$$

$\cancel{\frac{4}{\sqrt{2}}}$

$$= -mg\,\frac{\left(-\frac{4}{\sqrt{2}} + 1\right)}{2}$$

$\frac{4}{\sqrt{2}} = \frac{2\cdot 2}{\sqrt{2}} = 2\sqrt{2}$

$$= mg\left(\frac{2\sqrt{2}-1}{2}\right)$$

$\frac{mg}{\sqrt{2}}\left(2 - 1\right.$

$$= \frac{mg}{\sqrt{2}}\left(\frac{2\sqrt{2}-1}{\sqrt{2}}\right)$$

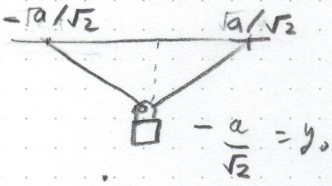
$$= \frac{mg}{\sqrt{2}}\left(2 - \frac{1}{\sqrt{2}}\right) > \frac{mg}{\sqrt{2}}$$

(as expected) for these ICs

Return to:

$$\lambda = \frac{m\left(2gy^3 + 3gya^2 - \frac{2E}{m}a^2\right)}{(a^2+2y^2)^2}$$

Consider different initial conditions



$\dot{x}_0 = 0$

$\dot{y}_0 = 0$

$x_0 = 0$

$y_0 = \frac{-a}{\sqrt{2}}$

$$E = \left[\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) + mgy\right]\Big|_0$$

$$= 0 - \frac{mga}{\sqrt{2}}$$

$$-\frac{2E}{m} = \sqrt{2}\,ga$$

Then

$$\lambda\Big|_{\text{lowest}} = \frac{m\left(2gy^3 + 3gya^2 + \sqrt{2}\,ga^3\right)}{(a^2+2y^2)^2}\Bigg|_{y=\frac{-a}{\sqrt{2}}}$$

$y_0 = -\frac{a}{\sqrt{2}}$

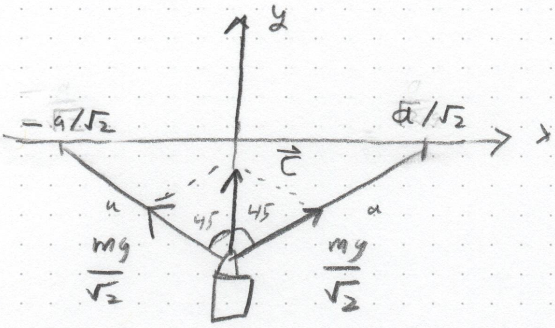
$$= \frac{m\left(\cancel{2}g\left(\frac{-a^3}{\cancel{2}\sqrt{2}}\right) - 3g\frac{a}{\sqrt{2}}a^2 + \sqrt{2}\,ga^3\right)}{\left(a^2 + \frac{\cancel{2}a^2}{\cancel{2}}\right)^2}$$

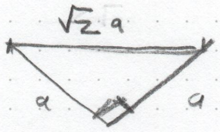
$$= \frac{mg}{2a}\,\frac{\left(-\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}} + \sqrt{2}\right)}{4}$$

$$= \frac{mg}{a}\,\frac{\left(-\frac{4}{\sqrt{2}} + \sqrt{2}\right)}{4} \qquad -2\sqrt{2}+\sqrt{2}$$

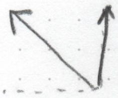
$$= \frac{mg}{a}\left(\frac{-\sqrt{2}}{4}\right)$$

$$\rightarrow T\Big|_{\text{lowest}} = -\frac{\lambda}{a}(a^2+2y^2) = -\frac{mg}{a^2}\left(\frac{-\sqrt{2}}{4}\right)\left(a^2 + \overbrace{\tfrac{\cancel{2}a^2}{\cancel{2}}}^{2}\right) = \boxed{\frac{mg}{\sqrt{2}}}$$





$a^2 + a^2 = 2a^2$ ✓



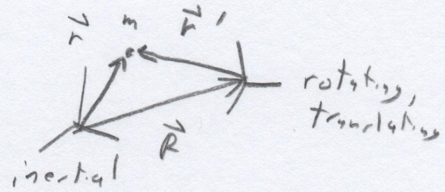
$\frac{mg}{\sqrt{2}} \frac{\sqrt{2}}{2} = \frac{mg}{2}$

$\boxed{\vec{C} = mg\,\hat{y}}$ (which balances gravity)

