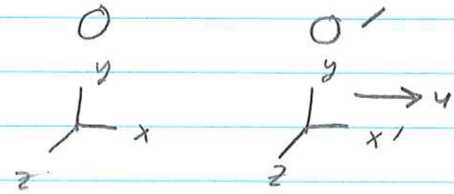
Exercise 11.1

a) $t' = \gamma(t - ux/c^2)$
$x' = \gamma(x - ut)$



Let v = velocity w.r.t O
v' = velocity w.r.t O'

$$\gamma = \frac{1}{\sqrt{1-(\frac{u}{c})^2}}$$

$$v = \frac{dx}{dt}$$

$$v' = \frac{dx'}{dt'}$$

$$dx' = \gamma(dx - u\,dt)$$
$$dt' = \gamma(dt - \frac{u}{c^2}dx)$$

Thus, $$v' = \frac{dx'}{dt'} = \frac{\cancel{\gamma}(dx - u\,dt)}{\cancel{\gamma}(dt - \frac{u}{c^2}dx)}$$

$$= \frac{\cancel{dt}\left(\frac{dx}{dt} - u\right)}{\cancel{dt}\left(1 - \frac{u}{c^2}\frac{dx}{dt}\right)}$$

$$= \frac{v-u}{1-\frac{vu}{c^2}}$$

so $$\boxed{v' = \frac{v-u}{1-\frac{vu}{c^2}}} \iff \boxed{v = \frac{v'+u}{1+\frac{uv'}{c^2}}}$$

b) $u = 100\text{ mph}$, $v' = 100\text{ mph}$

$$v = \frac{200\text{ mph}}{1+\frac{(100\text{ mph})^2}{c^2}} \approx 200\left(1-\left(\frac{100\text{ mph}}{c}\right)^2\right)$$

$$V - 200\,\text{mph} \approx -200\,\text{mph}\left(\frac{100\,\text{mph}}{c}\right)^2$$

$$= -200\,\text{mph}\left(\frac{100\,\text{mph}}{6.7\times10^8\,\text{mph}}\right)^2$$

$$\approx -200\,\text{mph}\left(\frac{10^{-6}}{6.7}\right)^2$$

$$\approx \boxed{-4.5\times10^{-12}\,\text{mph}}$$

$$\frac{\Delta V}{V} = \frac{V - 200\,\text{mph}}{200\,\text{mph}} = -\left(\frac{10^{-6}}{6.7}\right)^2$$

$$= \boxed{2.2\times10^{-14}}$$

so fractional difference is $\sim$ 1 part in 10^{14}

Exer. 11.2

Given: $\frac{u}{c} = \frac{k^2 - 1}{k^2 + 1}$, $k = \sqrt{\frac{1 + u/c}{1 - u/c}}$

$$k + \frac{1}{k} = \sqrt{\frac{1 + u/c}{1 - u/c}} + \sqrt{\frac{1 - u/c}{1 + u/c}}$$

$$= \frac{1 + \cancel{u/c} + 1 - \cancel{u/c}}{1 - (u/c)^2}$$

$$= \frac{2}{1 - (u/c)^2} = \boxed{2\gamma}$$

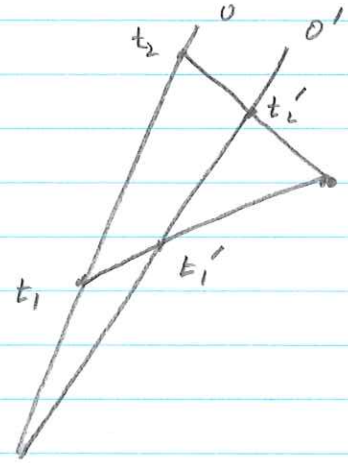
$$k - \frac{1}{k} = \sqrt{\frac{1 + u/c}{1 - u/c}} - \sqrt{\frac{1 - u/c}{1 + u/c}}$$

$$= \frac{\cancel{1} + u/c - (\cancel{1} - u/c)}{1 - (u/c)^2}$$

$$= \frac{2\, u/c}{1 - (u/c)^2}$$

$$= \boxed{2\gamma\left(\frac{u}{c}\right)}$$

Exer 11.3



$P \quad (ct, x), \ (ct', x')$

$t_1 = t - \frac{x}{c} \qquad t_1' = t' - \frac{x'}{c}$

$t_2 = t + \frac{x}{c} \qquad t_2' = t' + \frac{x'}{c}$

$t_1' = k t_1$

$t_2 = k t_2'$

$$t' - \frac{x'}{c} = k\left(t - \frac{x}{c}\right) \qquad (1)$$

$$t + \frac{x}{c} = k\left(t' + \frac{x'}{c}\right) \qquad (2)$$

$\underline{k(1) + (2):}$

$$\cancel{kt'} - k\frac{x'}{c} = k^2 t - k^2\frac{x}{c}$$

$$t + \frac{x}{c} = \cancel{kt'} + k\frac{x'}{c}$$

$$t + \frac{x}{c} - k\frac{x'}{c} = k^2 t - k^2\frac{x}{c} + k\frac{x'}{c}$$

$$t(1-k^2) = -(1+k^2)\frac{x}{c} + 2k\frac{x'}{c}$$

Recall: $\quad k + \frac{1}{k} = 2\gamma, \qquad k - \frac{1}{k} = 2\gamma\frac{v}{c}$

$$\rightarrow \quad \frac{k^2+1}{2k} = \gamma \qquad \frac{k^2-1}{2k} = \gamma\frac{v}{c}$$

Then,

$$\frac{x'}{c} = t\left(\frac{1-k^2}{2k}\right) + \left(\frac{1+k^2}{2k}\right)\frac{x}{c}$$

$$= -t\gamma\frac{u}{c} + \gamma\frac{x}{c}$$

$$= \gamma\left(\frac{x}{c} - \frac{tu}{c}\right)$$

$$\rightarrow \boxed{x' = \gamma(x-ut)}$$

<u>$k(1)-(2)$:</u>

$$kt' - \cancel{\frac{kx'}{c}} - t - \frac{x}{c} = k^2 t - k^2\frac{x}{c} - kt' - \cancel{\frac{kx'}{c}}$$

$$2kt' = (1+k^2)t + (1-k^2)\frac{x}{c}$$

$$t' = \left(\frac{1+k^2}{2k}\right)t + \left(\frac{1-k^2}{2k}\right)\frac{x}{c}$$

$$= \gamma t - \gamma\frac{u}{c}\frac{x}{c}$$

$$= \gamma\left(t - \frac{u}{c^2}x\right)$$

so $\boxed{t' = \gamma(t - ux/c^2)}$

Exercise (11.4)

$$t' = \gamma(t - ux/c^2)$$
$$x' = \gamma(x - ut)$$

Let $T' = T_0$ = proper time between two events with $x_1' = x_2'$.

equal
$$x_1' = \gamma(x_1 - ut_1)$$
$$x_2' = \gamma(x_2 - ut_2)$$

Thus
$$\cancel{\gamma}(x_1 - ut_1) = \cancel{\gamma}(x_2 - ut_2)$$
$$u\,\underbrace{(t_2 - t_1)}_{T} = x_2 - x_1$$

$$\rightarrow \quad T = \frac{x_2 - x_1}{u}$$

<u>Also:</u>
$$t_1' = \gamma(t_1 - ux_1/c^2)$$
$$t_2' = \gamma(t_2 - ux_2/c^2)$$

$$\begin{aligned}
T' &\equiv t_2' - t_1' \\
&= \gamma(t_2 - t_1) - \frac{\gamma u}{c^2}(x_2 - x_1) \\
&= \gamma T - \gamma\left(\frac{u}{c}\right)^2 T \\
&= \gamma T\left(1 - \left(\frac{u}{c}\right)^2\right) \\
&= \gamma T \gamma^{-2} \\
&= \gamma^{-1} T
\end{aligned}$$

Thus, $T = \gamma T' = \gamma T_0$

Exercise 11.5

Moving clock measures proper time T_0 1yr

Observer's clock measures $T = 1yr$

$u/c = 0.7$

Thus,

$$T = \gamma T_0$$

$$T_0 = \gamma^{-1} T$$

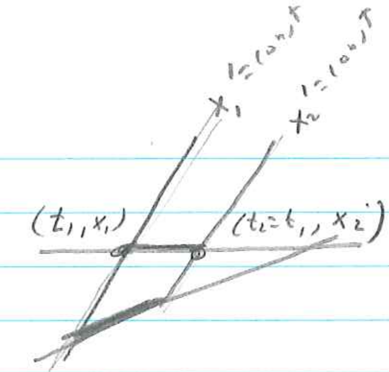
$$= \sqrt{1 - \left(\frac{u}{c}\right)^2} \quad T$$

$$= \sqrt{1 - (0.7)^2} \quad 1yr$$

$$= \sqrt{1 - 0.49} \quad 1yr$$

$$= \boxed{0.71 yr}$$

Exer (11.6)



$$t' = \gamma\left(t - ux/c^2\right)$$
$$x' = \gamma\left(x - ut\right)$$

$L' = L_0$ = proper length

$L \equiv x_2 - x_1$ measured at $t_1 = t_2$

equal $\begin{cases} t_1 = \gamma\left(t_1' + u x_1'/c^2\right) \\ t_2 = \gamma\left(t_2' + u x_2'/c^2\right) \end{cases}$

Thus, $\cancel{\gamma}\left(t_1' + u x_1'/c^2\right) = \cancel{\gamma}\left(t_2' + u x_2'/c^2\right)$

$$\frac{u}{c^2}\left(x_1' - x_2'\right) = t_2' - t_1'$$

$$\boxed{-\frac{uL'}{c^2} = t_2' - t_1'}$$

Also, $x_1 = \gamma\left(x_1' + u t_1'\right)$
$x_2 = \gamma\left(x_2' + u t_2'\right)$

$$\begin{aligned} L &\equiv x_2 - x_1 \\ &= \gamma\left[x_2' + u t_2' - x_1' - u t_1'\right] \\ &= \gamma\left[L' + u(t_2' - t_1')\right] \\ &= \gamma\left[L' - \frac{u^2}{c^2}L'\right] \\ &= \gamma\left(1 - \frac{u^2}{c^2}\right)L' \\ &= \gamma^{-1}L' \end{aligned}$$

Thus, $L = \gamma^{-1}L' = \gamma^{-1}L_0$

Exer 11.7

$$ct' = \gamma(ct - \vec{u}\cdot\vec{r}/c)$$

$$\vec{r}\,' = \gamma(\vec{r}_{\parallel} - \vec{u}t) + \vec{r}_{\perp}$$

$$\vec{r}_{\parallel} = \hat{u}(\hat{u}\cdot\vec{r}), \qquad \vec{r}_{\perp} = \vec{r} - \vec{r}_{\parallel}, \qquad \hat{u} = \vec{u}/u$$

<u>Consider</u>: $\vec{u} = u\hat{x}$ so $\hat{u} = \hat{x}$

Then $\vec{r}_{\parallel} = \hat{x}(\hat{x}\cdot\vec{r}) = x\hat{x}$

$\vec{r}_{\perp} = \vec{r} - x\hat{x}$

$= y\hat{y} + z\hat{z}$

Thus,

$$\boxed{ct' = \gamma(ct - x/c)}$$

$$\boxed{\begin{aligned} x' &= \gamma(x - ut) \\ y' &= y \\ z' &= z \end{aligned}}$$

Exer 11.8

$$t' = \gamma(t - ux/c^2)$$
$$x' = \gamma(x - ut)$$
$$y' = y$$
$$z' = z$$

$$v^{x'} = \frac{dx'}{dt'}$$

m

$$= \frac{\cancel{\gamma}(dx - u\,dt)}{\cancel{\gamma}(dt - u\,dx/c^2)}$$

$$= \frac{\cancel{dt}\left(\frac{dx}{dt} - u\right)}{\cancel{dt}\left(1 - \frac{u}{c^2}\frac{dx}{dt}\right)}$$

$$= \frac{v^x - u}{1 - \frac{uv^x}{c^2}}$$

$$v^{y'} = \frac{dy'}{dt'}$$

$$= \frac{dy}{\gamma(dt - u\,dx/c^2)}$$

$$= \frac{dy}{\gamma\,dt\left(1 - \frac{uv^x}{c^2}\right)}$$

$$= \frac{v^y}{\gamma\left(1 - \frac{uv^x}{c^2}\right)}$$

$$v^{z'} = \frac{dz'}{dt}$$

$$= \frac{dz}{\gamma\left(dt - u\,dx/c^2\right)}$$

$$= \frac{dz}{\gamma\,dt\left(1 - \frac{u v^x}{c^2}\right)}$$

$$= \frac{v^z}{\gamma\left(1 - \frac{u v^x}{c^2}\right)}$$

Exer (11.9)

Show $\sum_{\alpha',\beta'} \eta_{\alpha'\beta'} \Lambda^{\alpha'}{}_{\alpha} \Lambda^{\beta'}{}_{\beta} = \eta_{\alpha\beta}$

