10.1

Exercise: EOMs starting from action

$$S[y] = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \; \mathcal{L}(y, y_{,x}, y_{,t}, x, t)$$

Vary y:

$$\delta S = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left(\frac{\partial \mathcal{L}}{\partial y} \delta y + \frac{\partial \mathcal{L}}{\partial y_{,x}} \overset{\frac{\partial}{\partial x}(\delta y)}{\overset{\shortparallel}{\delta y_{,x}}} + \frac{\partial \mathcal{L}}{\partial y_{,t}} \overset{\frac{\partial}{\partial t}(\delta y)}{\overset{\shortparallel}{\delta y_{,t}}} \right)$$

$$= \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_{,x}}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_{,t}}\right)\right) \delta y$$

$$+ \int_{t_1}^{t_2} dt \left(\underset{0}{\cancel{\frac{\partial \mathcal{L}}{\partial y_{,x}} \delta y}} \right)\Bigg|_{x_1}^{x_2} + \int_{x_1}^{x_2} dx \left(\underset{0}{\cancel{\frac{\partial \mathcal{L}}{\partial y_{,t}} \delta y}} \right)\Bigg|_{t_1}^{t_2}$$

So $\delta S = 0 \quad \forall \delta y$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_{,x}}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_{,t}}\right) = 0$$

(10.2)

Exercise: $\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2} y_{,t}^2 - y_{,x}^2 - \mu^2 y^2 \right]$

$$0 = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_{,x}}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_{,t}}\right)$$

Now: $\frac{\partial \mathcal{L}}{\partial y} = -\mu^2 y$

$$\frac{\partial \mathcal{L}}{\partial y_{,x}} = -y_{,x}$$

$$\frac{\partial \mathcal{L}}{\partial y_{,t}} = \frac{1}{c^2} y_{,t}$$

Thus,
$$0 = -\mu^2 y + \frac{d}{dx}(y_{,x}) - \frac{1}{c^2}\frac{d}{dt}(y_{,t})$$
$$= -\mu^2 y + y_{,xx} - \frac{1}{c^2} y_{,tt}$$
$$= -\mu^2 y + \Box y$$

so $\boxed{\Box y = \mu^2 y}$

Exer 10.3

$$\mathcal{L} = \mathcal{L}(\varphi_I, \varphi_{I,\alpha}, x^\alpha)$$

$$\delta\mathcal{L} = \sum_I \left(\frac{\partial \mathcal{L}}{\partial \varphi_I} \delta\varphi_I + \sum_\alpha \frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \delta\varphi_{I,\alpha} \right)$$

$$= \sum_I \left(\frac{\partial \mathcal{L}}{\partial \varphi_I} \delta\varphi_I + \sum_\alpha \frac{d}{dx^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \delta\varphi_I \right) - \sum_\alpha \frac{d}{dx^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \right) \delta\varphi_I \right)$$

$$= \sum_I \left[\frac{\partial \mathcal{L}}{\partial \varphi_I} - \sum_\alpha \frac{d}{dx^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \right) \right] \delta\varphi_I$$

$$+ \sum_I \sum_\alpha \frac{d}{dx^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \delta\varphi_I \right)$$

$$= \sum_I \mathcal{E}_I \delta\varphi_I + \sum_\alpha \frac{d}{dx^\alpha}\Big(\underbrace{\sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \delta\varphi_I}_{\delta V^\alpha} \Big)$$

Exer 10.4:

$$\mathcal{L}' = \mathcal{L} + \sum_\alpha \frac{dV^\alpha}{dx^\alpha}$$

where $V^\alpha = V^\alpha(\varphi)$

Follows that

$$\mathcal{E}\left(\sum_\alpha \frac{dV^\alpha}{dx^\alpha}\right) = \sum_\alpha \mathcal{E}\left(\frac{dV^\alpha}{dx^\alpha}\right)$$

$$= \sum_\alpha \left[\frac{\partial}{\partial\varphi}\left(\frac{dV^\alpha}{dx^\alpha}\right) - \sum_\beta \frac{d}{dx^\beta}\left(\frac{\partial}{\partial\varphi_{,\beta}}\left(\frac{dV^\alpha}{dx^\alpha}\right)\right)\right]$$

Now: $$\frac{dV^\alpha}{dx^\alpha} = \frac{\partial V^\alpha}{\partial\varphi}\varphi_{,\alpha} + \underbrace{\cancel{\frac{\partial V^\alpha}{\partial x^\alpha}}}_{0}$$ no explicit dependence

$$\rightarrow \frac{\partial}{\partial\varphi}\left(\frac{dV^\alpha}{dx^\alpha}\right) = \frac{\partial^2 V^\alpha}{\partial\varphi^2}\varphi_{,\alpha}$$

$$\frac{\partial}{\partial\varphi_{,\beta}}\left(\frac{dV^\alpha}{dx^\alpha}\right) = \frac{\partial V^\alpha}{\partial\varphi}\delta_\alpha^\beta$$

Thus,

$$\mathcal{E}\left(\sum_\alpha \frac{dV^\alpha}{dx^\alpha}\right) = \sum_\alpha\left[\frac{\partial^2 V^\alpha}{\partial\varphi^2}\varphi_{,\alpha} - \sum_\beta \frac{d}{dx^\beta}\left(\frac{\partial V^\alpha}{\partial\varphi}\delta_\alpha^\beta\right)\right]$$

$$= \sum_\alpha\left[\frac{\partial^2 V^\alpha}{\partial\varphi^2}\varphi_{,\alpha} - \frac{\partial^2 V^\alpha}{\partial\varphi^2}\varphi_{,\alpha}\right]$$

$$= 0$$

~~Thu~~ $\rightarrow \boxed{\mathcal{E}(\mathcal{L}') = \mathcal{E}(\mathcal{L})}$

(10.5)

Exercise: Hamiltonian density for KG equation

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2} y_{,t}^2 - y_{,x}^2 - \mu^2 y^2\right]$$

$$\mathcal{H} = (\pi \dot{y} - \mathcal{L})\Big|_{\dot{y} = \dot{y}(y, y_{,x}, \pi, x, t)}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{1}{c^2} y_{,t} \quad \rightarrow \quad \dot{y} = c^2 \pi$$

Then,
$$\mathcal{H} = \pi(c^2\pi) - \frac{1}{2}\left[\frac{1}{c^2}(c^2\pi)^2 - y_{,x}^2 - \mu^2 y^2\right]$$

$$= c^2\pi^2 - \frac{1}{2}c^2\pi^2 + \frac{1}{2}y_{,x}^2 + \frac{1}{2}\mu^2 y^2$$

$$= \frac{1}{2}c^2\pi^2 + \frac{1}{2}(y_{,x}^2 + \mu^2 y^2)$$

(10.6)

Exercise:

For multiple fields

$$S[\varphi_I, \pi_I] = \int_{t_1}^{t_2} dt \int_V d^3x \left[\sum_I \pi_I \dot{\varphi}_I - \mathcal{H} \right]$$

Vary π_I:

$$\delta S = \int_{t_1}^{t_2} dt \int_V d^3x \left[\dot{\varphi}_I - \frac{\partial \mathcal{H}}{\partial \pi_I} \right] \delta \pi_I$$

$\delta S = 0 \quad \forall\, \delta \pi_I$

$$\rightarrow \boxed{\dot{\varphi}_I = \frac{\partial \mathcal{H}}{\partial \pi_I}}$$

Vary φ_I:

$$\delta S = \int_{t_1}^{t_2} dt \int_V d^3x \Big[\pi_I \underbrace{\delta \dot{\varphi}_I}_{\frac{\partial}{\partial t} \delta \varphi_I} - \frac{\partial \mathcal{H}}{\partial \varphi_I} \delta \varphi_I - \sum_i \frac{\partial \mathcal{H}}{\partial \varphi_{I,i}} \underbrace{\delta \varphi_{I,i}}_{\frac{\partial}{\partial x^i} \delta \varphi_I} \Big]$$

$$= \int_{t_1}^{t_2} dt \int_V d^3x \left[-\frac{d}{dt}(\pi_I) - \frac{\partial \mathcal{H}}{\partial \varphi_I} + \sum_i \frac{d}{dx^i}\left(\frac{\partial \mathcal{H}}{\partial \varphi_{I,i}} \right) \right] \delta \varphi_I$$

$+ \underbrace{\cancel{\text{Boundary terms}}}_{0}$

so $\delta S = 0 \quad \forall\, \delta \varphi_I$

$$\rightarrow \boxed{\dot{\pi}_I = -\frac{\partial \mathcal{H}}{\partial \varphi_I} + \sum_i \frac{d}{dx^i}\left(\frac{\partial \mathcal{H}}{\partial \varphi_{I,i}} \right)}$$

10.7

Exercise:

$$\mathcal{H}(\varphi_I, \varphi_{I,i}, \pi_I, x^i; t) = \left(\sum_I \pi_I \dot{\varphi}_I - \mathcal{L} \right) \Big|_{\dot{\varphi} = \dot{\varphi}(\varphi, \varphi_{,i}, \pi, x^i; t)}$$

$\mathcal{L} = \mathcal{L}(\varphi_I, \varphi_{I,i}, \dot{\varphi}_I, x^i; t)$

$$\boxed{\frac{\partial \mathcal{H}}{\partial \varphi_I}} = \sum_J \pi_J \frac{\partial \dot{\varphi}_J}{\partial \varphi_I} - \frac{\partial \mathcal{L}}{\partial \varphi_I} - \sum_J \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_J}}_{\pi_J} \frac{\partial \dot{\varphi}_J}{\partial \varphi_I}$$

$$= \cancel{\sum_J \pi_J \frac{\partial \dot{\varphi}_J}{\partial \varphi_I}} - \frac{\partial \mathcal{L}}{\partial \varphi_I} - \cancel{\sum_J \pi_J \frac{\partial \dot{\varphi}_J}{\partial \varphi_I}}$$

$$= \boxed{-\frac{\partial \mathcal{L}}{\partial \varphi_I}}$$

$$\boxed{\frac{\partial \mathcal{H}}{\partial \varphi_{I,i}}} = \cancel{\sum_J \pi_J \frac{\partial \dot{\varphi}_J}{\partial \varphi_{I,i}}} - \frac{\partial \mathcal{L}}{\partial \varphi_{I,i}} - \cancel{\sum_J \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_J}}_{\pi_J} \frac{\partial \dot{\varphi}_J}{\partial \varphi_{I,i}}}$$

$$= \boxed{-\frac{\partial \mathcal{L}}{\partial \varphi_{I,i}}}$$

$$\boxed{\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}}$$ (just want explicit dependence on t)

Recall: $$H(q,p,t) = \left(\sum_a p_a \dot{q}^a - L \right) \Big|_{\dot{q} = \dot{q}(q,p,t)}$$

$$\frac{\partial H}{\partial q^a} = \cancel{\sum_b p_b \frac{\partial \dot{q}^b}{\partial q^a}} - \frac{\partial L}{\partial q^a} - \cancel{\sum_b \underbrace{\frac{\partial L}{\partial \dot{q}^b}}_{p_b} \frac{\partial \dot{q}^b}{\partial q^a}}$$

$$= -\frac{\partial L}{\partial q^a}$$

10.8

Exercise: KG

$$\mathcal{H} = \frac{1}{2}c^2\pi^2 + \frac{1}{2}\left(y_{,x}^2 + \mu^2 y^2\right)$$

Hamilton's equations:

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial \pi} = c^2\pi$$

$$\dot{\pi} = -\frac{\partial \mathcal{H}}{\partial y} + \frac{d}{dx}\left(\frac{\partial \mathcal{H}}{\partial y_{,x}}\right)$$

$$= -\mu^2 y + \frac{d}{dx}\left(y_{,x}\right)$$

$$= -\mu^2 y + y_{,xx}$$

so $\boxed{\dot{y} = c^2\pi, \quad \dot{\pi} = -\mu^2 y + y_{,xx}}$

Recover 2nd order equation:

$$\ddot{y} = c^2\dot{\pi}$$

$$= c^2\left(-\mu^2 y + y_{,xx}\right)$$

$$\frac{1}{c^2}\ddot{y} = -\mu^2 y + y_{,xx}$$

$$\mu^2 y = y_{,xx} - \frac{1}{c^2}\ddot{y}$$

$$\boxed{\mu^2 y = \Box y}$$

~~[illegible]~~ $$H = \int_V dx\, \mathcal{H}$$

$$= \int_V dx\ \frac{1}{2}\left[c^2\pi^2 + y_{,x}^2 + \mu^2 y^2\right]$$

$\underline{\delta\pi}$:

$$\delta H = \int_V dx\ c^2\pi\,\delta\pi$$

$$\rightarrow \quad \frac{\delta H}{\delta\pi} = c^2\pi$$

Thus, $\boxed{\dot{y} = \frac{\delta H}{\delta\pi} = c^2\pi}$ (which agrees with earlier calculation)

$\underline{\delta y}$:

$$\delta H = \int_V dx\left[\ y_{,x}\,\delta y_{,x} + \mu^2 y\,\delta y\right]$$

$$= \int_V dx\left[-y_{,xx} + \mu^2 y\right]\delta y + \cancel{y_{,x}\,\delta y}\Big|_{\partial V}$$

$$\rightarrow \quad \frac{\delta H}{\delta y} = -y_{,xx} + \mu^2 y$$

Thus, $\boxed{\dot{\pi} = -\frac{\delta H}{\delta y} = y_{,xx} - \mu^2 y}$ (again agreed)