Problem: Fictitious forces from kinetic ~~energy~~ in non-inertial frame

2.7

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v} \cdot \vec{v}$$



$$\vec{v} = \vec{v}\,' + \vec{\omega} \times \vec{r}\,' + \left(\frac{d\vec{R}}{dt}\right)_f$$

Components: $\quad v^a = \dot{q}^a + \epsilon^{abc} \omega^b q^c + \left(\frac{dR^a}{dt}\right)_f$

where q^a = components of $\vec{r}\,'$

$\dot{} = \left(\frac{d}{dt}\right)_r$ [with respect to rotating frame]

$\vec{v}\,' = \dot{q}^a$

$$T = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

$$= \frac{1}{2} m \left[\vec{v}\,' + (\vec{\omega} \times \vec{r}\,') + \left(\frac{d\vec{R}}{dt}\right)_f \right] \cdot \left[\vec{v}\,' + (\vec{\omega} \times \vec{r}\,') + \left(\frac{d\vec{R}}{dt}\right)_f \right]$$

$$= \frac{1}{2} m \Big[\vec{v}\,' \cdot \vec{v}\,' + \vec{v}\,' \cdot (\vec{\omega} \times \vec{r}\,') + \vec{v}\,' \cdot \left(\frac{d\vec{R}}{dt}\right)_f$$

$$+ (\vec{\omega} \times \vec{r}\,') \cdot \vec{v}\,' + (\vec{\omega} \times \vec{r}\,') \cdot (\vec{\omega} \times \vec{r}\,') + (\vec{\omega} \times \vec{r}\,') \cdot \left(\frac{d\vec{R}}{dt}\right)_f$$

$$+ \left(\frac{d\vec{R}}{dt}\right) \cdot \vec{v}\,' + \left(\frac{d\vec{R}}{dt}\right)_f \cdot (\vec{\omega} \times \vec{r}\,') + \left(\frac{d\vec{R}}{dt}\right)_f \cdot \left(\frac{d\vec{R}}{dt}\right)_f \Big]$$

$$= \frac{1}{2} m \Big[\vec{v}\,' \cdot \vec{v}\,' + (\vec{\omega} \times \vec{r}\,') \cdot (\vec{\omega} \times \vec{r}\,') + \left(\frac{d\vec{R}}{dt}\right)_f \cdot \left(\frac{d\vec{R}}{dt}\right)_f$$

$$+ 2 \vec{v}\,' \cdot (\vec{\omega} \times \vec{r}\,') + 2 \vec{v}\,' \cdot \left(\frac{d\vec{R}}{dt}\right)_f + 2 \left(\frac{d\vec{R}}{dt}\right)_f \cdot (\vec{\omega} \times \vec{r}\,') \Big]$$

$\cancel{T = \frac{1}{2} m_{ab} v^a v^b}$

$\cancel{= \frac{1}{2} m \delta_{ab} \left(\dot{q}^a \dot{q}^b + (\vec{\omega} \times \vec{r})^a \dot{q}^b + \dot{R} \dot{q} \right)}$

$$T = \frac{1}{2} m \Big[\vec{v}\,' \cdot \vec{v}\,' + \underbrace{(\vec{\omega} \times \vec{r}\,') \cdot (\vec{\omega} \times \vec{r}\,')}_{\rightarrow \text{ centrifugal}} \boxed{+ \left(\frac{d\vec{R}}{dt}\right)_f \cdot \left(\frac{d\vec{R}}{dt}\right)_f}$$

independent of $\vec{r}\,'$ and $\vec{v}\,'$

$$+ \underbrace{2 \vec{v}\,' \cdot (\vec{\omega} \times \vec{r}\,')}_{\rightarrow \text{ coriolis \& Euler}} + \underbrace{2 \left(\frac{d\vec{R}}{dt}\right) \cdot (\vec{v}\,' + \vec{\omega} \times \vec{r}\,')}_{\rightarrow \text{ translational force}} \Big]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^a}\right) - \frac{\partial T}{\partial q^a} = G_a$$

$$T = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

$$\frac{\partial T}{\partial q^a} = m \vec{v} \cdot \left(\frac{\partial \vec{v}}{\partial q^a}\right) \; ; \; \frac{\partial T}{\partial \dot{q}^a} = m \vec{v} \cdot \left(\frac{\partial \vec{v}}{\partial \dot{q}^a}\right)$$