RHS = $\mathrm{diag}(-1,1,1,1)$

LHS = $\sum_{\alpha',\beta'} \Lambda^{\alpha'}{}_{\alpha} \eta_{\alpha'\beta'} \Lambda^{\beta'}{}_{\beta}$

$= \sum_{\alpha',\beta'} (\Lambda^T)_{\alpha}{}^{\alpha'} \eta_{\alpha'\beta'} \Lambda^{\beta'}{}_{\beta}$

$= \boxed{\Lambda^T}\ \boxed{\eta}\ \boxed{\Lambda}$

$$\eta\Lambda = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma u_x/c & -\gamma u_y/c & -\gamma u_z/c \\ -\frac{\gamma u_x}{c} & 1+(\gamma-1)\frac{u_x^2}{u^2} & (\gamma-1)\frac{u_x u_y}{u^2} & (\gamma-1)\frac{u_x u_z}{u^2} \\ -\frac{\gamma u_y}{c} & (\gamma-1)\frac{u_x u_y}{u^2} & 1+(\gamma-1)\frac{u_y^2}{u^2} & (\gamma-1)\frac{u_y u_z}{u^2} \\ -\frac{\gamma u_z}{c} & (\gamma-1)\frac{u_x u_z}{u^2} & (\gamma-1)\frac{u_y u_z}{u^2} & 1+(\gamma-1)\frac{u_z^2}{u^2} \end{bmatrix}$$

$$= \begin{bmatrix} -\gamma & \gamma u_x/c & \gamma u_y/c & \gamma u_z/c \\ -\frac{\gamma u_x}{c} & 1+(\gamma-1)\frac{u_x^2}{u^2} & (\gamma-1)\frac{u_x u_y}{u^2} & (\gamma-1)\frac{u_x u_z}{u^2} \\ -\frac{\gamma u_y}{c} & (\gamma-1)\frac{u_x u_y}{u^2} & 1+(\gamma-1)\frac{u_y^2}{u^2} & (\gamma-1)\frac{u_y u_z}{u^2} \\ -\frac{\gamma u_z}{c} & (\gamma-1)\frac{u_x u_z}{u^2} & (\gamma-1)\frac{u_y u_z}{u^2} & 1+(\gamma-1)\frac{u_z^2}{u^2} \end{bmatrix}$$

$$\Lambda^T \overset{=\Lambda}{}$$

$$\Lambda^T \eta \Lambda = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+\frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1+\frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1+\frac{(\gamma-1)\beta_z^2}{\beta^2} \end{bmatrix} \begin{bmatrix} -\gamma & \gamma\beta_x & \gamma\beta_y & \gamma\beta_z \\ -\gamma\beta_x & 1+\frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1+\frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1+\frac{(\gamma-1)\beta_z^2}{\beta^2} \end{bmatrix}$$

$\underline{0\text{-}0}$:

$$-\gamma^2 + \gamma^2\beta_x^2 + \gamma^2\beta_y^2 + \gamma^2\beta_z^2 = -\gamma^2\left(1-\beta_x^2-\beta_y^2-\beta_z^2\right)$$
$$= -\gamma^2(1-\beta^2)$$
$$= -\gamma^2\gamma^{-2}$$
$$= \boxed{-1}$$

$\underline{0\text{-}1}$:

$$\gamma^2\beta_x - \gamma\beta_x\left(1+\frac{(\gamma-1)\beta_x^2}{\beta^2}\right) - \gamma\beta_y\frac{(\gamma-1)\beta_x\beta_y}{\beta^2} - \gamma\beta_z\frac{(\gamma-1)\beta_x\beta_z}{\beta^2}$$

$$= \gamma^2\beta_x - \gamma\beta_x - \gamma(\gamma-1)\frac{\beta_x^3}{\beta^2} - \gamma(\gamma-1)\frac{\beta_x\beta_y^2}{\beta^2} - \gamma(\gamma-1)\frac{\beta_x\beta_z^2}{\beta^2}$$

$$= \gamma(\gamma-1)\beta_x - \gamma(\gamma-1)\frac{\beta_x}{\cancel{\beta^2}}\left[\underbrace{\beta_x^2+\beta_y^2+\beta_z^2}_{\cancel{\beta^2}}\right]$$

$$= \gamma(\gamma-1)\beta_x - \gamma(\gamma-1)\beta_x$$

$$= 0$$

similarly $\underline{0\text{-}2} = 0$, $\underline{0\text{-}3} = 0$

$\underline{1-0}$: $\gamma^2\beta_x - \gamma\beta_x\left(1+\frac{(\gamma-1)\beta_x^2}{\beta^2}\right) - \gamma\frac{(\gamma-1)\beta_x\beta_y^2}{\beta^2} - \gamma\frac{(\gamma-1)\beta_x\beta_z^2}{\beta^2}$

$= \gamma(\gamma-1)\beta_x - \frac{\gamma(\gamma-1)}{\cancel{\beta^2}}\beta_x\left(\cancel{\beta_x^2+\beta_y^2+\beta_z^2}\right)$

$= 0$

$\underline{1-1}$: $-\gamma^2\beta_y^2 + \left(1+\frac{(\gamma-1)\beta_x^2}{\beta^2}\right)^2 + (\gamma-1)^2\frac{\beta_x^2\beta_y^2}{\beta^4} + (\gamma-1)^2\frac{\beta_x^2\beta_z^2}{\beta^4}$

$= -\gamma^2\beta_x^2 + 1 + \boxed{\frac{(\gamma-1)^2\beta_x^2\beta_x^2}{\beta^4}} + \frac{2(\gamma-1)\beta_x^2}{\beta^2} \boxed{+ (\gamma-1)^2\frac{\beta_x^2\beta_y^2}{\beta^4}}$
$\boxed{+ (\gamma-1)^2\frac{\beta_x^2\beta_z^2}{\beta^4}}$

$= -\gamma^2\beta_y^2 + 1 + \frac{2(\gamma-1)\beta_x^2}{\beta^2} + \frac{(\gamma-1)^2\beta_x^2}{\beta^4}(\underbrace{\beta_x^2+\beta_y^2+\beta_z^2}_{\beta^2})$

$= -\gamma^2\beta_x^2 + 1 + 2(\gamma-1)\frac{\beta_x^2}{\beta^2} + (\gamma-1)^2\frac{\beta_x^2}{\beta^2}$

$= -\gamma^2\beta_x^2 + 1 + \frac{\beta_x^2}{\beta^2}(\gamma-1)[\underbrace{2+\gamma-1}_{\gamma+1}]$

$= -\gamma^2\beta_x^2 + 1 + \frac{\beta_x^2}{\beta^2}(\underbrace{\gamma^2-1}_{\gamma^2\beta^2})$

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$\gamma^2 = \frac{1}{1-\beta^2}$

$\gamma^2 - \gamma^2\beta^2 = 1$

$\gamma^2 - 1 = \gamma^2\beta^2$

$= \cancel{-\gamma^2\beta_x^2} + 1 + \cancel{\gamma^2\beta_x^2}$

$= \boxed{1}$

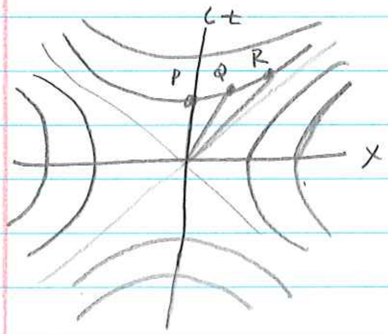
$\underline{\text{Similarly}}$: $\underline{2-2} = 1$, $\underline{3-3} = 1$, etc.

Thus,

$$\Lambda^T \eta \Lambda = \boxed{\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}$$

$$= \eta$$

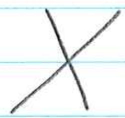
Exer 11.10



$$ds^2 = -c^2dt^2 + dx^2$$

$$const = -c^2dt^2 + dx^2$$

$$\boxed{-c^2t^2 + x^2 = const}$$

const = 0 $\rightarrow$ 

const < 0 

const > 0 

Exer 11.11

$$a^{\alpha'} = \sum_{\alpha} \Lambda^{\alpha'}{}_{\alpha}\, a^{\alpha}$$

For a Lorentz boost in x-direction:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta_x & 0 & 0 \\ -\gamma\beta_x & 1\cdot\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\beta_x = \frac{u_x}{c} = \frac{u}{c}$

$u_y = 0,\ u_z = 0$

Then,

$$a' = \Lambda a$$

$$= \begin{bmatrix} \gamma & -\gamma\beta_x & 0 & 0 \\ -\gamma\beta_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a^0 \\ a^x \\ a^y \\ a^z \end{bmatrix}$$

$$= \begin{bmatrix} \gamma(a^0 - a^x\beta_x) \\ \gamma(a^x - a^0\beta_x) \\ a^y \\ a^z \end{bmatrix}$$

$$= \begin{bmatrix} \gamma(a^0 - a^x u/c) \\ \gamma(a^x - a^0 u/c) \\ a^y \\ a^z \end{bmatrix} = \begin{bmatrix} a^{t'} \\ a^{x'} \\ a^{y'} \\ a^{z'} \end{bmatrix}$$

Exer 11.12

Prove: $-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$
$= -a^{0'} b^{0'} + a^{1'} b^{1'} + a^{2'} b^{2'} + a^{3'} b^{3'}$

$$a^{\alpha'} = \sum_{\beta} \Lambda^{\alpha'}{}_{\beta}\, a^{\beta} \quad , \quad b^{\alpha'} = \sum_{\beta} \Lambda^{\alpha'}{}_{\beta}\, b^{\beta}$$

~~Th~~ Thus

$$a \cdot b = \sum_{\alpha,\beta} a^{\alpha} b^{\beta} \eta_{\alpha\beta}$$

$$= \sum_{\alpha,\beta} a^{\alpha} b^{\beta} \sum_{\mu',\nu'} \Lambda^{\mu'}{}_{\alpha} \Lambda^{\nu'}{}_{\beta}\, \eta_{\mu'\nu'}$$

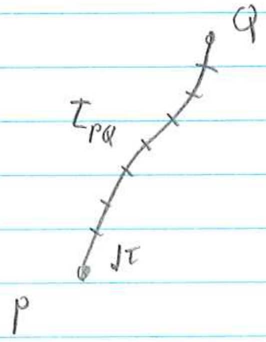
$$= \sum_{\mu',\nu'} \left(\sum_{\alpha} a^{\alpha} \Lambda^{\mu'}{}_{\alpha} \right) \left(\sum_{\beta} b^{\beta} \Lambda^{\nu'}{}_{\beta} \right) \eta_{\mu'\nu'}$$

$$= \sum_{\mu',\nu'} a^{\mu'} b^{\nu'} \eta_{\mu'\nu'}$$

so $\sum_{\alpha,\beta} a^{\alpha} b^{\beta} \underbrace{\eta_{\alpha\beta} = \sum_{\mu',\nu'} a^{\mu'} b^{\nu'} \eta_{\mu'\nu'}}_{\text{equal} = \mathrm{diag}(-1,1,1,1)}$

$$-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 = -a^{0'} b^{0'} + a^{1'} b^{1'} + a^{2'} b^{2'} + a^{3'} b^{3'}$$

Exer 11.13



$$\tau_{pq}[x^\alpha] = \int_p^q d\tau$$

$$= \frac{1}{c}\int_p^q \sqrt{-ds^2}$$

$$= \int_{\sigma_p}^{\sigma_q} \frac{d\sigma}{c}\sqrt{-\sum_{\alpha,\beta} \eta_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}}$$

$$= \int_{\sigma_p}^{\sigma_q} d\sigma\, L(x^\alpha, \dot{x}^\alpha, \sigma)$$

$$0 = \delta\tau_{pq} \iff \text{EL equations}$$

$$0 = \frac{\partial L}{\partial x^\alpha} - \frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{dx^\alpha}{d\sigma}\right)}\right) \qquad \alpha = 0,1,2,3$$

where $$L = \frac{1}{c}\sqrt{-\sum_{\alpha,\beta}\eta_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}} = \frac{d\tau}{d\sigma}$$

$$\frac{\partial L}{\partial x^\alpha} = 0$$

$$\frac{\partial L}{\partial\left(\frac{dx^\alpha}{d\sigma}\right)} = \frac{1}{c}\left(\frac{1}{\cancel{2}}\right)\frac{1}{\sqrt{}}(-\cancel{2})\,\eta_{\alpha\beta}\frac{dx^\beta}{d\sigma}$$

$$= -\frac{1}{c\sqrt{}}\,\eta_{\alpha\beta}\frac{dx^\beta}{d\sigma}$$

$$= -\frac{1}{c}\frac{1}{c}\frac{d\sigma}{d\tau}\,\eta_{\alpha\beta}\frac{dx^\beta}{d\sigma}$$

$$= -\frac{1}{c^2}\,\eta_{\alpha\beta}\frac{dx^\beta}{d\tau}$$

Thus,

$$0 = 0 - \frac{d}{d\sigma}\left(-\frac{1}{c^2} \eta_{\alpha\beta} \frac{dx^\beta}{d\tau} \right)$$

$$= \frac{1}{c^2} \frac{d}{d\sigma}\left(\eta_{\alpha\beta} \frac{dx^\beta}{d\tau} \right)$$

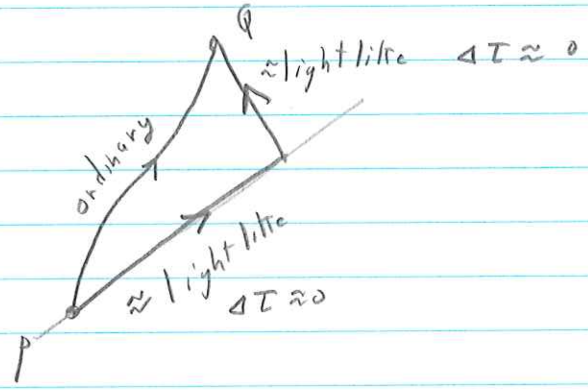
$$\Rightarrow \eta_{\alpha\beta} \frac{dx^\beta}{d\tau} = \text{constant}$$

$$\frac{dx^\beta}{d\tau} = \text{const}$$

$$\boxed{x^\beta(\tau) = x_0 + v_0^\beta \tau} \quad \text{straight line}$$

Extremal curve; this is a maximum because you can always make τ_{PQ} shorter by moving along a path that is closer to a null line.