(10.9)

Exercise: Rewrite continuity equation in terms of $\mathcal{L}$

$$0 = \frac{d\mathcal{H}}{dt} + \sum_i \frac{d}{dx^i}\left(\sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,i}}\,\dot{\varphi}_I\right)$$

Use:

$$\mathcal{H} = \left(\sum_I \pi_I \dot{\varphi}_I - \mathcal{L}\right)\Bigg|_{\dot{\varphi} = \dot{\varphi}(\varphi,\, \varphi_{,i},\, x^i,\, t)}$$

~~NOTES~~

Check:

$$\sum_\beta \frac{d}{dx^\beta}\left(\sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,\beta}}\,\dot{\varphi}_I - \delta_{t\beta}\mathcal{L}\right)$$

$$= \frac{d}{dt}\Big(\sum_I \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_I}}_{\pi_I}\,\dot{\varphi}_I - \mathcal{L}\Big) + \sum_i \frac{d}{dx^i}\left(\sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,i}}\,\dot{\varphi}_I\right)$$

$$= \frac{d\mathcal{H}}{dt} + \sum_i \frac{d}{dx^i}\left(\sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,i}}\,\dot{\varphi}_I\right)$$

$$= 0$$

Thus, continuity equation can be written as

$$\boxed{\sum_\beta \frac{d}{dx^\beta}\left(\sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,\beta}}\,\dot{\varphi}_I - \delta_{t\beta}\mathcal{L}\right) = 0}$$

$\boxed{\text{Example}}$ (10.7)

KG field, time translation

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}\varphi_{,t}^2 - (\vec{\nabla}\varphi)(\vec{\nabla}\varphi) - \mu^2\varphi^2 \right]$$

Time translation:

$$\varphi(x) \to \varphi(\vec{r}, t+\epsilon) = \varphi(\vec{r}, t) + \epsilon \frac{\partial\varphi}{\partial t}(\vec{r}, t) + \ldots$$

Thus, $\left.\frac{d\varphi(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \frac{\partial\varphi}{\partial t} \equiv \varphi_{,t}$

$$\left.\frac{d\mathcal{L}}{d\epsilon}\right|_{\epsilon=0} = \frac{\partial\mathcal{L}}{\partial\varphi}\left.\frac{d\varphi}{d\epsilon}\right|_{\epsilon=0} + \frac{\partial\mathcal{L}}{\partial\varphi_{,t}}\left.\frac{d}{d\epsilon}(\varphi_{,t})\right|_{\epsilon=0} + \frac{\partial\mathcal{L}}{\partial\varphi_{,i}}\left.\frac{d}{d\epsilon}(\varphi_{,i})\right|_{\epsilon=0}$$

$$= -\mu^2\varphi\varphi_{,t} + \frac{1}{c^2}\varphi_{,t}\varphi_{,tt} - \varphi_{,i}\varphi_{,it}$$

$$= \frac{d\mathcal{L}}{dt}$$

NOTE: $\frac{d\mathcal{L}}{dt} = \cancel{\frac{\partial\mathcal{L}}{\partial t}}^{\,0} + \frac{\partial\mathcal{L}}{\partial\varphi}\varphi_{,t} + \frac{\partial\mathcal{L}}{\partial\varphi_{,t}}\varphi_{,tt} + \frac{\partial\mathcal{L}}{\partial\varphi_{,i}}\varphi_{,it}$

$$= -\mu^2\varphi\varphi_{,t} + \frac{1}{c^2}\varphi_{,t}\varphi_{,tt} - \varphi_{,i}\varphi_{,it}$$

$$\boxed{\text{Thus, } \left.\frac{d\mathcal{L}}{d\epsilon}\right|_{\epsilon=0} = \frac{dW^\alpha}{dx^\alpha}, \quad W^\alpha = (\mathcal{L}, \vec{0}) = \delta^\alpha_t\mathcal{L}}$$

$$\to J^\alpha = \frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\left.\frac{d\varphi}{d\epsilon}\right|_{\epsilon=0} - \delta^\alpha_t\mathcal{L}$$

$$= \frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\varphi_{,t} - \delta^\alpha_t\mathcal{L}$$

$$J^0 = \frac{\partial \mathcal{L}}{\partial \varphi_{,t}} \varphi_{,t} - \mathcal{L}$$

$$= \frac{1}{c^2} \varphi_{,t}^2 - \frac{1}{2}\left[\frac{1}{c^2} \varphi_{,t}^2 - \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi - \mu^2 \varphi^2 \right]$$

$$= \frac{1}{2}\left[\frac{1}{c^2} \varphi_{,t}^2 + \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi + \mu^2 \varphi^2 \right]$$

$$J^i = \frac{\partial \mathcal{L}}{\partial \varphi_{,i}} \varphi_{,t}$$

$$= - \varphi_{,i}\, \varphi_{,t}$$

$$Q = \int_{\text{all space}} dV \, J^0$$

$\boxed{\text{Example}}$ 10.8 KG field, spatial translation

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}\varphi_{,t}^2 - \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi - \mu^2\varphi^2\right]$$

Spatial translation:

$$\varphi(\vec{r},t) \to \varphi(x^i+\epsilon n^i, t) = \varphi(\vec{r},t) + \epsilon n^i \varphi_{,i}(\vec{r},t)$$
$$= \varphi(\vec{r},t) + \epsilon\,\vec{n}\cdot\vec{\nabla}\varphi(\vec{r},t)$$

Thus, $$\left.\frac{d\varphi}{d\epsilon}\right|_{\epsilon=0} = \vec{n}\cdot\vec{\nabla}\varphi$$

$$\left.\frac{d\mathcal{L}}{d\epsilon}\right|_{\epsilon=0} = \frac{\partial\mathcal{L}}{\partial\varphi}\vec{n}\cdot\vec{\nabla}\varphi + \frac{\partial\mathcal{L}}{\partial\varphi_{,t}}\vec{n}\cdot\vec{\nabla}\varphi_{,t} + \frac{\partial\mathcal{L}}{\partial\varphi_{,i}}\vec{n}\cdot\vec{\nabla}\varphi_{,i}$$
$$= \vec{n}\cdot\left(\frac{\partial\mathcal{L}}{\partial\varphi}\vec{\nabla}\varphi + \frac{\partial\mathcal{L}}{\partial\varphi_{,t}}\vec{\nabla}\varphi_{,t} + \frac{\partial\mathcal{L}}{\partial\varphi_{,i}}\vec{\nabla}\varphi_{,i}\right)$$
$$= \vec{n}\cdot\vec{\nabla}\mathcal{L}$$
$$= \vec{\nabla}\cdot(\vec{n}\mathcal{L})$$
$$= \frac{dW^\alpha}{dx^\alpha} \quad \text{where} \quad W^\alpha = (0, \vec{n}\mathcal{L})$$