<u>Now:</u> $\boxed{\frac{\partial \vec{v}}{\partial q^a}}$; $\frac{\partial v^b}{\partial q^a} = \frac{\partial}{\partial q^a} \left[\dot{q}^b + \epsilon^{bcd} \omega^c q^d + \left(\frac{dR^b}{dt}\right)_f \right]$

$$= \epsilon^{bca} \omega^c$$

thus,
$$\frac{\partial T}{\partial q^a} = m\, v_b\, \epsilon^{bca} \omega_c$$
$$= -m\, \epsilon^{acb} \omega_c v_b$$
$$= -m (\vec{\omega} \times \vec{v})^a$$
$$= -m \left(\vec{\omega} \times \left[\vec{v}\,' + (\vec{\omega} \times \vec{r}\,') + \left(\frac{d\vec{R}}{dt}\right)_f \right]\right)^a$$

$$\frac{\partial T}{\partial \vec{r}\,'} = \frac{\partial T}{\partial \vec{q}} = -m (\vec{\omega} \times \vec{v}\,') - m\, \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') - m\, \vec{\omega} \times \left(\frac{d\vec{R}}{dt}\right)_f$$

$\boxed{\dfrac{\partial \vec{v}}{\partial \dot{q}^a}}$: $\dfrac{\partial v^b}{\partial \dot{q}^a} = \dfrac{\partial}{\partial \dot{q}^a}\left[\dot{q}^b + \epsilon^{bcd}\omega^c q^d + \left(\dfrac{dR^b}{dt}\right)_f \right]$

$= \delta_a^b$

Thus, $\dfrac{\partial T}{\partial \dot{q}^a} = m v_b \delta_a^b = m v_a = m\left[\vec{v}' + (\vec{\omega}\times\vec{r}') + \left(\dfrac{d\vec{R}}{dt}\right)_f \right]$,

$$\rightarrow \boxed{\frac{\partial T}{\partial \dot{\vec{q}}} = m\left[\vec{v}' + (\vec{\omega}\times\vec{r}') + \left(\frac{d\vec{R}}{dt}\right)_f\right]} = \frac{\partial T}{\partial \vec{v}'}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\vec{q}}}\right) = m\left[\vec{a}' + \dot{\vec{\omega}}\times\vec{r}' + \vec{\omega}\times\vec{v}' + \frac{d}{dt}\bigg|_R \left(\frac{d\vec{R}}{dt}\right)_f\right]$$

$\uparrow$ wrt non-inertial fram

$$\frac{d}{dt}\bigg|_R \left(\frac{d\vec{R}}{dt}\right)_f = \frac{d}{dt}\bigg|_f \left(\frac{d\vec{R}}{dt}\right)_f - \vec{\omega}\times\left(\frac{d\vec{R}}{dt}\right)_f$$

$$= \left(\frac{d^2R}{dt}\right)_f - \vec{\omega}\times\left(\frac{d\vec{R}}{dt}\right)_f$$

$$= m\left[\vec{a}' + (\dot{\vec{\omega}}\times\vec{r}') + (\vec{\omega}\times\vec{v}') - \vec{\omega}\times\left(\frac{d\vec{R}}{dt}\right)_f + \left(\frac{d^2R}{dt}\right)_f\right]$$

$$\frac{d}{dt}\bigg|_{\text{rotatn}} \left(\frac{\partial T}{\partial \vec{v}'}\right) - \frac{\partial T}{\partial \vec{r}'} = \vec{G}$$

Thus,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\vec{q}}}\right) - \frac{\partial T}{\partial \vec{q}}$$

cancel

$$= m \Big\{ \vec{a}\,' + (\dot{\vec{\omega}} \times \vec{r}\,') + \underbrace{(\vec{\omega} \times \vec{v}\,')}_{\text{combine}} - \cancel{\vec{\omega} \times \left(\frac{d\vec{R}}{dt}\right)_f} + \left(\frac{d^2\vec{R}}{dt^2}\right)_f$$

$$+ (\vec{\omega} \times \vec{v}\,') + \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') + \cancel{\vec{\omega} \times \left(\frac{d\vec{R}}{dt}\right)_f} \Big\}$$

$$= m \Big\{ \vec{a}\,' + (\dot{\vec{\omega}} + \vec{r}\,') + 2(\vec{\omega} \times \vec{v}\,') + \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') + \left(\frac{d^2\vec{R}}{dt^2}\right)_f \Big\}$$