Exercise 11.14



Exer 11.15

$$u^\alpha \equiv \frac{dx^\alpha}{d\tau} \qquad \text{(definition)}$$

$$u \cdot u = \sum_{\alpha,\beta} \eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$$

$$= \frac{\sum_{\alpha,\beta} \eta_{\alpha\beta} dx^\alpha dx^\beta}{d\tau^2}$$

$$= \frac{ds^2}{d\tau^2}$$

$$= \frac{-c^2 \cancel{d\tau^2}}{\cancel{d\tau^2}} \qquad \text{using } c\,d\tau \equiv \sqrt{-ds^2}$$

$$= \boxed{-c^2}$$

$$u^\alpha = \frac{dx^\alpha}{d\tau}$$

$$= \left(\frac{c\,dt}{d\tau}, \frac{dx^i}{d\tau}\right)$$

$$= \frac{dt}{d\tau}\left(c, \frac{dx^i}{dt}\right)$$

$$= \frac{dt}{d\tau}\left(c, v^i\right)$$

Now: $-c^2 = u \cdot u = \left(\frac{dt}{d\tau}\right)^2 \left(-c^2 + v^2\right)$

$$\rightarrow \left(\frac{dt}{d\tau}\right)^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1-\left(\frac{v}{c}\right)^2} = \gamma^2$$

so $\boxed{\frac{dt}{d\tau} = \gamma}$

Thus, $\boxed{u^\alpha = \gamma(c, v^i)}$

Exer (11.16)

$$a^\alpha = \frac{du^\alpha}{d\tau} \quad , \quad u^\alpha = \frac{dx^\alpha}{d\tau} = \gamma(c, v^i)$$

Thus,

$$a^\alpha = \frac{d}{d\tau}\left[\gamma(c, v^i)\right]$$

$$= \left(\frac{dt}{d\tau}\right)\frac{d}{dt}\left[\gamma(c, v^i)\right]$$

$$= \gamma\left[\frac{d\gamma}{dt}(c, v^i) + \gamma(0, a^i)\right]$$

Now: $\frac{d\gamma}{dt} = \frac{d}{dt}\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$ $\qquad v^2 = \sum_i (v_i)^2$

$$= -\frac{1}{\cancel{2}}\frac{1}{\left(1-\left(\frac{v}{c}\right)^2\right)^{3/2}} \quad -\cancel{2}\left(\frac{1}{c^2}\right)\vec{v}\cdot\vec{a}$$

$$= \gamma^3\frac{\vec{v}\cdot\vec{a}}{c^2}$$

$$\rightarrow \quad a^\alpha = \gamma\left[\gamma^3\frac{\vec{v}\cdot\vec{a}}{c^2}(c, v^i) + \gamma(0, a^i)\right]$$

$$= \gamma^2\left[\gamma^2\frac{\vec{v}\cdot\vec{a}}{c^2}(c, v^i) + (0, a^i)\right]$$

$$= \gamma^2\left(\gamma^2\frac{\vec{v}\cdot\vec{a}}{c}, \quad a^i + \gamma^2\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)v^i\right)$$

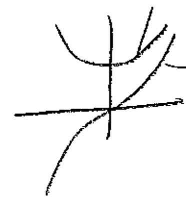
11.17

Exercise const acceleration

$$ct = a^{-1} c^2 \sinh\left(\frac{a\tau}{c}\right)$$

$$x = a^{-1} c^2 \cosh\left(\frac{a\tau}{c}\right)$$

$$\left.\begin{aligned} y &= \text{const} \\ z &= \text{const} \end{aligned}\right\} \text{ignore}$$



$$\cancel{c}\frac{dt}{d\tau} = \cancel{c}\cosh\left(\frac{a\tau}{c}\right) \rightarrow \frac{dt}{d\tau} = \cosh\left(\frac{a\tau}{c}\right) = \gamma$$

$$\frac{dx}{d\tau} = c\sinh\left(\frac{a\tau}{c}\right)$$

$$\frac{dy}{d\tau} = 0, \quad \frac{dz}{d\tau} = 0$$

$$u^\alpha = c \left[\cosh\left(\frac{a\tau}{c}\right), \sinh\left(\frac{a\tau}{c}\right), 0, 0 \right]$$

$$a^\alpha = \frac{du^\alpha}{d\tau}$$

$$= a \left[\sinh\left(\frac{a\tau}{c}\right), \cosh\left(\frac{a\tau}{c}\right), 0, 0 \right]$$

$$|\mathbf{a}|^2 = \eta_{\alpha\beta} a^\alpha a^\beta$$

$$= -(a^0)^2 + (a^1)^2$$

$$= a^2 \left[-\sinh^2\left(\frac{a\tau}{c}\right) + \cosh^2\left(\frac{a\tau}{c}\right) \right]$$

$$= a^2$$

Exer: 11.18

a) $p = m u$

$p \cdot p = m^2 u \cdot u = \boxed{-m^2 c^2}$

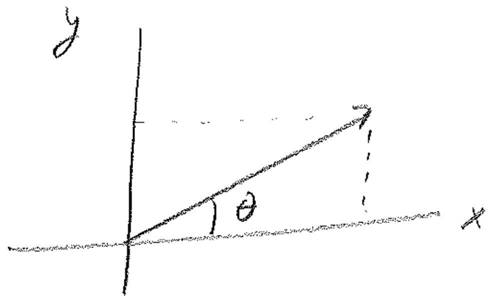
b) $p = \left(\frac{E}{c}, p^i\right)$

$-m^2 c^2 = p \cdot p = -\frac{E^2}{c^2} + |\vec{p}|^2$

so $\frac{E^2}{c^2} = m^2 c^2 + |\vec{p}|^2$

$\boxed{E^2 = m^2 c^4 + |\vec{p}|^2 c^2}$

Exer: 11.19



$$\vec{k} = \frac{2\pi}{\lambda}\left(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right)$$

$$= \frac{\omega}{c}\left(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right)$$

$$c = f\lambda = 2\pi f\left(\frac{\lambda}{2\pi}\right)$$

$$= \omega\left(\frac{\lambda}{2\pi}\right)$$

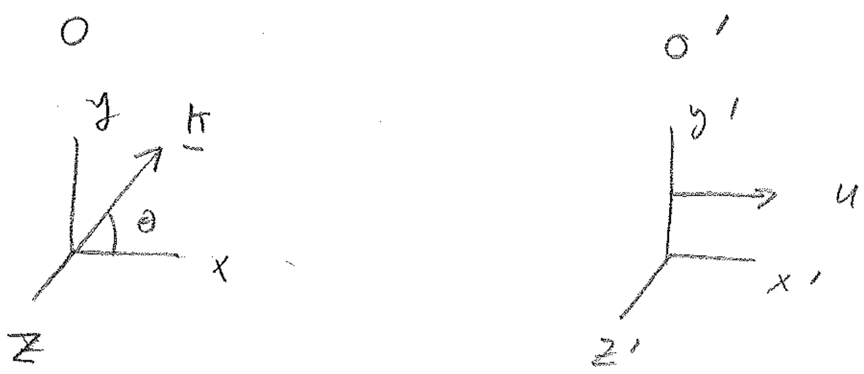
$$\frac{2\pi}{\lambda} = \frac{\omega}{c}$$

$$p^\alpha = \left(\frac{\hbar\omega}{c}, \hbar k^i\right)$$

$$= \boxed{\frac{\hbar\omega}{c}, \frac{\hbar\omega}{c}\cos\theta, \frac{\hbar\omega}{c}\sin\theta, 0}$$

$$= \frac{\hbar\omega}{c}\left(1, \cos\theta, \sin\theta, 0\right)$$

Transverse Doppler effect Exercise (11.20)



photon emitted in O in xy-plane

$$k^x = \omega \cos\theta$$
$$k^y = \omega \sin\theta$$

ω, θ : measured w.r.t O.

$$p^\alpha = (\hbar\omega, \hbar \vec{k}) = \hbar k^\alpha$$
$$= \hbar\omega \left[\, 1 \,|\, \cos\theta \,|\, \sin\theta \,|\, 0 \,\right]$$

$$k^{\alpha'} = \Lambda^{\alpha'}{}_{\alpha} k^\alpha$$

$$k^{t'} = \gamma (k^t - u k^x)$$
$$k^{x'} = \gamma (k^x - u k^t)$$
$$k^{y'} = k^y$$
$$k^{z'} = k^z \quad , \quad \gamma = \frac{1}{\sqrt{1-u^2}}$$

$$p^{\alpha'} = \hbar\omega' \left[\, 1 \,|\, \cos\theta' \,|\, \sin\theta' \,|\, 0 \,\right]$$

$f = \frac{1}{\Delta t}$ ↑ proper time

$\underbrace{\Delta t}_{\text{w proper}} = \gamma^{-1} \underbrace{\Delta t'}_{\text{w not proper}}$

$$f' = \frac{1}{\Delta t'} = \frac{1}{\gamma \Delta t}$$
$$= \frac{1}{\gamma} f$$
$$= \sqrt{1-u^2}\, f$$

$\cancel{w' \quad \gamma}$

$$k^{\alpha} = [\, w \mid w\cos\theta \mid w\sin\theta \mid 0 \,]$$

$$k^{\alpha'} = [\, w' \mid w'\cos\theta' \mid w'\sin\theta' \mid 0 \,]$$

$$w' = \gamma(w - u\, w\cos\theta) = \gamma w(1 - u\cos\theta)$$

$$w'\cos\theta' = \gamma(w\cos\theta - u w) = \gamma w(\cos\theta - u)$$

$$w'\sin\theta' = w\sin\theta$$

Thus $\boxed{w' = \dfrac{w}{\sqrt{1-u^2}}(1 - u\cos\theta)}$

&

$$\cos\theta' = \gamma\frac{w}{w'}(\cos\theta - u) = \gamma\frac{1}{\gamma(1-u\cos\theta)}(\cos\theta - u) = \boxed{\left(\frac{\cos\theta - u}{1 - u\cos\theta}\right)}$$

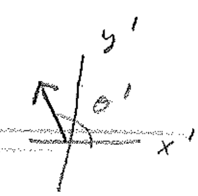
① ~~suppose~~ $\cancel{\theta = 0}$: $\cancel{w' = \frac{w}{\sqrt{1-u^2}}(1-u) = w\sqrt{\frac{1-u}{1+u}}}$

$\cancel{(\cos\theta = 1)}$ $\cancel{\cos\theta' = \frac{1-u}{1-u} = 1 \rightarrow \theta' = 0}$

② suppose $\theta = \frac{\pi}{2}$: $w' = \frac{w}{\sqrt{1-u^2}} \cdot 1 = \frac{w}{\sqrt{1-u^2}}$