Thus, $$J^\alpha = \frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\left.\frac{d\varphi}{d\epsilon}\right|_{\epsilon=0} - W^\alpha$$
$$= \frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\vec{n}\cdot\vec{\nabla}\varphi - (0, \vec{n}\mathcal{L})$$

NOTE:

$$\delta\varphi = \epsilon\,\vec{n}\cdot\vec{\nabla}\varphi$$

$$\delta\varphi_{,t} \sim \frac{\partial}{\partial t}(\delta\varphi) = \epsilon\frac{\partial}{\partial t}(\vec{n}\cdot\vec{\nabla}\varphi) = \epsilon\,\vec{n}\cdot\vec{\nabla}\varphi_{,t}$$

$$\delta\varphi_{,i} \sim \frac{\partial}{\partial x^i}(\delta\varphi) = \epsilon\frac{\partial}{\partial x^i}(\vec{n}\cdot\vec{\nabla}\varphi) = \epsilon\,\vec{n}\cdot\vec{\nabla}\varphi_{,i}$$

$$J^\alpha = \sum_i n^i \left(\frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}} \varphi_{,i} - \delta_i^\alpha \mathcal{L} \right) = \sum_i n^i T_{i\alpha}$$

$$J^0 = \frac{\partial \mathcal{L}}{\partial \varphi_{,t}} \vec{n} \cdot \vec{\nabla} \varphi = \frac{1}{c^2} \varphi_{,t} \vec{n} \cdot \vec{\nabla} \varphi$$

$$J^j = \frac{\partial \mathcal{L}}{\partial \varphi_{,j}} \vec{n} \cdot \vec{\nabla} \varphi - n^j \mathcal{L}$$

$$Q = \int_{\text{all space}} dV \, J^0$$

$$= \sum_i n^i \int_{\text{all space}} dV \frac{\partial \mathcal{L}}{\partial \varphi_{,t}} \varphi_{,i}$$

$$= \sum_i n^i \int_{\text{all space}} dV \frac{1}{c^2} \varphi_{,t} \varphi_{,i}$$

$$= \frac{1}{c^2} \int_{\text{all space}} dV \, \varphi_{,t} \vec{n} \cdot \vec{\nabla} \varphi .$$

$$J^0 = \frac{1}{c^2} \varphi_{,t} \vec{n} \cdot \vec{\nabla} \varphi$$

$$J^j = -\varphi_{,j} \vec{n} \cdot \vec{\nabla} \varphi - n^j \frac{1}{2} \left(\frac{1}{c^2} \varphi_{,t}^2 - \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi - \mu^2 \varphi^2 \right)$$

Exercise (10.10) KG field, rotation

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}\varphi_{,t}^2 - \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi - \mu^2\varphi^2\right]$$

Rotation:

$$\vec{r} \rightarrow \vec{r} + \lambda\,\hat{n}\times\vec{r}$$
$$x^i + \lambda\,\epsilon^{ijk} n_j x_k$$

$$\varphi(\vec{r},t) \rightarrow \varphi(\vec{r}+\lambda\hat{n}\times\vec{r},t) = \varphi(\vec{r},t) + \lambda(\hat{n}\times\vec{r})\cdot\vec{\nabla}\varphi(\vec{r},t)$$

Thus, $\left(\frac{d\varphi}{d\lambda}\right)_{\lambda=0} = (\hat{n}\times\vec{r})\cdot\vec{\nabla}\varphi$

$$\left.\frac{d\mathcal{L}}{d\lambda}\right|_{\lambda=0} = \frac{\partial\mathcal{L}}{\partial\varphi}\left.\frac{d\varphi}{d\lambda}\right|_{\lambda=0} + \frac{\partial\mathcal{L}}{\partial\varphi_{,t}}\left.\frac{d\varphi_{,t}}{d\lambda}\right|_{\lambda=0} + \frac{\partial\mathcal{L}}{\partial\varphi_{,i}}\left.\frac{d\varphi_{,i}}{d\lambda}\right|_{\lambda=0}$$

$$= \frac{\partial\mathcal{L}}{\partial\varphi}(\hat{n}\times\vec{r})\cdot\vec{\nabla}\varphi + \frac{\partial\mathcal{L}}{\partial\varphi_{,t}}(\hat{n}\times\vec{r})\cdot\vec{\nabla}\varphi_{,t} + \frac{\partial\mathcal{L}}{\partial\varphi_{,i}}(\hat{n}\times\vec{r})\cdot\vec{\nabla}\varphi_{,i}$$

$$= (\hat{n}\times\vec{r})\cdot\vec{\nabla}\mathcal{L}$$
$$= \vec{\nabla}\cdot\left((\hat{n}\times\vec{r})\mathcal{L}\right) - \mathcal{L}\,\underbrace{\vec{\nabla}\cdot(\hat{n}\times\vec{r})}_{0}$$
$$= \vec{\nabla}\cdot\left((\hat{n}\times\vec{r})\mathcal{L}\right) = \frac{dW^\alpha}{dx^\alpha} \quad \text{where } W^\alpha = (0,(\hat{n}\times\vec{r})\mathcal{L})$$

$$\vec{\nabla}\cdot(\hat{n}\times\vec{r}) = \partial_i(\epsilon^{ijk} n_j x_k)$$
$$= \epsilon^{ijk} n_j \underbrace{\partial_i x_k}_{\delta_{ik}} = 0 \quad (\text{since } \epsilon^{ijk}\delta_{ik} = 0$$

$$j^\alpha = \sum_i (\hat{n}\times\vec{r})^i \left(\frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}} \varphi_{,i} - \delta_i^\alpha \mathcal{L} \right)$$

$$= \sum_i (\hat{n}\times\vec{r})^i \, T_{i\alpha}$$

$$J^0 = \frac{\partial \mathcal{L}}{\partial \varphi_{,t}} (\hat{n}\times\vec{r}) \cdot \vec{\nabla}\varphi = \frac{1}{c^2} \varphi_{,t} \, (\hat{n}\times\vec{r}) \cdot \vec{\nabla}\varphi$$

$$J^j = \frac{\partial \mathcal{L}}{\partial \varphi_{,j}} (\hat{n}\times\vec{r}) \cdot \vec{\nabla}\varphi - (\hat{n}\times\vec{r})^j$$

<u>Recall</u>: $T_{i0} = -p_i = \left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) \varphi_{,i}$