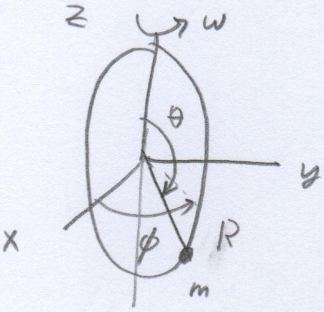
---

so $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\vec{q}}}\right) - \frac{\partial T}{\partial \vec{q}} = \vec{G}$ iff

$$\boxed{m\vec{a}\,' = \vec{G} - \overbrace{m(\dot{\vec{\omega}} \times \vec{r}\,')}^{\text{Euler}} - \overbrace{2m(\vec{\omega} \times \vec{v}\,')}^{\text{Coriolis}} - \underbrace{m\,\vec{\omega} \times (\vec{\omega} \times \vec{r}\,')}_{\text{centrifugal}} - \underbrace{m\left(\frac{d^2\vec{R}}{dt^2}\right)_f}_{\text{translational}}}$$

2.8

Problem: Redo rotating hoop using Lagrange's equations of the 2nd kind



$\dot{\phi} = w = \text{const} \rightarrow \phi = wt + \phi_0$

Only one generalized coord: θ

$T = \frac{1}{2} m R^2 (\dot{\theta}^2 + \sin^2\theta \, w^2)$

$U = mgz = mgR\cos\theta$

$$L = T - U$$
$$= \frac{1}{2} m R^2 (\dot{\theta}^2 + \sin^2\theta \, w^2) - mgR\cos\theta$$

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$$
$$= mR^2 \sin\theta\cos\theta \, w^2 + mgR\sin\theta - \frac{d}{dt}(mR^2\dot{\theta})$$
$$= mR^2 \sin\theta\cos\theta \, w^2 + mgR\sin\theta - mR^2\ddot{\theta}$$

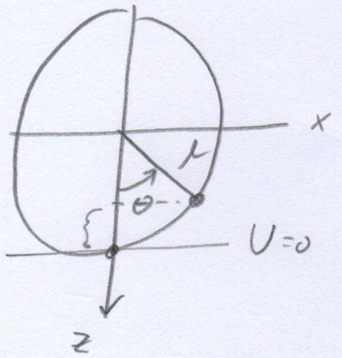
$$\rightarrow \quad mR^2\ddot{\theta} = mR^2\sin\theta\cos\theta \, w^2 + mgR\sin\theta$$

$\div$ by mR^2

$$\boxed{\begin{aligned} \ddot{\theta} &= \sin\theta\cos\theta \, w^2 + \frac{g}{R}\sin\theta \\ &= \sin\theta\left(w^2\cos\theta + \frac{g}{R}\right) \end{aligned}}$$

2.9

Problem: Redo planar pendulum problem using Lagrange's equation of 2nd kind



generalized coord θ

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = mgl(1-\cos\theta) \leftarrow \text{when } \theta = 0, \; U = 0$$

when $\theta = \pi/2$
$U = mgl$

$$L = T - U = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)$$

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$$

$$= -mgl\sin\theta - \frac{d}{dt}\left(ml^2\dot{\theta}\right)$$

$$= -mgl\sin\theta - ml^2\ddot{\theta}$$

so $ml^2\ddot{\theta} = -mgl\sin\theta$

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta} \quad \text{(equation of motion)}$$

Energy conserved since L indep. of t: $\frac{\partial L}{\partial t} = 0$

$$E = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L$$

$$= \dot{\theta}\, ml^2\dot{\theta} - \frac{1}{2} ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

$$\boxed{= \frac{1}{2} ml^2\dot{\theta}^2 + mgl(1-\cos\theta)}$$

At bottom $(\theta = 0)$: $E = \frac{1}{2} ml^2 \dot{\theta}_0^2$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

Multiply by $\dot{\theta}$:

$$\dot{\theta}\ddot{\theta} = -\frac{g}{l}\sin\theta\,\dot{\theta}$$

$$\frac{1}{2}\frac{d}{dt}\left(\dot{\theta}^2\right) = +\frac{g}{l}\frac{d}{dt}(\cos\theta)$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta\right) = 0$$

$$\rightarrow \quad \frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta = \text{const}$$

$$\underbrace{\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = \text{const}}$$

differs from E by the constant term mgl.