$(\cos\theta = 0)$

$$\cos\theta' = \frac{-u}{1} \rightarrow \theta' > \frac{\pi}{2}$$



$$\therefore \cos\theta' = \frac{\cos\theta - u}{1 - u\cos\theta}$$

Thus $\rightarrow$ $\cos\theta' - u\cos\theta\cos\theta' = \cos\theta - u$

$$\cos\theta' + u = \cos\theta\left[1 + u\cos\theta'\right]$$

so $\boxed{\cos\theta = \dfrac{\cos\theta' + u}{1 + u\cos\theta'}}$

$$\rightarrow \quad \omega' = \frac{\omega}{\sqrt{1-u^2}}\left(1 - u\cos\theta\right)$$

$$= \frac{\omega}{\sqrt{1-u^2}}\left(1 - \frac{u(\cos\theta' + u)}{1 + u\cos\theta'}\right)$$

$$= \frac{\omega}{\sqrt{1-u^2}}\left(\frac{1}{1+u\cos\theta'}\right)\left[1 + \cancel{u\cos\theta'} - \cancel{u\cos\theta'} - u^2\right]$$

$$= \omega\,\frac{\sqrt{1-u^2}}{(1 + u\cos\theta')}$$

so $\boxed{\omega' = \omega\,\dfrac{\sqrt{1-u^2}}{1+u\cos\theta'}}$

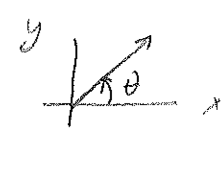
1) suppose $\theta' = 0$ $\rightarrow$ $\omega' = \omega\sqrt{\dfrac{1-u^2}{1+u}} = \boxed{\omega\sqrt{\dfrac{1-u}{1+u}} = \omega'}$

Also $\cos\theta = \dfrac{1+u}{1+u} = 1 \rightarrow \theta = 0$

~~2)~~

2) suppose $\theta' = \pi/2$ $\rightarrow$ $\boxed{\omega' = \omega\sqrt{1-u^2}}$ (consistent with time dilation)

Also $\cos\theta = \dfrac{0+u}{1} = u$



$$w' \sin\theta' =$$

$$= \frac{w}{\sqrt{1-u^2}} \left(1 - u\cos\theta\right) \sin\theta'$$

$$= \frac{w}{\sqrt{1-u^2}} \left(1 - u\cos\theta\right) \sqrt{1-\cos^2\theta'}$$

$$= \frac{w}{\sqrt{1-u^2}} \left(1 - u\cos\theta\right) \sqrt{1 - \frac{(\cos\theta - u)^2}{(1-u\cos\theta)^2}}$$

$$= \frac{w}{\sqrt{1-u^2}} \cancel{\left(1 - u\cos\theta\right)} \sqrt{\frac{(1-u\cos\theta)^2 - (\cos\theta-u)^2}{\cancel{(1-u\cos\theta)^2}}}$$

$$= \frac{w}{\sqrt{1-u^2}} \sqrt{1 - \cancel{2u\cos\theta} + u^2\cos^2\theta - \cos^2\theta - u^2 + \cancel{2u\cos\theta}}$$

$$= \frac{w}{\sqrt{1-u^2}} \sqrt{(1-u^2) - \cos^2\theta\,(1-u^2)}$$

$$= \frac{w}{\cancel{\sqrt{1-u^2}}} \sqrt{\cancel{(1-u^2)}(1-\cos^2\theta)}$$

$$= w\sin\theta$$

Thus, $w'\sin\theta' = w\sin\theta$

Exer (11.21)

To show: $\frac{1}{c}\frac{dE}{dt} = \vec{F}\cdot\vec{v}/c$

for $E = \gamma mc^2$, $\vec{F} = \frac{d\vec{p}}{dt}$, $\vec{p} = \gamma m\vec{v}$, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\text{LHS} = \frac{1}{c}\frac{dE}{dt}$$

$$= \frac{1}{c} mc^2 \frac{d\gamma}{dt}$$

$$= \frac{1}{c} m \cancel{c^2} \left(\cancel{-\frac{1}{2}}\right) \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \cancel{\frac{-2}{c^2}} \vec{v}\cdot\vec{a}$$

$$= \frac{m\gamma^3}{c} \vec{v}\cdot\vec{a}$$

$$\gamma^2 = \frac{1}{1-\left(\frac{v}{c}\right)^2}$$

$$\gamma^2 - \gamma^2\left(\frac{v}{c}\right)^2 = 1$$

$$\gamma^2 = 1 + \gamma^2 \frac{v^2}{c^2}$$

$$\text{RHS} = \vec{F}\cdot\vec{v}/c$$

$$= \frac{1}{c}\left(\frac{d}{dt}\vec{p}\right)\cdot\vec{v}$$

$$= \frac{1}{c}\frac{d}{dt}(\gamma m\vec{v})\cdot\vec{v}$$

$$= \frac{m}{c}\left(\frac{d\gamma}{dt}\vec{v} + \gamma\vec{a}\right)\cdot\vec{v}$$

$$= \frac{m}{c}\left(\gamma^3 \frac{\vec{v}\cdot\vec{a}}{c^2} v^2 + \gamma\,\vec{v}\cdot\vec{a}\right)$$

$$= \frac{\gamma m}{c}\vec{v}\cdot\vec{a}\left(\underbrace{\gamma^2\left(\frac{v^2}{c^2}\right)+1}_{\gamma^2}\right)$$

$$= \frac{\gamma^3 m}{c}\vec{v}\cdot\vec{a}$$

So LHS = RHS.

Exer. 11.22

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\gamma m \vec{v}\right)$$

$$= m\left(\frac{d\gamma}{dt}\vec{v} + \gamma \frac{d\vec{v}}{dt}\right)$$

$$= m\left(\gamma^3 \frac{\vec{v}\cdot\vec{a}}{c^2}\vec{v} + \gamma\vec{a}\right.$$

$$= \gamma m\left[\gamma^2\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)\vec{v} + \vec{a}\right]$$

using

$$\frac{d\gamma}{dt} = \frac{\gamma^3\,\vec{v}\cdot\vec{a}}{c^2}$$

from Exer. 11.21

Exercise 11.23

See Exercise 11.13 where we showed

$$0 = \frac{\partial L}{\partial x^\alpha} \quad , \quad \frac{\partial L}{\partial\left(\frac{dx^\alpha}{d\sigma}\right)} = -\frac{1}{c^2}\, \eta_{\alpha\beta} \frac{dx^\beta}{d\tau}$$

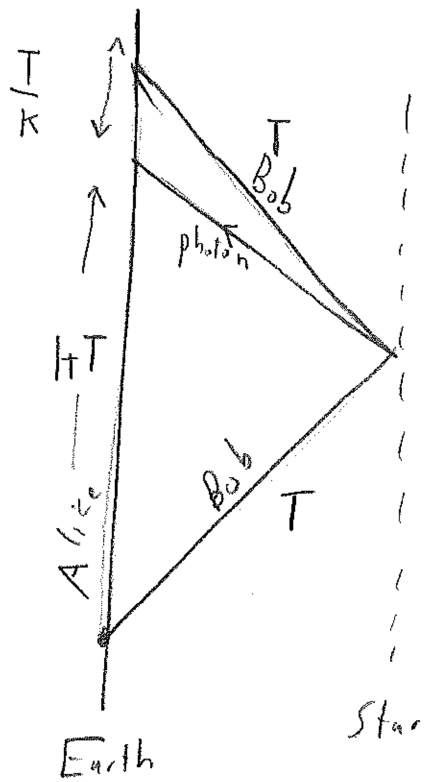
$$\rightarrow \quad 0 = -\frac{d}{d\sigma}\left(-\frac{1}{c^2}\, \eta_{\alpha\beta} \frac{dx^\beta}{d\tau} \right)$$

multiply by $\frac{d\sigma}{d\tau}$: $\Rightarrow \quad 0 = \frac{d}{d\tau}\left(\eta_{\alpha\beta} \frac{dx^\beta}{d\tau} \right)$

$$= \eta_{\alpha\beta} \frac{d^2 x^\beta}{d\tau^2}$$

so $\boxed{\frac{d^2 x^\beta}{d\tau^2} = 0}$

Prob 11.1: Twin paradox



Bob: Ages by $2T$

Alice: Ages by $kT + \frac{T}{k} = T\left(k + \frac{1}{k}\right)$

$$= T\,2\gamma \qquad (\text{see Exer. 11.2})$$

$$= 2T\gamma$$

$$= 2T \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$> 2T$$

Thus, Alice will be older than Bob by a multiplicative factor of $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$

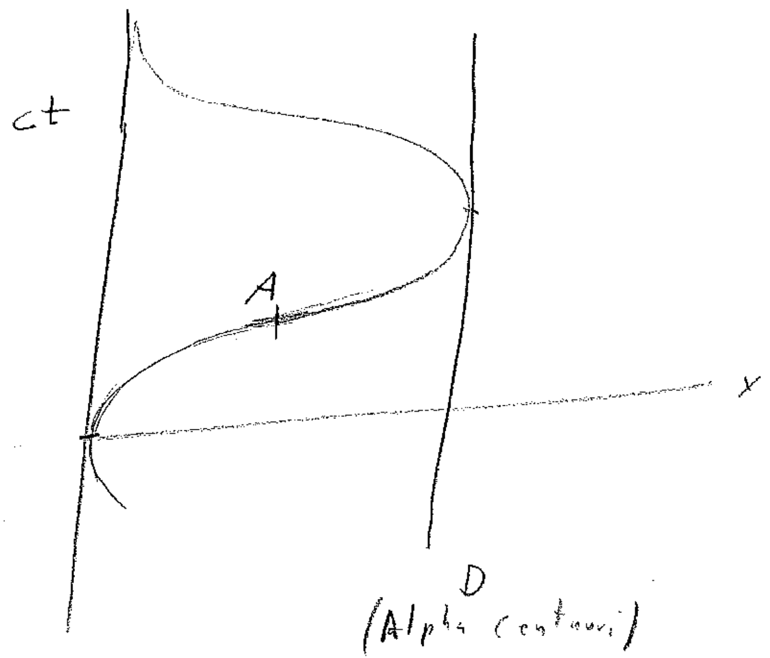
Problem: Const acceleration twin paradox

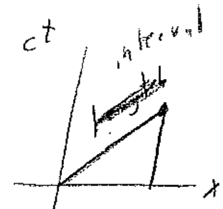
$$ct = a^{-1} c^2 \sinh\left(\frac{a\tau}{c}\right)$$

$$x = a^{-1} c^2 \cosh\left(\frac{a\tau}{c}\right) \boxed{+ x_0}$$

what about this?

Take $a = g$



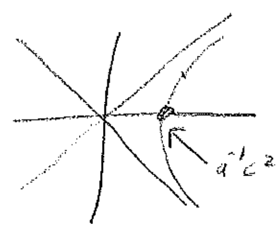


$$\text{interval}^2 = -c^2t^2 + x^2$$

$$= -a^{-2}c^4 \sinh^2 + a^{-2}c^4 \cosh$$

$$= a^{-2}c^4$$

(same value everywhere on curve)



$$x(\tau = 0) = 0 = a^{-1} c^2 \underbrace{\cosh\left(\frac{a\,0}{c}\right)}_{1} + x_0$$

$$0 = a^{-1}c^2 + x_0$$

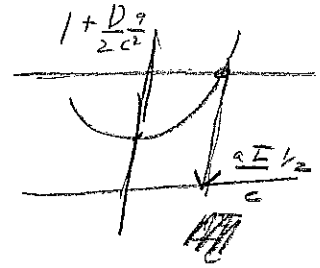
$$\rightarrow \quad x_0 = -a^{-1}c^2 = -\frac{c^2}{g}$$

$$\boxed{x(\tau) = a^{-1}c^2\left(\cosh\left(\frac{a\tau}{c}\right) - 1\right)}$$

when $x(\tau) = \frac{D}{2}$, I decelerate (change a to $-g$)

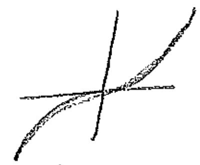
$$\frac{D}{2} = a^{-1}c^2\left(\cosh\left(\frac{a\tau_{\frac{1}{2}}}{c}\right) - 1\right)$$

$$\frac{D}{2}\frac{a}{c^2} = \cosh\left(\frac{a\tau}{c}\right) - 1$$



$$\cosh\left(\frac{a\tau}{c}\right) = \left(\frac{D}{2}\right)\frac{a}{c^2} + 1$$

$$\boxed{\tau_{\frac{1}{2}} = \frac{c}{a}\cosh^{-1}\left[\frac{D}{2}\frac{a}{c^2}+1\right]}$$



Arrive at Alpha centauri at time

$$\tau_{alpha} = 2\,\tau_{\frac{1}{2}}$$

$$= \left(\frac{2c}{a}\right)\cosh^{-1}\left[\frac{D}{2}\frac{a}{c^2}+1\right]$$

Arrive back at Earth

$$\tau_{return} = 4\,\tau_{\frac{1}{2}}$$

$$= \frac{4c}{a}\cosh^{-1}\left[\frac{D}{2}\frac{a}{c^2}+1\right]$$

$$= 2.2684\times10^{8}\ \text{s}$$

$$= \boxed{7.19\ \text{yr}}$$

$c = 3\times10^{8}$ m/s

$a = 9.8$ m/s^2

$D = 4.4$ lyr $= 4.4 * c * 365\times24\times3600$

$= 4.16\times10^{16}$ m

To find Alice's age:

At A, $\tau_{\frac{1}{2}} = \frac{c}{a} \cosh^{-1}\left[\frac{D}{2}\frac{a}{c^2}+1\right]$

$$\rightarrow \quad t_{\frac{1}{2}} = a^{-1} c \sinh\left[\frac{a}{c}\tau_{\frac{1}{2}}\right]$$

$$= a^{-1} c \sinh\left[\cosh^{-1}\left(\frac{D}{2}\frac{a}{c^2}+1\right)\right]$$

Thus,

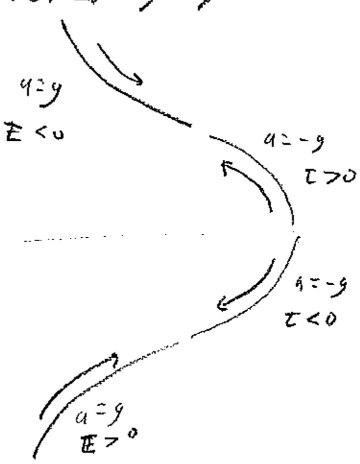
$$\Delta t_{Alice} = 4 t_{\frac{1}{2}}$$

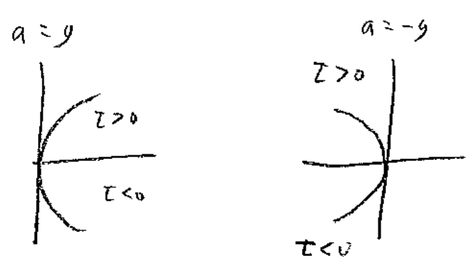
$$= 4 a^{-1} c \sinh\left[\cosh^{-1}\left(\frac{D}{2}\frac{a}{c^2}+1\right)\right]$$

$$= \boxed{12{,}07\,yr}$$

For a trip to the center of the Milky Way galaxy and back ($D = 10$ kpc $= 10{,}000 \times 3.26$ light yr)

$$\boxed{\begin{aligned} \tau_{Bob} &= 40{,}5\ yrs \\ \Delta t_{Alice} &= 65204\ yr \end{aligned}}$$





```
% calculate elapsed time for alice and bob for const
% proper acceleration a=g to alpha centauri (or to
% the center of milky way)
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
close all

% constants
c = 3e8; % m/s
yr = 365*24*60*60; % in sec
pc = 3.26*c*yr; % in m
kpc = 1000*pc;
g = 9.8; % m/s

% distance to star (or galaxy) wrt person on Earth
D = 4.4*c*yr; % distance to alpha centauri in m
%D = 10*kpc; % distance to center of galaxy

% calculate time to get 1/2-way there (outbound)
a = g;
tau_half = (c/a)*acosh((D/2)*(a/c^2)+1);
t_half = (c/a)*sinh((a/c)*tau_half);

% total time is 4 * 1/2-way outbound time
TB = 4*tau_half/yr;
TA = 4*t_half/yr;

fprintf('bob   ages by %f yrs\n', TB);
fprintf('alice ages by %f yrs\n', TA);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% calculate world line
N = 250;
tau = linspace(0, tau_half, N);

% part 1
a = g;
ct_1 = (c^2/a)*sinh(a*tau/c);
x_1  = (c^2/a)*(cosh(a*tau/c) - 1);

% part 2
a = -g;
ct_2 = (c^2/a)*sinh(a*(-tau)/c) + c*2*t_half;
x_2  = (c^2/a)*(cosh(a*(-tau)/c) - 1) + D;
% need to flip since time is running backward
ct_2 = fliplr(ct_2);
x_2  = fliplr(x_2);

% part 3
a = -g;
ct_3 = (c^2/a)*sinh(a*tau/c) + c*2*t_half;
x_3  = (c^2/a)*(cosh(a*tau/c) - 1) + D;

% part 4
a = g;
ct_4 = (c^2/a)*sinh(a*(-tau)/c) + c*4*t_half;
x_4  = (c^2/a)*(cosh(a*tau/c) - 1);
% need to flip since time is running backward
ct_4 = fliplr(ct_4);
x_4  = fliplr(x_4);

% concatenate the parts
ct = [ct_1 ct_2 ct_3 ct_4];
x  = [x_1  x_2  x_3  x_4];

% plot figure
figure()
plot(x/(c*yr),ct/(c*yr),'k')
axis equal
xlabel('x (light-years)')
ylabel('t (years)')
axis off
print('-depsc2','twinparadox.eps')

return
```

11.3

Problem:

Pole and barn paradox

~~(Pole and barn)~~

pole: 20 ft: in rest frame

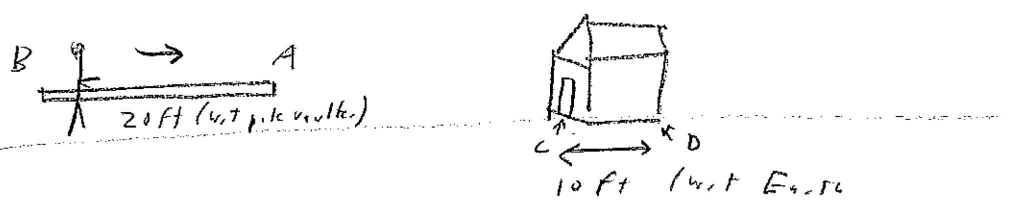
barn: 10 ft in rest frame

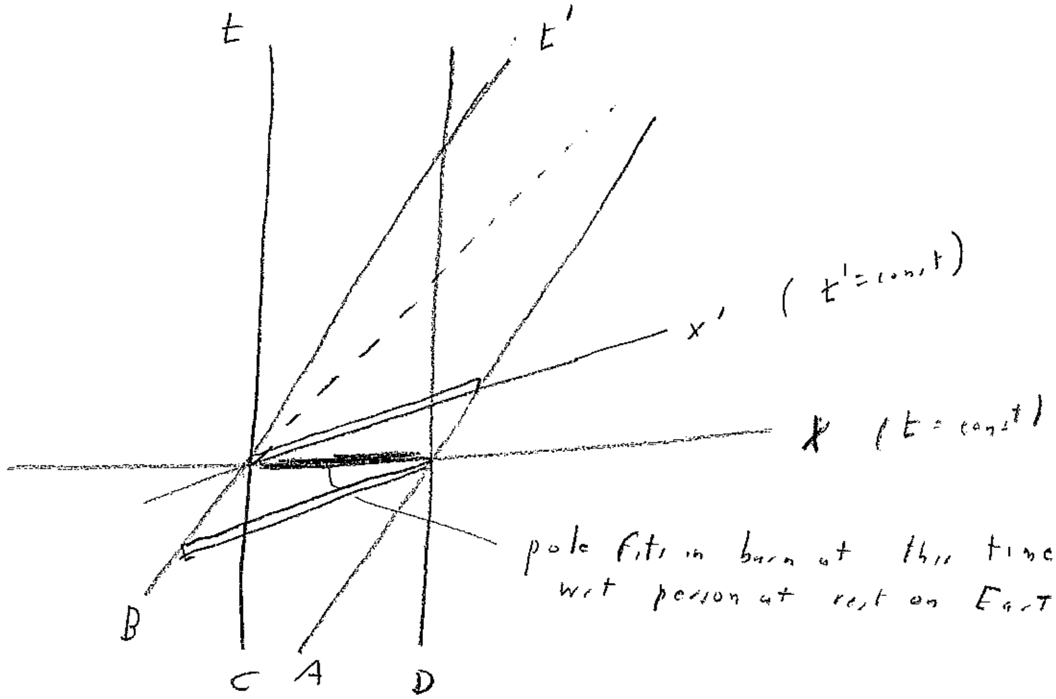
$\frac{v}{c}$ such that $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = 2$

Thus, w.r.t Earth pole has length 10 ft, same as barn.

W.r.t pole vaulter, barn has length 5 FT.

Spectator on Earth see pole as fitting within barn at some ~~instant~~ of time. Pole vaulter says that's impossible





11.4 Problem
$(\vec{F} = m\vec{g})$

$$\frac{dx}{c\,dt} = \frac{gt/c}{\sqrt{1 + g^2t^2/c^2}}$$

where g = const [unit of acceleration]

$[g]: \frac{L}{T^2}$

$\left[\frac{gt}{c}\right] = \frac{\frac{L}{T^2}T}{\frac{L}{T}} = 1$

a) $$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{\frac{g^2t^2}{c^2}}{\left(1 + \frac{g^2t^2}{c^2}\right)}}}$$

$$= \frac{1}{\sqrt{\frac{1 + \cancel{\frac{g^2t^2}{c^2}} - \cancel{\frac{g^2t^2}{c^2}}}{1 + \frac{g^2t^2}{c^2}}}}$$

$$= \sqrt{1 + \frac{g^2t^2}{c^2}}$$

b) $$\gamma = \frac{dt}{d\tau}$$

$$\frac{dx}{d\tau} = \frac{dt}{d\tau}\frac{dx}{dt}$$

$$= \gamma \frac{dx}{dt}$$

$$= \cancel{\sqrt{1 + \frac{g^2t^2}{c^2}}}\ \frac{gt}{\cancel{\sqrt{1 + g^2t^2/c^2}}}$$

$$= gt$$

c) $$\frac{dt}{d\tau} = \sqrt{1 + \frac{g^2 t^2}{c^2}}$$

$$\int \frac{dt}{\sqrt{1 + \frac{g^2t^2}{c^2}}} = \int d\tau = \tau$$

$$LHS = \int \frac{dt}{\sqrt{1+\frac{g^2t^2}{c^2}}} = \frac{1}{\frac{g}{c}} \sinh^{-1}\left(\frac{gt}{c}\right) + \text{const}$$

$$RHS = \tau$$

Thus, $$\frac{g\tau}{c} + \text{const} = \sinh^{-1}\left(\frac{gt}{c}\right)$$

Thus, $$\frac{gt}{c} = \sinh\left(\frac{g}{c}\tau\right)$$

$$\boxed{ct = \frac{c^2}{g}\sinh\left(\frac{g}{c}\tau\right)}$$

d) $$\frac{dx}{d\tau} = gt = c\sinh\left(\frac{g}{c}\tau\right)$$

$$\boxed{x(\tau) = x_0 + \frac{c^2}{g}\cosh\left[\frac{g}{c}\tau\right]}$$

e) Calculate components of 4-force

$$f^\alpha = ma^\alpha = \frac{m\,du^\alpha}{d\tau}$$

③

Take

$$ct = \frac{c^2}{g}\sinh\left(\frac{g\tau}{c}\right)$$

$$x = \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right)$$

Then

$$u^\alpha = \frac{dx^\alpha}{d\tau}$$

$$= \frac{d}{d\tau}\left(ct,\ x,\ 0,\ 0\right)$$

$$= \left(c\cosh\left(\frac{g\tau}{c}\right),\ c\sinh\left(\frac{g\tau}{c}\right),\ 0,\ 0\right)$$

$$a^\alpha = \frac{du^\alpha}{d\tau}$$

$$= \left(g\sinh\left(\frac{g\tau}{c}\right),\ g\cosh\left(\frac{g\tau}{c}\right),\ 0,\ 0\right)$$

Thus,

$$F^\alpha = m a^\alpha$$

$$= mg\left(\sinh\left(\frac{g\tau}{c}\right),\ \cosh\left(\frac{g\tau}{c}\right),\ 0,\ 0\right)$$

$$\gamma\vec{F} = f^i$$

$$= mg\left(\cosh\left(\frac{g\tau}{c}\right),\ 0,\ 0\right)$$

$$\vec{F} = \frac{mg}{\gamma}\left(\cosh\left(\frac{g\tau}{c}\right),\ 0,\ 0\right)$$

$$= \frac{mg}{\sqrt{1+\frac{g^2t^2}{c^2}}}\left(\cosh\left(\frac{g\tau}{c}\right),\ 0,\ 0\right)$$

Now:

$$\gamma = \sqrt{1 + \frac{g^2 t^2}{c^2}}$$

$$= \sqrt{1 + \frac{\cancel{g^2}}{\cancel{c^2}} \frac{\cancel{c^2}}{\cancel{g^2}} \sinh^2\left(\frac{g\tau}{c}\right)}$$

$$= \sqrt{1 + \sinh^2\left(\frac{g\tau}{c}\right)}$$

$$= \cosh\left(\frac{g\tau}{c}\right)$$

$\cosh^2 - \sinh^2 = 1$

$1 + \sinh^2 = \cosh^2$

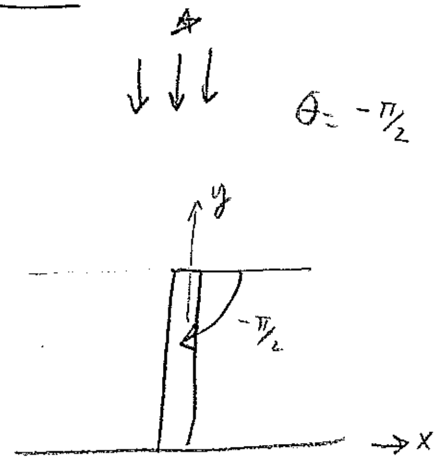
Thus, $\boxed{\vec{F} = mg \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}$