$(\vec{A}\times\vec{B})\cdot\vec{C}$
$= (\vec{B}\times\vec{C})\cdot\vec{A}$

The $J^0 = \sum_i (\hat{n}\times\vec{r})^i \, T_{i0}$

$$= -\sum_i (\hat{n}\times\vec{r})^i p_i$$

$$= -\epsilon^{ijk} n_j x_k p_i$$

$$= -\epsilon^{jki} n_j x_k p_i$$

$$= -\hat{n}\cdot(\vec{r}\times\vec{p})$$

$$Q = \int_{\text{all space}} dV \, \overbrace{\sum_i (\hat{n}\times\vec{r})^i \, T_{i0}}^{J^0} = -\int_{\text{all space}} dV \; \hat{n}\cdot(\vec{r}\times\vec{p})$$

$$= -\hat{n}\cdot\left(\int_{\text{all space}} dV \; \vec{\ell}\right)$$

$$= -\hat{n}\cdot(\text{Total angular momentum})$$

10.9 EXAMPLE 10.9

Problem: complex-valued FTG

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}|\varphi_{,t}|^2 - |\varphi_{,x}|^2 - \mu^2|\varphi|^2\right]$$

a) $\delta\varphi$:
$$0 = \frac{\partial \mathcal{L}}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{,x}}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{,t}}\right)$$
$$= \left(-\mu^2\varphi^* - \frac{d}{dx}\left(-\varphi^*_{,x}\right) - \frac{d}{dt}\left(\frac{1}{c^2}\varphi^*_{,t}\right)\right)\frac{1}{2}$$
$$= -\mu^2\varphi^* + \varphi^*_{,xx} - \frac{1}{c^2}\varphi^*_{,tt}$$
$$= -\mu^2\varphi^* + \Box\varphi^*$$

so $\boxed{\Box\varphi^* = \mu^2\varphi^*}$

$\delta\varphi^*$:
$$0 = \frac{\partial \mathcal{L}}{\partial \varphi^*} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \varphi^*_{,x}}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \varphi^*_{,t}}\right)$$
$$\vdots$$
$$\boxed{\Box\varphi = \mu^2\varphi}$$

complex conjugates

b) obviously $|\varphi|^2 = \varphi\varphi^*$ is invariant under $\varphi \to \varphi' = e^{i\lambda}\varphi$ since

$$|\varphi'|^2 = \cancel{e^{i\lambda}}\varphi\,\cancel{e^{-i\lambda}}\varphi^*$$
$$= |\varphi|^2$$

c) $J_a^{\alpha} = \frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}} (i\varphi) + \frac{\partial \mathcal{L}}{\partial \varphi^*_{,\alpha}} (-i\varphi^*)$

$\delta\varphi = \Phi_a \delta\omega^a$

$\varphi' = \varphi e^{i\lambda}$
$= \varphi(1 + i\lambda)$
$= \varphi + \delta\varphi$

so $\delta\varphi = i\varphi$, $\delta\varphi^* = -i\varphi^*$

$\left(\frac{d}{d\lambda} \varphi_\lambda\right)\Big|_{\lambda=0}$
$= \frac{d}{d\lambda}(\varphi e^{i\lambda})\Big|_{\lambda=0}$
$= i\varphi$

Now:

$\frac{\partial \mathcal{L}}{\partial \varphi_{,t}} = \frac{1}{2}\frac{1}{c^2} \varphi^*_{,t}$

$\frac{\partial \mathcal{L}}{\partial \varphi_{,x}} = -\frac{1}{2} \varphi^*_{,x}$

$\frac{\partial \mathcal{L}}{\partial \varphi^*_{,t}} = \frac{1}{2}\frac{1}{c^2} \varphi_{,t}$

$\frac{\partial \mathcal{L}}{\partial \varphi^*_{,x}} = -\frac{1}{2} \varphi_{,x}$

$\rightarrow J_a^t = \frac{1}{2}\frac{1}{c^2} \varphi^*_{,t} i\varphi + \frac{1}{2}\frac{1}{c^2} \varphi_{,t}(-i\varphi^*)$

$= \frac{1}{2}\frac{1}{c^2} i(\varphi^*_{,t}\varphi - \varphi_{,t}\varphi^*)$

$$\boxed{J_a^t = \frac{i}{2c^2}\left(\varphi^*_{,t}\varphi - \varphi_{,t}\varphi^*\right)}$$

$$\boxed{J_a^x = \frac{-i}{2}\left(\varphi^*_{,x}\varphi - \varphi_{,x}\varphi^*\right)}$$

d) Check $\partial_\alpha J^\alpha = 0$ $\Box$

$$\partial_\alpha J^\alpha = \partial_t J^t + \partial_x J^x$$

$$= \frac{i}{2c^2}\left(\varphi^*_{,tt}\varphi + \cancel{\varphi^*_{,t}\varphi_{,t}} - \varphi_{,tt}\varphi^* - \cancel{\varphi_{,t}\varphi^*_{,t}} \right)$$

$$+ \frac{-i}{2}\left(\varphi^*_{,xx}\varphi + \cancel{\varphi^*_{,x}\varphi_{,x}} - \varphi_{,xx}\varphi^* - \cancel{\varphi_{,x}\varphi^*_{,x}} \right)$$

$$= \frac{i}{2}\Big[\left(\frac{1}{c^2}\varphi^*_{,tt} - \varphi^*_{,xx} \right)\varphi$$

$$- \left(\frac{1}{c^2}\varphi_{,tt} - \varphi_{,xx} \right)\varphi^* \Big]$$

$$= \frac{i}{2}\left[-(\Box\varphi^*)\varphi + (\Box\varphi)\varphi^* \right]$$

$$= \frac{i}{2}\left[-\mu^2\varphi^*\varphi + \mu^2\varphi\varphi^* \right]$$

$$= 0$$

e) Conserved charge is integral of J^0.

$$\frac{i}{2c^2}\int_{-\infty}^{\infty} dx \left(\varphi^*\varphi_{,t} - \varphi\varphi^*_{,t} \right) \equiv Q$$

~~DD~~ $\boxed{\text{OLD}}$

Exercise:

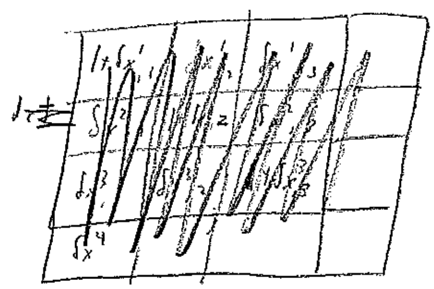
$$d^4x' = \det\left(\frac{\partial x'^\alpha}{\partial x^\beta}\right) d^4x$$

Now:

$$x'^\alpha = x^\alpha + \delta x^\alpha$$

so
$$\frac{\partial x'^\alpha}{\partial x^\beta} = \frac{\partial}{\partial x^\beta}\left[x^\alpha + \delta x^\alpha\right]$$
$$= \frac{\partial x^\alpha}{\partial x^\beta} + \frac{\partial}{\partial x^\beta}(\delta x^\alpha)$$
$$= \delta^\alpha_\beta + \delta x^\alpha{}_{,\beta}$$