V_0 = initial velocity
$= l\dot{\theta}_0$

Thus,
$$E = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

$$\frac{1}{2}m\underbrace{l^2\dot{\theta}_0^2}_{V_0^2} = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

$$\boxed{\frac{1}{2}mV_0^2 = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)}$$

$$\dot{\theta} = 0 \quad \rightarrow \quad \frac{1}{2}mV_0^2 = mgl(1-\cos\theta_{max})$$

$$\frac{V_0^2}{2gl} = 1-\cos\theta_{max} \quad \rightarrow \quad \cos\theta_{max} = 1 - \frac{V_0^2}{2gl}$$

$$\boxed{\theta_{max} = a\cos\left[1 - \frac{V_0^2}{2gl}\right]}$$

At top of circle, $\dot{\theta}$ must be $\geq \dot{\theta}_{min}$ value needed for

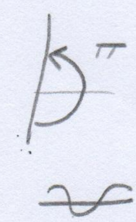
$$\underbrace{\cancel{m}\,\dot{\theta}_{min}^{2}\, l}_{\text{centripetal force}} = \underbrace{\cancel{m} g}_{\text{gravity}}$$

$$\left[F = m\omega^2 r = \frac{mv^2}{r} \right]$$

thus, $\boxed{\dot{\theta}_{min} = \sqrt{\frac{g}{l}}}$

substitute for $\dot{\theta}$ and solve for V_0

$$\frac{1}{2} m V_0^2 = \frac{1}{2} m l^2 \dot{\theta}_{min}^2 + mgl(1-\underset{\theta = \pi}{\cos\theta})$$



$$= \frac{1}{2} m l^2 \frac{g}{l} + 2mgl$$

$$= \frac{1}{2} mgl + 2mgl$$

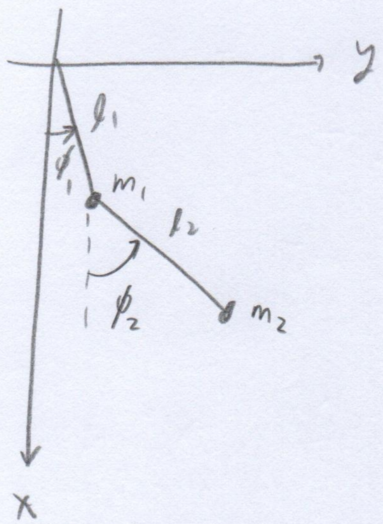
$$= \frac{5}{2} mgl$$

$\rightarrow \quad v_0^2 = 5gl$

$$\boxed{V_0 = \sqrt{5gl}}$$

2,10

Problem: Redo double pendulum problem using Lagrange's equations of 2nd kind



$$x_1 = l_1 \cos\phi_1$$
$$y_1 = l_1 \sin\phi_1$$
$$x_2 = x_1 + l_2 \cos\phi_2$$
$$= l_1 \cos\phi_1 + l_2 \cos\phi_2$$
$$y_2 = y_1 + l_2 \sin\phi_2$$
$$= l_1 \sin\phi_1 + l_2 \sin\phi_2$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_1 [l_1^2 \sin^2\phi_1 \, \dot{\phi}_1^2 + l_1^2 \cos^2\phi_1 \, \dot{\phi}_1^2]$$

$$+ \frac{1}{2} m_2 [(-l_1 \sin\phi_1 \dot{\phi}_1 - l_2 \sin\phi_2 \dot{\phi}_2)^2 + (l_1 \cos\phi_1 \dot{\phi}_1 + l_2 \cos\phi_2 \dot{\phi}_2)^2]$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 [l_1^2 \sin^2\phi_1 \dot{\phi}_1^2 + \boxed{l_2^2 \sin^2\phi_2 \dot{\phi}_2^2} + 2 l_1 l_2 \sin\phi_1 \sin\phi_2 \dot{\phi}_1 \dot{\phi}_2 + l_1^2 \cos^2\phi_1 \dot{\phi}_1^2 + \boxed{l_2^2 \cos^2\phi_2 \dot{\phi}_2^2} + 2 l_1 l_2 \cos\phi_1 \cos\phi_2 \dot{\phi}_1 \dot{\phi}_2]$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 [l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2]$$

$$U = -m_1 g x_1 - m_2 g x_2$$

$$= -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$

Thus,

$$L = T - U$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 \right]$$

$$+ m_1 g l_1 \cos\phi_1 + m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$

EOMs:

ϕ_1:

$$0 = \frac{\partial L}{\partial \phi_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_1}\right)$$

$$= -m_2 l_1 l_2 \sin(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 - \sin\phi_1 (m_1 g l_1 + m_2 g l_1)$$

$$- \frac{d}{dt}\left[m_1 l_1^2 \dot{\phi}_1 + m_2 l_1^2 \dot{\phi}_1 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_2 \right]$$

$$= \cancel{-m_2 l_1 l_2 \sin(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2} - \sin\phi_1 (m_1 g l_1 + m_2 g l_2)$$

$$- m_1 l_1^2 \ddot{\phi}_1 - m_2 l_1^2 \ddot{\phi}_1 + m_2 l_1 l_2 \sin(\phi_1 - \phi_2)(\cancel{\dot{\phi}_1} - \dot{\phi}_2) \dot{\phi}_2$$

$$- m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \ddot{\phi}_2$$

$$\boxed{0 = -\sin\phi_1 (m_1 + m_2) g l_1 - (m_1 + m_2) l_1^2 \ddot{\phi}_1 - m_2 l_1 l_2 \left[\sin(\phi_1 - \phi_2) \dot{\phi}_2^2 + \cos(\phi_1 - \phi_2) \ddot{\phi}_2 \right]}$$

$\div \ (m_1+m_2)\, l_1$:

$$0 = -g\sin\phi_1 - l_1\ddot{\phi}_1 - \left(\frac{m_2}{m_1+m_2}\right) l_2 \left[\sin(\phi_1-\phi_2)\,\dot{\phi}_2^2 + \cos(\phi_1-\phi_2)\,\ddot{\phi}_2\right]$$

$$\rightarrow \boxed{g\sin\phi_1 = -l_1\ddot{\phi}_1 - \frac{m_2 l_2}{m_1+m_2}\left[\ddot{\phi}_2\cos(\phi_1-\phi_2) + \dot{\phi}_2^2\sin(\phi_1-\phi_2)\right]}$$

(which agrees with prob 1.4, part b)

$\underline{\phi_2}$: $\quad 0 = \dfrac{\partial L}{\partial \phi_2} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\phi}_2}\right)$

$$= -m_2 l_1 l_2 \sin(\phi_1 - \phi_2)(-1)\,\dot{\phi}_1 \dot{\phi}_2 - m_2 g l_2 \sin\phi_2$$

$$- \frac{d}{dt}\left(m_2 l_2^2 \dot{\phi}_2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2)\dot{\phi}_1 \right)$$

$$= \cancel{m_2 l_1 l_2 \sin(\phi_1 - \phi_2)\dot{\phi}_1 \dot{\phi}_2} - m_2 g l \sin\phi_2$$

$$- m_2 l_2^2 \ddot{\phi}_2 + m_2 l_1 l_2 \sin(\phi_1 - \phi_2)(\dot{\phi}_1 - \cancel{\dot{\phi}_2})\dot{\phi}_1$$

$$- m_2 l_1 l_2 \cos(\phi_1 - \phi_2)\ddot{\phi}_1$$

$$= -m_2 g l_2 \sin\phi_2 - m_2 l_2^2 \ddot{\phi}_2 + m_2 l_1 l_2 \sin(\phi_1 - \phi_2)\dot{\phi}_1^2$$

$$- m_2 l_1 l_2 \cos(\phi_1 - \phi_2)\ddot{\phi}_1$$

$$\boxed{0 = -m_2 g l_2 \sin\phi_2 - m_2 l_2^2 \ddot{\phi}_2 + m_2 l_1 l_2 \sin(\phi_1 - \phi_2)\dot{\phi}_1^2 - m_2 l_1 l_2 \cos(\phi_1 - \phi_2)\ddot{\phi}_1}$$

$\downarrow$

$\underline{\div \text{ by } m_2 l_2}$

$$0 = -g \sin\phi_2 - l_2 \ddot{\phi}_2 + l_1 \sin(\phi_1 - \phi_2)\dot{\phi}_1^2 - l_1 \cos(\phi_1 - \phi_2)\ddot{\phi}_1$$

$$\rightarrow \boxed{g \sin\phi_2 = -l_2 \ddot{\phi}_2 + l_1 \dot{\phi}_1^2 \sin(\phi_1 - \phi_2) - l_1 \ddot{\phi}_1 \cos(\phi_1 - \phi_2)}$$

(agrees with Prob 1.4 part (b))