(1)

11.5

Problem: Stellar aberration

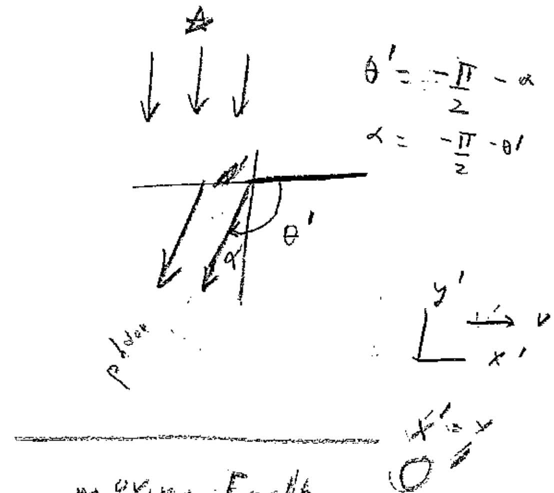
$\theta = -\pi/2$



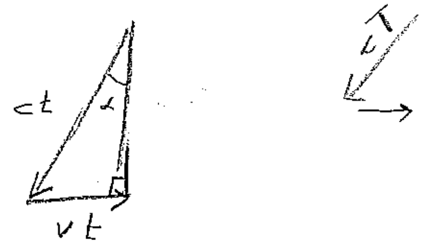
stationary earth
(rest frame for starlight)

$\theta' = -\frac{\pi}{2} - \alpha$

$\alpha = -\frac{\pi}{2} - \theta'$



moving Earth



~~Thus,~~

$$+ \sin\alpha = \frac{vt}{ct} = \beta$$

$$p^{\alpha} = \frac{\hbar\omega}{c}(1, 0, -1, 0) \quad \text{w.r.t. } \mathcal{O}$$

$$p^{\alpha'} = \frac{\hbar\omega'}{c}(1, \cos\theta', \sin\theta', 0)$$

~~$\hbar\omega$~~ Now:

$$p^{0'} = \gamma(p^0 - \beta p^1)$$
$$p^{x'} = \gamma(p^1 - \beta p^0)$$
$$p^{y'} = p^y$$
$$p^{z'} = p^z$$

Thus,

$$\frac{\hbar\omega'}{c} = \gamma\frac{\hbar\omega}{c}$$
$$\frac{\hbar\omega'}{c}\cos\theta' = \gamma\left(-\beta\frac{\hbar\omega}{c}\right) = -\beta\gamma\frac{\hbar\omega}{c}$$
$$\frac{\hbar\omega'}{c}\sin\theta' = -\frac{\hbar\omega}{c}$$
$$0 = 0$$

$\cos\theta'$

$= \cos\left(\frac{-\pi}{2} - \alpha\right)$

$= \cos\cancel{\left(\frac{-\pi}{2}\right)}\cos\alpha + \sin\left(\frac{-\pi}{2}\right)\sin\alpha$

$= -\sin\alpha$

Thus

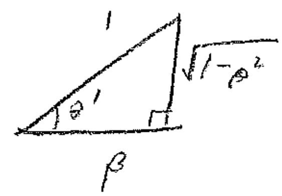
$\sin\alpha = -\cos\theta'$
$= \beta$

so $\boxed{\cos\theta' = -\beta}$

$\omega' = \gamma\omega$

$\omega'\cos\theta' = -\beta\gamma\omega \rightarrow \boxed{\cos\theta' = -\beta}$

$\omega'\sin\theta' = -\omega$



Thus, $\boxed{\tan\theta' = \frac{1}{\beta\gamma} = \frac{\sqrt{1-\beta^2}}{\beta}}$

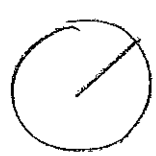
$\underline{\theta'}$: angle wrt x'-axis.

wrt y' axis, angle $= \alpha$

where $\boxed{\sin\alpha = \beta}$

---

plug in numbers:



$6400\text{ km} = R_E$

$R_{ES} = 1\text{ AU}$

$\sin\alpha = \frac{V_{tot}}{c} = \frac{3.04\times10^4\text{ m/s}}{3\times10^8\text{ m/s}}$

$= 10^{-4}$

so $\boxed{\alpha = 10^{-4}\text{ rad}}$

$V_{orbit} = \frac{2\pi R_{ES}}{1\text{yr}}$

$= \frac{2\pi(1\text{AU})}{365\cdot24\cdot3600}$

$= \boxed{3\times10^4\ \frac{\text{m}}{\text{s}}}$

$V_{rotation} = \frac{2\pi R_E}{1\text{day}}$

$= \frac{2\pi(6400\text{ km})}{24\cdot3600\text{ s}}$

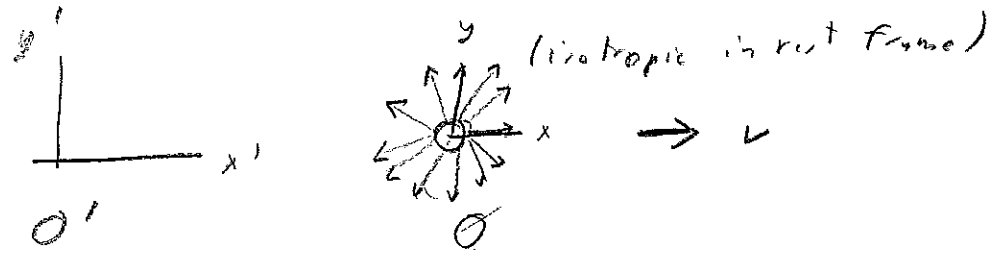
$= \boxed{465\text{ m/s}}$

$V_{tot} = 3.04\times10^4\text{ m/s}$

①

(11.6)

Problem: Relativistic beaming



(a) From Example 11.4:

$$\rightarrow \cos\theta' = \frac{\cos\theta + \beta}{1+\beta\cos\theta}$$

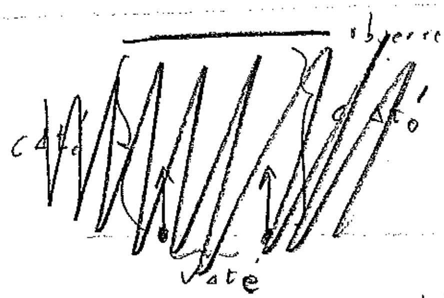
$$\rightarrow \quad \omega' = \gamma\omega(1+\beta\cos\theta)$$

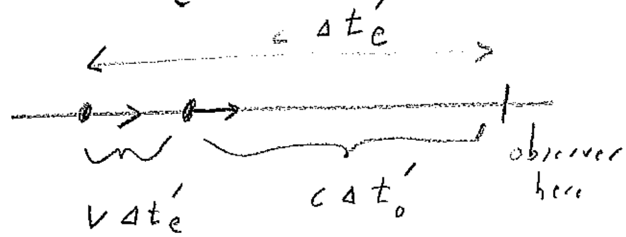
or $\omega' = \dfrac{\omega}{\gamma(1-\beta\cos\theta')}$

[$\omega = \gamma\omega'(1-\beta\cos\theta')$

$\rightarrow \omega' = \dfrac{\omega}{\gamma(1-\beta\cos\theta')}$]

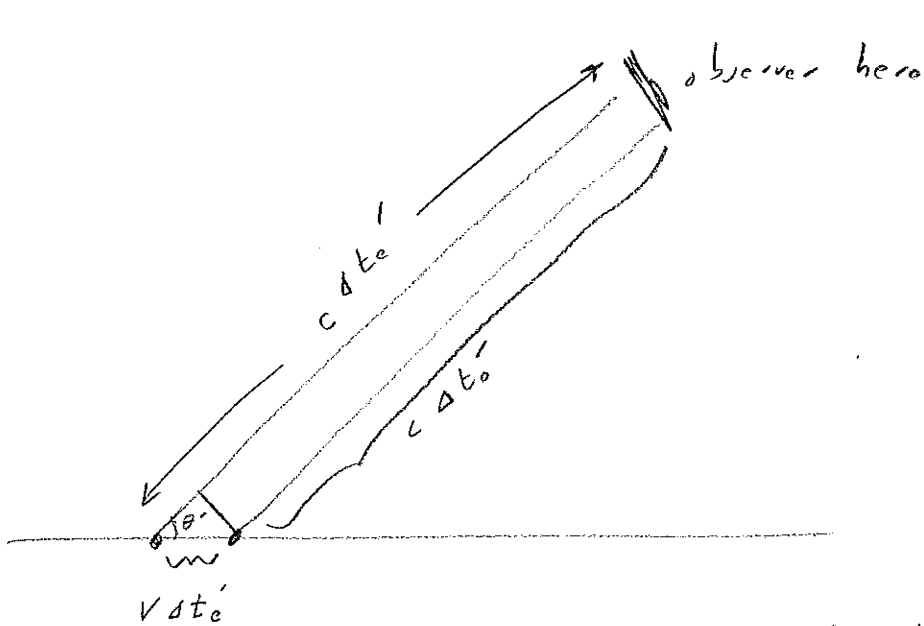
(b)

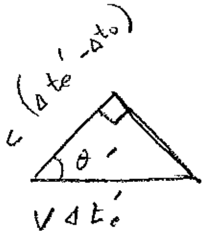




Thus, $c\Delta t_0' + v\Delta t_e' = c\Delta t_e'$

$$\Delta t_0' = \left(1-\frac{v}{c}\right)\Delta t_e'$$





$$\cos\theta' = \frac{c(\Delta t_e' - \Delta t_o')}{v\,\Delta t_e'}$$

$$v\cos\theta'\,\Delta t_e' = c\,\Delta t_e' - c\,\Delta t_o'$$

$$\boxed{\Delta t_o' = \Delta t_e'\left(1 - \frac{v}{c}\cos\theta'\right)}$$

(c) $\Delta t_e'$, $\underbrace{\Delta t_e}_{\text{proper time (source frame)}}$

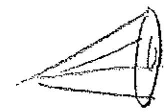
$$\boxed{\gamma = \frac{dt}{d\tau} = \frac{\Delta t_e'}{\Delta t_e}}$$

$$T = \gamma \underbrace{T_0}_{\text{proper time}} \rightarrow \gamma = \frac{T}{T_0} \text{ — proper time}$$

③

(d) $$\# = \mathcal{F}(\theta)\, R^2 2\pi \sin\theta\, \Delta\theta\, \Delta t_o$$

$$= \mathcal{F}'(\theta')\, R^2 2\pi \sin\theta'\, \Delta\theta'\, \Delta t_o'$$



$\dfrac{\#}{\text{area}\cdot\text{time}}$

Then,

$$\mathcal{F}'(\theta')\, \cancel{R^2}\, \cancel{2\pi} \sin\theta'\, \Delta\theta' \Delta t_o' = \mathcal{F}_* \cancel{R^2}\, \cancel{2\pi} \sin\theta\, \Delta\theta\, \Delta t_o$$

$$\mathcal{F}'(\theta') = \mathcal{F}_* \frac{\sin\theta\, d\theta}{\sin\theta'\, d\theta'} \boxed{\frac{\Delta t_e}{\Delta t_e' (1-\beta\cos\theta')}}$$

relation between emitted and observed.

$$= \mathcal{F}_* \frac{\sin\theta\, d\theta}{\sin\theta'\, d\theta'} \frac{dt_e}{dt_e'} \frac{1}{1-\beta\cos\theta'}$$