Thus
$$\det\left(\frac{\partial x'^\alpha}{\partial x^\beta}\right) = \det\left(\delta^\alpha_\beta + \delta x^\alpha{}_{,\beta}\right)$$

$=$  ~~(3×3 example)~~

$$= \epsilon^{\alpha\beta\gamma\delta}\, T_{0\alpha}\, T_{1\beta}\, T_{2\gamma}\, T_{3\delta}$$
$$= \epsilon^{\alpha\beta\gamma\delta}\left(\delta^0_\alpha + \delta x^0{}_{,\alpha}\right)\left(\delta^1_\beta + \delta x^1{}_{,\beta}\right)\left(\delta^2_\gamma + \delta x^2{}_{,\gamma}\right)\left(\delta^3_\delta + \delta x^3{}_{,\delta}\right)$$
$$= 1 + \underbrace{O(1)}_{\text{linear in } \delta x^\alpha{}_{,\beta}} + \underbrace{\cdots}_{\text{ignore higher-order terms}}$$

~~$1 + \delta x^\alpha{}_{,\beta}$~~

$$= 1 + \left(\delta x^0{}_{,\alpha}\,\epsilon^{\alpha 123}{}_{123} + \delta x^1{}_{,\beta}\,\epsilon^{0\beta 23} + \delta x^2{}_{,\gamma}\,\epsilon^{01\gamma 3} + \delta x^3{}_{,\delta}\,\epsilon^{012\delta}\right)$$
$$= 1 + \left(\delta x^0{}_{,0} + \delta x^1{}_{,1} + \delta x^2{}_{,2} + \delta x^3{}_{,3}\right)$$
$$= 1 + \sum_\alpha \delta x^\alpha{}_{,\alpha}$$

There~~fore $d(d^4x)$~~ =

Thus,

$$d^4x' = \Big(1 + \sum_\alpha \delta x^\alpha{}_{,\alpha} + \underbrace{2^{nd}\text{ order}}_{\text{ignore}}\Big)\, d^4x$$

$$= d^4x + \Big(\sum_\alpha \delta x^\alpha{}_{,\alpha}\Big) d^4x$$

$$= d^4x + \delta(d^4x)$$

So $\boxed{\delta(d^4x) = \Big(\sum_\alpha \delta x^\alpha{}_{,\alpha}\Big)\, d^4x}$

OLD

Exercise: Verify δS expression

$$\delta S = \iint d^4x \sum_\alpha \frac{d}{dx^\alpha}\left(\sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \underbrace{\delta_0 \varphi_I}_{\text{fields}} + \mathcal{L}\, \delta x^\alpha \right)$$

Now: $\delta_0 \varphi_I = \delta \varphi_I - \sum_\beta \varphi_{I,\beta}\, \delta x^\beta$

$$\rightarrow \delta S = \iint d^4x \sum_\alpha \frac{d}{dx^\alpha}\left[\sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \delta \varphi_I - \sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \left(\sum_\beta \varphi_{I,\beta}\, \delta x^\beta \right) + \mathcal{L}\, \delta x^\alpha \right]$$

Also: $\delta x^\alpha \equiv \sum_a X_a{}^\alpha\, \delta \omega^a$

$\delta \varphi_I \equiv \sum_a \Phi_{Ia}\, \delta \omega^a$

$$\rightarrow \boxed{\delta S} = \iint d^4x \sum_a \sum_\alpha \frac{d}{dx^\alpha}\left[\sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \Phi_{Ia}\, \delta \omega^a - \sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \sum_\beta \varphi_{I,\beta} X_a{}^\beta + \mathcal{L} X_a{}^\alpha \right] \delta \omega^a$$

$$= \iint d^4x \sum_a \sum_\alpha \frac{d}{dx^\alpha}\left[\sum_\beta \left(\mathcal{L}\, \delta_{\alpha\beta} - \sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \varphi_{I,\beta} \right) X_a{}^\beta + \sum_I \frac{\partial \mathcal{L}}{\partial \varphi_{I,\alpha}} \Phi_{Ia} \right] \delta \omega^a$$

(10.1)

Problem: PBs for KGH theory

$$\{F, G\} = \int_V d^3x \left(\frac{\delta F}{\delta \varphi(\vec{r},t)} \frac{\delta G}{\delta \pi(\vec{r},t)} - \frac{\delta F}{\delta \pi(\vec{r},t)} \frac{\delta G}{\delta \varphi(\vec{r},t)} \right)$$

(a) Fundamental PBs:

$$\{\varphi(\vec{r},t), \varphi(\vec{r}',t)\} = \int_V d^3y \left(\underbrace{\frac{\delta \varphi(\vec{r},t)}{\delta \varphi(\vec{y},t)}}_{=\delta(\vec{r}-\vec{y})} \cancelto{0}{\frac{\delta \varphi(\vec{r}',t)}{\delta \pi(\vec{y},t)}} - \cancelto{0}{\frac{\delta \varphi(\vec{r},t)}{\delta \pi(\vec{y},t)}} \underbrace{\frac{\delta \varphi(\vec{r}',t)}{\delta \varphi(\vec{y},t)}}_{=\delta(\vec{r}'-\vec{y})} \right)$$

$$= \boxed{0}$$

$$\{\pi(\vec{r},t), \pi(\vec{r}',t)\} = \boxed{0} \quad \text{since } \frac{\delta \pi}{\delta \varphi} = 0 \text{ in both terms}$$

$$\{\varphi(\vec{r},t), \pi(\vec{r}',t)\} = \int_V d^3y \left(\frac{\delta \varphi(\vec{r},t)}{\delta \varphi(\vec{y},t)} \frac{\delta \pi(\vec{r}',t)}{\delta \pi(\vec{y},t)} - 0 \right)$$

$$= \int_V d^3x \; \delta(\vec{r}-\vec{y})\, \delta(\vec{r}'-\vec{y})$$

$$= \delta(\vec{r}-\vec{r}')$$

(b)
$$\{\varphi(\vec{r},t), H\} = \int_V d^3y \left[\underbrace{\frac{\delta \varphi(\vec{r},t)}{\delta \varphi(\vec{y},t)}}_{\delta(\vec{r},\vec{y})} \frac{\delta H}{\delta \pi(\vec{y},t)} - \cancelto{0}{\frac{\delta \varphi(\vec{r},t)}{\delta \pi(\vec{r},t)}} \frac{\delta H}{\delta \varphi(\vec{y},t)} \right]$$

$$= \frac{\delta H}{\delta \pi(\vec{r},t)} = \dot{\varphi}(\vec{r},t)$$

$$\{\pi(\vec{r},t), H\} = \int d^3y \left[\underbrace{\cancel{\frac{\delta \pi}{\delta \varphi}}}_{0} \frac{\delta H}{\delta \pi} - \underbrace{\frac{\delta \pi(\vec{r},t)}{\delta \pi(\vec{y},t)}}_{\delta(\vec{r}-\vec{y})} \frac{\delta H}{\delta \varphi(\vec{y},t)} \right]$$

$$= -\frac{\delta H}{\delta \varphi(\vec{r},t)}$$

$$= \dot{\pi}(\vec{r},t)$$

Thus, $\{\varphi, H\} = \dot{\varphi}$, $\{\pi, H\} = \dot{\pi}$

(10.2)

0

Problem: KG stress-energy

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}\varphi_{,t}^2 - \varphi_{,x}^2 - \mu^2\varphi^2\right]$$

(a) $$T_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial \varphi_{,\beta}}\varphi_{,\alpha} - \delta_{\alpha\beta}\mathcal{L}$$

$$T_{tt} = \frac{\partial \mathcal{L}}{\partial \varphi_{,t}}\varphi_{,t} - \mathcal{L}$$

$$= \frac{1}{c^2}\varphi_{,t}\varphi_{,t} - \frac{1}{2}\left[\frac{1}{c^2}\varphi_{,t}^2 - \varphi_{,x}^2 - \mu^2\varphi^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{c^2}\varphi_{,t}^2 + \varphi_{,x}^2 + \mu^2\varphi^2\right]$$

$$T_{xx} = \frac{\partial \mathcal{L}}{\partial \varphi_{,x}}\varphi_{,x} - \mathcal{L}$$

$$= -\varphi_{,x}\varphi_{,x} - \frac{1}{2}\left[\frac{1}{c^2}\varphi_{,t}^2 - \varphi_{,x}^2 - \mu^2\varphi^2\right]$$

$$= -\frac{1}{2}\left[\frac{1}{c^2}\varphi_{,t}^2 + \varphi_{,x}^2\right] + \frac{1}{2}\mu^2\varphi^2$$

$$= -\frac{1}{2}\left[\frac{1}{c^2}\varphi_{,t}^2 + \varphi_{,x}^2 - \mu^2\varphi^2\right]$$

$$T_{tx} = \frac{\partial \mathcal{L}}{\partial \varphi_{,x}}\varphi_{,t}$$

$$= -\varphi_{,x}\varphi_{,t}$$

$$T_{xt} = \frac{\partial \mathcal{L}}{\partial \varphi_{,t}}\varphi_{,x}$$

$$= \frac{1}{c^2}\varphi_{,t}\varphi_{,x}$$

(b) Conserved current for time-translation symmetry

Example 10.7

$$J_t^{\alpha} = - T_{t\alpha}$$

so $J_t^{t} = - T_{tt} = \frac{1}{2}\left[\frac{1}{c^2}\varphi_{,t}^2 + \varphi_{,x}^2 + \mu^2\varphi^2\right]$

$$J_t^{x} = - T_{tx} = \varphi_{,x}\varphi_{,t}$$

(c) Similarly

$$J_x^{\alpha} = - T_{x\alpha}$$

so $J_x^{t} = - T_{xt} = -\frac{1}{c^2}\varphi_{,t}\varphi_{,x}$

$$J_x^{x} = - T_{xx} = +\frac{1}{2}\left[\frac{1}{c^2}\varphi_{,t}^2 + \varphi_{,x}^2 - \mu^2\varphi^2\right]$$

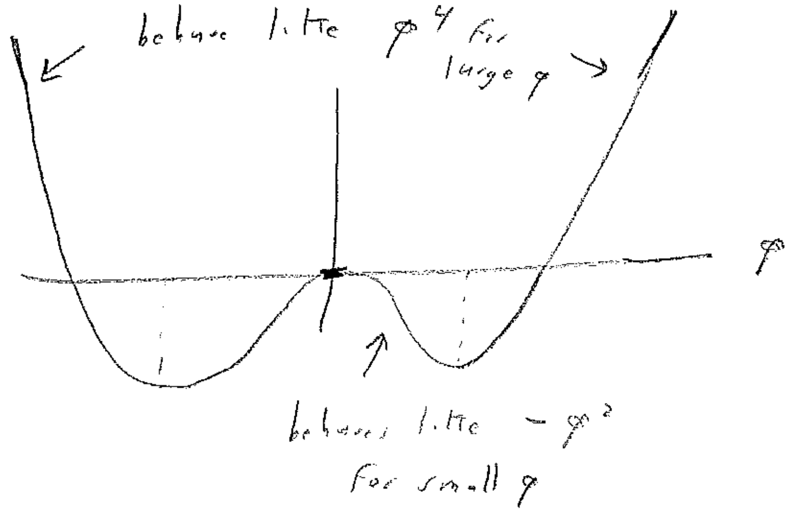
(10.3)

Problem: Self-interacting KG field (real 1-d)

a) $\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}\varphi_{,t}^2 - \varphi_{,x}^2 \boxed{-\mu^2}\varphi^2\right] - V(\varphi)$

Set this to zero ☆

$V(\varphi) = -\frac{1}{2}a^2\varphi^2 + \frac{1}{4}b^2\varphi^4$



Extreme values:

$$0 = \frac{dV}{d\varphi} = -a^2\varphi + b^2\varphi^3$$
$$= -a^2\varphi\left(1 - \frac{b^2}{a^2}\varphi^2\right)$$

Then, $\boxed{\varphi = 0}$, $\boxed{\varphi = \pm\left|\frac{a}{b}\right|}$

b) $$0 = \frac{\partial\mathcal{L}}{\partial\varphi} - \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\varphi_{,x}}\right) - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\varphi_{,t}}\right)$$

$$\frac{\partial\mathcal{L}}{\partial\varphi} = -\mu^2\varphi + a^2\varphi - b^2\varphi^3$$
$$= -(\mu^2 - a^2)\varphi - b^2\varphi^3$$

$$\frac{\partial\mathcal{L}}{\partial\varphi_{,x}} = -\varphi_{,x}\,, \qquad \frac{\partial\mathcal{L}}{\partial\varphi_{,t}} = \frac{1}{c^2}\varphi_{,t}$$

Thus,

$$0 = -(\mu^2 - a^2)\varphi - b^2\varphi^3 + \underbrace{\varphi_{,xx} - \frac{1}{c^2}\varphi_{,tt}}_{\Box^2\varphi}$$