<u>Now:</u>

$$\cos\theta' = \frac{\beta + \cos\theta}{1+\beta\cos\theta} \iff \cos\theta = \frac{-\beta + \cos\theta'}{1-\beta\cos\theta'}$$

$$-\sin\theta'\, d\theta' = \frac{-\sin\theta\, d\theta\,(1+\beta\cos\theta) + \beta\sin\theta\, d\theta\,(\beta+\cos\theta)}{(1+\beta\cos\theta)^2}$$

$$= -\sin\theta\, d\theta\, \frac{[\,(1+\beta\cancel{\cos\theta}) - \beta(\beta+\cancel{\cos\theta})\,]}{(1+\beta\cos\theta)^2}$$

$$= -\sin\theta\, d\theta\, \frac{1-\beta^2}{(1+\beta\cos\theta)^2}$$

$$= -\sin\theta\, d\theta\, \frac{1}{\gamma^2(1+\beta\cos\theta)^2}$$

<u>so</u> $$\frac{\sin\theta\, d\theta}{\sin\theta'\, d\theta'} = \gamma^2(1+\beta\cos\theta)^2$$

$$\mathcal{F}'(\theta') = \mathcal{F}_* \ \gamma^2 (1+\beta\cos\theta)^2 \frac{1}{1-\beta\cos\theta'} \ \underbrace{\frac{dt_c}{dt_c'}}_{\frac{1}{\gamma}}$$

proper time

$$= \mathcal{F}_* \frac{\gamma(1+\beta\cos\theta)^2}{1-\beta\cos\theta'}$$

<u>Now:</u>

$$\cos\theta = \frac{-\beta + \cos\theta'}{1-\beta\cos\theta'}$$

$$\rightarrow \quad 1+\beta\cos\theta = 1 + \beta\left(\frac{-\beta+\cos\theta'}{1-\beta\cos\theta'}\right)$$

$$= \frac{1-\cancel{\beta\cos\theta'} - \beta^2 + \cancel{\beta\cos\theta'}}{1-\beta\cos\theta'}$$

$$= \frac{1-\beta^2}{1-\beta\cos\theta'}$$

$$= \frac{1}{\gamma^2(1-\beta\cos\theta')}$$

Thus,

$$\boxed{\mathcal{F}'(\theta')} = \mathcal{F}_* \frac{\gamma}{1-\beta\cos\theta'} \ \frac{1}{\gamma^4(1-\beta\cos\theta')^2}$$

$$\boxed{= \mathcal{F}_* \frac{1}{\gamma^3 (1-\beta\cos\theta')^3}}$$

Energy flux:

$$\mathcal{E}'(\theta') = \mathcal{F}'(\theta')\ \hbar\omega'$$

$$= \mathcal{F}_* \frac{1}{\gamma^3(1-\beta\cos\theta')^3}\ \hbar \frac{\omega}{\gamma(1-\beta\cos\theta')}$$

$$= \mathcal{F}_*\ \hbar\omega \frac{1}{\gamma^4(1-\beta\cos\theta')^4}$$

$$= \mathcal{E}_* \frac{1}{\gamma^4(1-\beta\cos\theta')^4}$$

Specific intensity: $\quad I = \dfrac{dE}{d\omega\, d\Omega\, dt}$

$$I'_{\omega'}(\theta') = \frac{\mathcal{E}'(\theta')}{d\omega'}$$

$$= \mathcal{E}_* \frac{1}{\gamma^4(1-\beta\cos\theta')^4} \left(\frac{d\omega}{d\omega'}\right) \frac{1}{d\omega}$$

$$= I_\omega \frac{1}{\gamma^4(1-\beta\cos\theta')^4} \frac{d\omega}{d\omega'}$$

Now: $\quad \omega' = \dfrac{\omega}{\gamma(1-\beta\cos\theta')} \rightarrow \dfrac{d\omega'}{d\omega} = \dfrac{1}{\gamma(1-\beta\cos\theta')}$

so
$$\boxed{I'_{\omega'}(\theta') = I_\omega \frac{1}{\gamma^4(1-\beta\cos\theta')^4}\ \gamma(1-\beta\cos\theta') = I_\omega \frac{1}{\gamma^3(1-\beta\cos\theta')^3}}$$

e) Beaming Factor:

$$\frac{F'(\theta'=0)}{F} = \frac{1}{\gamma^3(1-\beta)^3}\,\frac{(1+\beta)^3}{(1+\beta)^3}$$

$$= \frac{(1+\beta)^3}{\gamma^3(1-\beta^2)^3}$$

$$= \frac{(1+\beta)^3}{\gamma^3\left(\frac{1}{\gamma^6}\right)}$$

$$= \boxed{\gamma^3(1+\beta)^3}$$

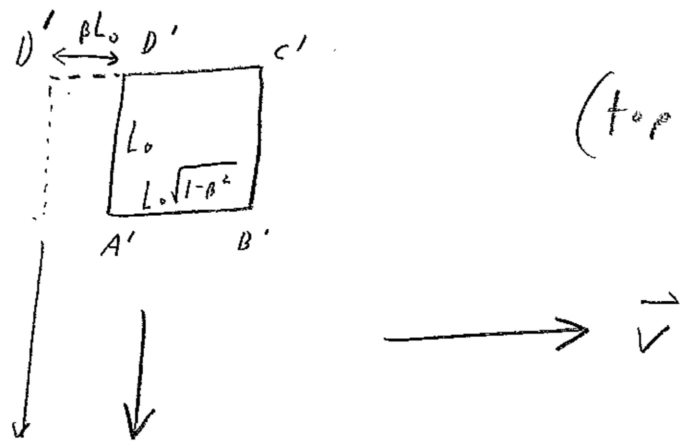
WAB

BM.

Problem: Terrell rotation

(11.8)

cube with proper edge length L_0 moving with speed v wrt stationary observer



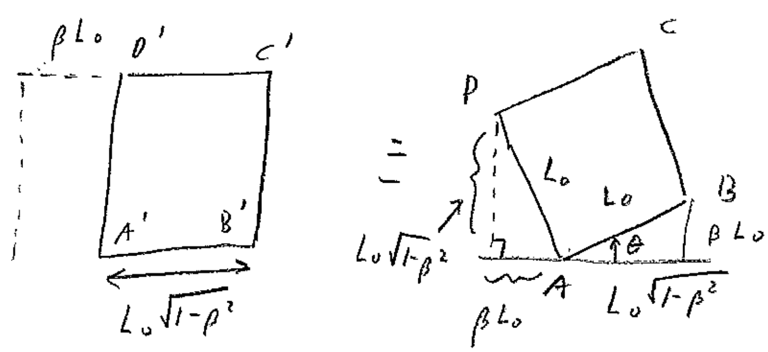
(top view)

light to observer's camera

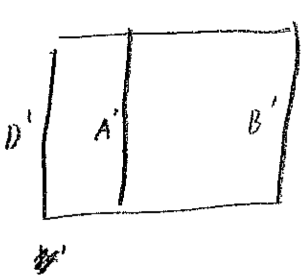
light leaves D' a time $\Delta t = \frac{L_0}{c}$ earlier

Thus $\underset{D' \qquad D'}{\overset{v\Delta t}{\longleftrightarrow}} = \frac{vL_0}{c} = \beta L_0$

(top)



(Front)



$$\sin\theta = \frac{\beta L_0}{L_0} = \beta$$

$$\theta = \sin^{-1}\beta$$

~~[illegible]~~

Generating Figures

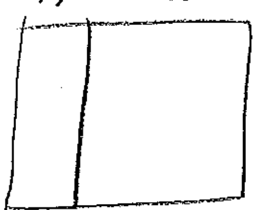
200 units in Keynote

$\beta = \boxed{\frac{5}{13}} = 0.38 \approx 0.4$

$$\sqrt{1-\beta^2} = 0.92 = \sqrt{1 - \frac{25}{169}} = \sqrt{\frac{169-25}{169}} = \sqrt{\frac{144}{169}} = \boxed{\frac{12}{13}}$$

262

77 185



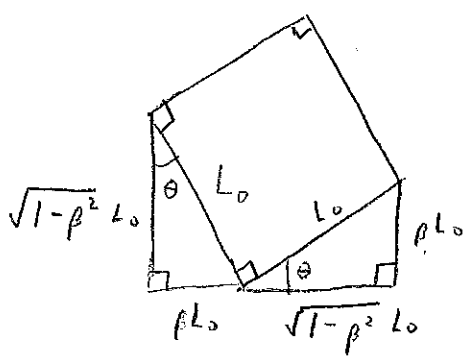
$\beta L_0 = \frac{5}{13} L_0$ $\quad \sqrt{1-\beta^2}\, L_0 = \frac{12 L_0}{13}$

$\frac{17}{13} L_0$

$L_0 = 200$

$\theta = \sin^{-1}(\beta)$

$= 22.6°$



$\sin\theta = \beta$

$\theta = \sin^{-1}\beta$

11.9

Problem: Collisions / disintegration in SR

$$\sum_I \gamma_I m_I \vec{v}_I = \text{const} \quad , \qquad \sum_I \gamma_I m_I c^2 = \text{const}$$

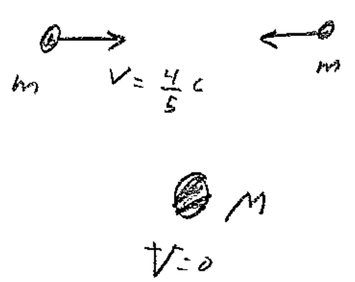
a)
$$p_I^\alpha = \left(\frac{E_I}{c} \, , \, \gamma_I m_I \vec{v} \right)$$
$$= \left(\frac{\gamma_I m_I}{c} \, , \, \gamma_I m_I \vec{v} \right)$$

Thus, $\sum_I p_I^\alpha = \text{const} \leftarrow$ 4-vector

$$\Leftrightarrow \left(\sum_I \frac{\gamma_I m_I}{c} \, , \, \sum_I \gamma_I m_I \vec{v} \right) = \text{const}$$

$$\sum_I \frac{\gamma_I m_I}{c} = \text{const} \, , \quad \sum_I \gamma_I m_I \vec{v} = \text{const}$$

b)



Before: $\vec{P}_{tot} = 0$

$$E_{tot} = 2\gamma m c^2 = \frac{2mc^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

After: $\vec{p} = 0$

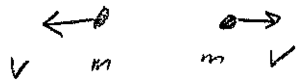
$$E_{tot} = Mc^2$$

Thus,
$$M\cancel{c^2} = \frac{2m\cancel{c^2}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$M = 2m\gamma$$
$$= 2m \frac{1}{\sqrt{1-\left(\frac{4}{5}\right)^2}}$$
$$= 2m \frac{1}{\sqrt{\frac{25-16}{25}}} = 2m\sqrt{\frac{25}{9}} = \boxed{\frac{10m}{3}}$$

c)

before $\vec{p} = \vec{0}$, $E_{tot} = Mc^2$



after $\vec{p}_{tot} = \vec{0}$, $E_{tot} = 2\gamma m c^2$

Thus, $2\gamma m\cancel{c^2} = M\cancel{c^2}$

$$2m \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = M$$

$$\frac{2m}{M} = \sqrt{1-\left(\frac{v}{c}\right)^2}$$

$$\frac{4m^2}{M^2} = 1-\left(\frac{v}{c}\right)^2$$

$$\boxed{\frac{v}{c} = \sqrt{1-\frac{4m^2}{M^2}}}$$

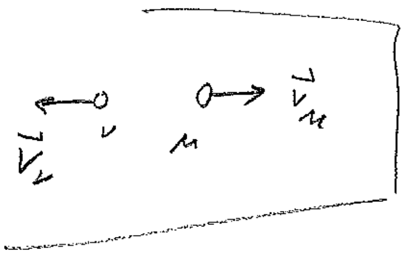
Decay:

d) $\pi^- \rightarrow \mu^- + \nu$ (massless)
(at rest)

Initially : $p_\pi^\alpha = (m_\pi c, \vec{0})$

After : $p_\mu^\alpha + p_\nu^\alpha = (\gamma_\mu m_\mu c, \gamma_\mu m_\mu \vec{V_\mu}) + (\frac{E_\nu}{c}, \vec{p_\nu})$

$= (\gamma_\mu m_\mu c + \frac{E_\nu}{c}, \gamma_\mu m_\mu \vec{V_\mu} + \vec{p_\nu})$



Equate components :

$m_\pi c = \gamma_\mu m_\mu c + \frac{E_\nu}{c}$

$\vec{p_\nu} = -\gamma_\mu m_\mu \vec{V_\mu} \rightarrow p_\nu = -\gamma_\mu m_\mu V_\mu$

Now: $\frac{E_\nu}{c} = |p_\nu| = \gamma_\mu m_\mu V_\mu$

Thus,
$$\gamma_\mu m_\mu V_\mu = m_\pi c - \gamma_\mu m_\mu c$$
$$\gamma_\mu m_\mu (V_\mu + c) = m_\pi c$$
$$\gamma_\mu m_\mu \left(1 + \frac{V_\mu}{c}\right) = m_\pi$$

$\left(\frac{E}{c}, \vec{p}\right)$

$$\frac{m_\mu \left(1 + \frac{V_\mu}{c}\right)}{\sqrt{1 - \left(\frac{V_\mu}{c}\right)^2}} = m_\pi$$
$$m_\mu \sqrt{\frac{1 + \frac{V_\mu}{c}}{1 - \frac{V_\mu}{c}}} = m_\pi$$

$$\rightarrow \frac{m_\mu^2}{m_\pi^2} = \frac{1 - \frac{V_\mu}{c}}{1 + \frac{V_\mu}{c}}$$
$$m_\mu^2 \left(1 + \frac{V_\mu}{c}\right) = m_\pi^2 \left(1 - \frac{V_\mu}{c}\right)$$
$$\frac{V_\mu}{c}(m_\mu^2 + m_\pi^2) = m_\pi^2 - m_\mu^2 \rightarrow \boxed{\frac{V_\mu}{c} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}}$$

11.10

Problem: Maxwell's equations in terms of $(\phi, \vec{A})$.

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

source-free:

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0 \quad ✓$$

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right) = -\vec{\nabla}\cdot\vec{\nabla}\phi - \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\frac{\partial\vec{B}}{\partial t} \quad ✓$$

Remaining equations:

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \quad \text{and} \quad \vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$$

$$\begin{aligned}
\frac{\rho}{\epsilon_0} &= \vec{\nabla}\cdot\vec{E} \\
&= \vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right) \\
&= -\nabla^2\phi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) \\
&= -\left(\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\right) - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) \\
&= -\Box^2\phi - \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right)
\end{aligned}$$

$$\cancel{\Box^2\phi = \frac{\rho}{\epsilon_0}}$$

$$\boxed{\Box^2\phi = -\frac{\rho}{\epsilon_0} - \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right)}$$

$$\boxed{\Box^2\phi + \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\rho/\epsilon_0}$$

$$\mu_0 \vec{J} = \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$= \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right)$$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) \boxed{-\nabla^2\vec{A}} + \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{\nabla}\phi) \boxed{+\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}}$$

$$= -\Box^2\vec{A} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right)$$

Thus, $$\boxed{\Box^2\vec{A} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\mu_0\vec{J}}$$

<u>NOTE:</u> Let $\alpha \equiv \vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} \neq 0$

Make a gauge transformation:

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda$$

$$\phi \to \phi' = \phi - \frac{\partial\Lambda}{\partial t}$$

$$0 = \vec{\nabla}\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial\phi'}{\partial t}$$

$$= \vec{\nabla}\cdot(\vec{A} + \vec{\nabla}\Lambda) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\phi - \frac{\partial\Lambda}{\partial t}\right)$$

$$= \vec{\nabla}\cdot\vec{A} + \nabla^2\Lambda + \frac{1}{c^2}\frac{\partial\phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t}$$

$$= \Box^2\Lambda + \left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right)$$

$$= \Box^2\Lambda + \alpha$$

so $(\vec{A}', \phi')$ satisfies the Lorentz gauge condition iff $\boxed{\Box^2\Lambda = -\alpha}$

11.10 / 11.11 A little of each

Prob: Maxwell's equations in 4-vector notation

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \quad , \quad \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{J}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad , \quad \vec{\nabla}\cdot\vec{B} = 0$$

a) $$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \quad , \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0 \quad ✓$$

$$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right)$$

$$= -\underbrace{\cancel{\vec{\nabla}\times\vec{\nabla}\phi}}_{0} - \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$= -\frac{\partial\vec{B}}{\partial t} \quad ✓$$

b) Define: $A^\alpha = (\phi/c, \vec{A})$, $A_\alpha = (-\frac{\phi}{c}, \vec{A})$

$j^\alpha = (\rho c, \vec{J})$

$x^\alpha = (ct, \vec{x})$ [inertial coords]

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

$$= \epsilon_{ijk}(\nabla\times\vec{A})_k$$

$$= \epsilon_{ijk} B_k$$

$$F_{0i} = \partial_0 A_i - \partial_i A_0$$

$$= \frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{\partial}{\partial x^i}\left(\frac{-\phi}{c}\right)$$

$$= \frac{1}{c}\frac{\partial A_i}{\partial t} + \frac{1}{c}\frac{\partial\phi}{\partial x^i}$$

$$= -\frac{1}{c}\left[-\frac{\partial\phi}{\partial x^i} - \frac{\partial A_i}{\partial t}\right]$$

$$= -\frac{1}{c}E_i$$

$[A] = [B]L$

$[E] = \frac{[B]L}{T} = \frac{[A]}{T}$

$[E] = \frac{[\phi]}{L}$

so $\frac{[A]}{T} = \frac{[\phi]}{L}$

so $[A] = \frac{[\phi]}{L/T} = \frac{[\phi]}{c}$

Thus, $\left(F_{ij} = \epsilon_{ijk} B_k\right) \epsilon_{ijl}$

$$
\begin{aligned}
F_{ij}\,\epsilon_{ijl} &= \epsilon_{ijk}\,\epsilon_{ijl}\, B_k \\
&= \left(\delta_{jj}\delta_{kl} - \delta_{jl}\delta_{jk}\right) B_k \\
&= \left(3\delta_{kl} - \delta_{kl}\right) B_k \\
&= 2 B_l
\end{aligned}
$$

$$\rightarrow \quad \boxed{B_i = \frac{1}{2}\,\epsilon_{ijk}\, F_{jk}}$$

Also, $\boxed{E_i = -c\, F_{0i}}$

Alternatively:

$$\boxed{\begin{aligned} F_{0i} &= -\frac{1}{c} E_i \\ F_{ij} &= \epsilon_{ijk}\, B_k \end{aligned}}$$

---

c) Source-Free maxwell equations

$$\left(0 = \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}\right) \epsilon^{\alpha\beta\gamma\delta}$$

$\rightarrow$ RHS $= \oslash \left(\partial_\alpha F_{\beta\gamma}\right)\epsilon^{\alpha\beta\gamma\delta}$ since F is antisymmetric

$$
\begin{aligned}
&= \oslash \quad \partial_\alpha\left(\partial_\beta A_\gamma - \partial_\gamma A_\beta\right)\epsilon^{\alpha\beta\gamma\delta} \\
&= \oslash \left(\partial_\alpha \partial_\beta A_\gamma\right)\epsilon^{\alpha\beta\gamma\delta} = 0
\end{aligned}
$$

since $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$

(so ~~##~~ $dF = 0$)

~~[illegible]~~: write in terms of $\vec{E}, \vec{B}$

$\underline{\delta=0}$: $0 = \epsilon^{0ijk}\, \partial_i F_{jk}$

$= \epsilon^{ijk}\, \partial_i F_{jk}$

$= \partial_i \left(\epsilon^{ijk} F_{jk}\right)$

$= \partial_i \left(2 B_i\right)$

$= 2\, \vec{\nabla}\cdot\vec{B} \quad \rightarrow \quad \boxed{\vec{\nabla}\cdot\vec{B} = 0}$

$\underline{\delta=i}$ $\quad 0 = \epsilon^{i\alpha\beta\gamma}\, \partial_\alpha F_{\beta\gamma}$

$= \epsilon^{i0jk}\, \partial_0 F_{jk} + \epsilon^{ijk0}\, \partial_j F_{k0} + \epsilon^{ik0j}\, \partial_k F_{0j}$

$= \epsilon^{i0jk} \left[\partial_0 \left(\epsilon_{jkl} B_l\right) + \frac{1}{c} \partial_j E_k + \partial_k \left(-\frac{1}{c} E_j\right)\right]$

$= \epsilon^{i0jk} \left[\epsilon_{jkl} \frac{1}{c} \frac{\partial B_l}{\partial t} + \frac{1}{c}\left(\partial_j E_k - \partial_k E_j\right)\right]$

$= \epsilon^{i0jk} \left(\frac{1}{c}\right) \left[\epsilon_{jkl} \frac{\partial B_l}{\partial t} + \epsilon_{jkl} (\vec{\nabla}\times\vec{E})_l \right]$

$= \epsilon^{i0jk} \left(\frac{1}{c}\right) \epsilon_{jkl} \left[\frac{\partial \vec{B}}{\partial t} + (\vec{\nabla}\times\vec{E}) \right]_l$

so $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$

---

d) $F^{\alpha\beta} = \eta^{\alpha\mu} \eta^{\beta\nu} F_{\mu\nu}$ $\qquad \eta^{\alpha\beta} = (-1, 1, 1, 1)$

$\qquad \eta_{\alpha\beta} = (-1, 1, 1, 1)$

$F^{ij} = \eta^{i\mu} \eta^{j\nu} F_{\mu\nu}$ ~~=~~

$= \eta^{ik} \eta^{jl} F_{kl}$

$= \delta^{ik} \delta^{jl} F_{kl}$

$= F_{ij}$

$$F^{00} = \eta^{0\mu}\eta^{0\nu} F_{\mu\nu}$$
$$= \eta^{00}\eta^{00} F_{00}$$
$$= (-1)(-1) F_{00}$$
$$= F_{00}$$

$$F^{0i} = \eta^{0\mu}\eta^{i\nu} F_{\mu\nu}$$
$$= \eta^{00}\eta^{ij} F_{0j}$$
$$= (-1)\delta^{ij} F_{0j}$$
$$= -F_{0i}$$

so $F^{ij} = F_{ij}$, $F^{00} = F_{00}$, $F^{0i} = -F_{0i}$

Now: $\partial_\alpha F^{\alpha\beta} = -\mu_0 j^\beta$

$\underline{\beta = 0}$: $RHS = -\mu_0 j^0 = -\mu_0 \rho c$

$$LHS = \partial_\alpha F^{\alpha 0} = \partial_0 \cancel{F^{00}}^{0} + \partial_i F^{i0}$$
$$= -\partial_i F_{i0}$$
$$= -\frac{1}{c}\partial_i E_i \qquad c^2 = \frac{1}{\epsilon_0\mu_0}$$

so $\frac{1}{c}\vec{\nabla}\cdot\vec{E} = \rho c \mu_0 \rightarrow \vec{\nabla}\cdot\vec{E} = \rho c^2 \mu_0 = \frac{\rho}{\epsilon_0}$

$\underline{\beta = i}$: $RHS = -\mu_0 j^i =$

$$LHS = \partial_\alpha F^{\alpha i} = \partial_0 F^{0i} + \partial_j F^{ji}$$
$$= -\frac{1}{c}\frac{\partial}{\partial t} F_{0i} + \partial_j F_{ji}$$
$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}E_i\right) + \partial_j\left(\epsilon_{jik} B_k\right)$$

Units:

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}$$

$$= \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}$$

$$c^2 \left(\vec{\nabla} \times \vec{B}\right) = \frac{\partial \vec{E}}{\partial t} + \frac{\vec{j}}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$[\rho] = [\epsilon_0] \frac{[E]}{L}$$

$$[\vec{j}] = [\epsilon_0] \frac{[E]}{T}$$

$$= [\epsilon_0] \frac{[E]}{L} \frac{L}{T}$$

$$= [\rho]\, c$$

$$\frac{L^2}{T^2} \frac{1}{L} [B] = \frac{[E]}{T}$$

$$\frac{L}{T} [B] = [E]$$

$$c\, [B] = [E]$$

$$\text{LHS} = \frac{1}{c^2} \left(\frac{\partial E_i}{\partial t}\right) - \epsilon_{ijk}\, \partial_j B_k$$

$$= \left[\frac{1}{c^2} \left(\frac{\partial \vec{E}}{\partial t}\right) - \left(\vec{\nabla} \times \vec{B}\right) \right]_i$$

Thus, $$-\mu_0 \vec{j} = \frac{1}{c^2} \left(\frac{\partial \vec{E}}{\partial t}\right) - \vec{\nabla} \times \vec{B}$$

$$\boxed{\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}}$$

(11.12)

Problem: Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + \mu_0 j^\alpha A_\alpha \equiv \mathcal{L}(A_\alpha, A_{\alpha,\beta}, x^\alpha)$$

field, derivatives, $(\vec{x}, t)$

$$L = \int d^3x\, \mathcal{L}$$

EOM:

$$0 = \frac{\partial \mathcal{L}}{\partial A_\alpha} - \partial_\beta \left(\frac{\partial \mathcal{L}}{\partial A_{\alpha,\beta}} \right)$$

Now: $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$

Thus

$$\mathcal{L} = -\frac{1}{4} \eta^{\mu\alpha} \eta^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + \mu_0 j^\alpha A_\alpha$$

$$= \cancel{-\frac{1}{4} \eta^{\mu\alpha} \eta^{\nu\beta} F_{\mu\nu} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)}$$

$$= \cancel{-\frac{1}{2} \eta^{\mu\alpha} \eta^{\nu\beta} F_{\mu\nu} (\partial_\alpha A_\beta) + \mu_0 j^\alpha A_\alpha}$$

$$= \cancel{-\frac{1}{2} \eta^{\mu\alpha} \eta^{\nu\beta} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta) + \mu_0 j^\alpha A_\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} = \mu_0 j^\alpha$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} = -\frac{1}{2} F^{\mu\nu} \frac{\partial F_{\mu\nu}}{\partial(\partial_\beta A_\alpha)}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= -\frac{1}{2} F^{\mu\nu} \left(\delta_\mu^\beta \delta_\nu^\alpha - \delta_\nu^\beta \delta_\mu^\alpha \right)$$

$$= -\frac{1}{2} \left(F^{\beta\alpha} - F^{\alpha\beta} \right)$$

$$= F^{\alpha\beta}$$

Thus, $0 = \mu_0 j^\alpha - \partial_\beta F^{\alpha\beta} \rightarrow$

$$\partial_\beta F^{\alpha\beta} = \mu_0 j^\alpha$$

$$\boxed{\partial_\beta F^{\beta\alpha} = -\mu_0 j^\alpha}$$