Thus, $$\boxed{\Box^2\varphi = b^2\varphi^3 + (\mu^2 - a^2)\varphi}$$

c) $\varphi = \text{const} \rightarrow \Box\varphi = 0$

Thus $$0 = b^2\varphi^3 + (\mu^2 - a^2)\varphi$$
$$= \varphi\left(b^2\varphi^2 + (\mu^2 - a^2)\right)$$

$$\boxed{\varphi = 0} \qquad \varphi^2 = \frac{a^2 - \mu^2}{b^2}$$

$$\boxed{\varphi = \pm\frac{\sqrt{a^2 - \mu^2}}{|b|}}$$

$$\boxed{(\text{set } \mu = 0)} \rightarrow \boxed{\varphi = \pm\left|\frac{a}{b}\right|}$$

d) $$\mathcal{H} = \left(\Pi\dot{\varphi} - \mathcal{L}\right)\Big|_{\dot{\varphi} = \dot{\varphi}(\text{other quantities})}$$

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \frac{1}{c^2}\varphi_{,t} \rightarrow \dot{\varphi} = c^2\Pi$$

Thus, $$\mathcal{H} = \Pi c^2\Pi - \frac{1}{2}\left[\frac{1}{c^2}(c^2\Pi)^2 - \varphi_{,x}^2\right] + V(\varphi)$$
$$= \frac{1}{2}\left[c^2\Pi^2 + \varphi_{,x}^2\right] + V(\varphi)$$

③

$$H = \int dx \left[\frac{1}{2} \left[c^2 \pi^2 + \varphi_{,x}^2 \right] + V(\varphi) \right]$$

e) $$\delta H = \int dx \left[c^2 \pi \, \delta\pi + \varphi_{,x} \, \delta \varphi_{,x} + \frac{\partial V}{\partial \varphi} \delta\varphi \right]$$

$$= \int dx \left[c^2 \pi \, \delta\pi - \varphi_{,xx} \, \delta\varphi + \frac{\partial V}{\partial \varphi} \delta\varphi \right]$$

$$+ \text{Boundary term} \left(\varphi_{,x} \delta\varphi \right) \Big|_{\infty}$$

ignore

~~δH~~

$$= \int dx \left[c^2 \pi \, \delta\pi + \left(-\varphi_{,xx} + \frac{\partial V}{\partial \varphi} \right) \delta\varphi \right]$$

$\varphi = \text{const} \rightarrow \pi = 0$

$\rightarrow \varphi_{,xx}$

$$= \int dx \left(\frac{\partial V}{\partial \varphi} \right) \delta\varphi$$

$$\uparrow$$

but $\left. \dfrac{\partial V}{\partial \varphi} \right| = 0$ from part (a)

$\varphi = 0, \pm\sqrt{\frac{a}{b}}$

Thus, $\boxed{\delta H = 0}$

(10.4)

<u>Problem:</u> Extend self-interacting KG field to complex ~~...~~ fields by taking

$$V(\varphi) = -\frac{1}{2} a^2 \varphi\varphi^* + \frac{1}{4} b^2 \varphi^2(\varphi^*)^2$$

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}\varphi^*_{,t}\varphi_{,t} - \varphi^*_{,x}\varphi_{,x}\right] - V(\varphi)$$

a) Again $\mathcal{L}$ invariant under $\varphi \to e^{i\lambda}\varphi = \varphi'$

<u>EOM:</u>

$$\frac{\partial \mathcal{L}}{\partial \varphi} = +\frac{1}{2}a^2\varphi^* - \frac{1}{2}b^2\varphi(\varphi^*)^2$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_{,t}} = \frac{1}{2c^2}\varphi^*_{,t}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_{,x}} = -\frac{1}{2}\varphi_{,x}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \varphi} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{,t}}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{,x}}\right)$$

$$= \frac{1}{2}a^2\varphi^* - \frac{1}{2}b^2\varphi(\varphi^*)^2 \underbrace{- \frac{1}{2c^2}\varphi^*_{,tt} + \frac{1}{2}\varphi_{,xx}}_{\Box\varphi^*}$$

Thus, $\Box\varphi^* = -\frac{1}{2}a^2\varphi^*\left(1 - \frac{b^2}{a^2}|\varphi|^2\right)$

similarly $\Box\varphi = -\frac{1}{2}a^2\varphi\left(1 - \frac{b^2}{a^2}|\varphi|^2\right)$

Constant solutions now have

$$\boxed{|\varphi|^2 = \frac{a^2}{b^2}} \, , \quad \boxed{\varphi = 0}$$

$$\rightarrow \boxed{\varphi = \frac{a}{b} e^{i\theta}}$$

(10.5)

Problem: Sound waves in a gas

(a) $$\mathcal{L} = \frac{1}{2}\left(\mu_0 \dot{\eta}^2 - \gamma P_0 (\vec{\nabla}\cdot\vec{\eta})^2\right)$$

where η^i : displacement of gas molecules

μ_0 : mean mass ~~density~~ of gas molecule

P_0 : mean pressure

γ : ratio of specific heats , $\dot{\eta}^2 = \delta^{ij}\dot{\eta}_i\dot{\eta}_j$

$(\vec{\nabla}\cdot\vec{\eta})^2 = (\delta^{ij}\eta_{i,j})^2$

$$0 = \frac{\partial \mathcal{L}}{\partial \eta_i} - \frac{d}{dx^j}\left(\frac{\partial \mathcal{L}}{\partial \eta_{i,j}}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}_i}\right)$$

$$= 0 - \frac{d}{dx^j}\left(-\gamma P_0 (\vec{\nabla}\cdot\vec{\eta})\delta_{ij}\right) - \frac{d}{dt}\left(\mu_0 \dot{\eta}_i\right)$$

$$= \gamma P_0 \frac{\partial}{\partial x^i}(\vec{\nabla}\cdot\vec{\eta}) - \mu_0 \ddot{\eta}_i$$

$$\boxed{\frac{-\mu_0}{\gamma P_0}\ddot{\vec{\eta}} + \vec{\nabla}(\vec{\nabla}\cdot\vec{\eta}) = 0}$$

μ_0

(b) For deviation away from μ_0 $(\mu = \mu_0(1+\sigma))$

$\sigma = -\vec{\nabla}\cdot\vec{\eta}$ (see next page)

6

Take divergence of above equation

$$-\frac{\mu_0}{\gamma P_0}\frac{d^2}{dt^2}\underbrace{(-\vec{\nabla}\cdot\vec{\eta})}_{\sigma} - \nabla^2\underbrace{(\vec{\nabla}\cdot\vec{\eta})}_{-\sigma} = 0$$

$$\boxed{\frac{-\mu_0}{\gamma P_0}\ddot{\sigma} + \nabla^2\sigma = 0}, \quad \boxed{\text{where } C_s = \sqrt{\frac{\gamma P_0}{\mu_0}}}$$

Continuity equation : μ = mass density

$$\frac{\partial \mu}{\partial t} = -\vec{\nabla}\cdot(\mu \vec{v})$$

$$\vec{v} = \frac{\Delta \vec{s}}{\Delta t}$$

Thus, $$\frac{\partial \mu}{\partial t} = -\vec{\nabla}\cdot\left(\mu \frac{\Delta \vec{s}}{\Delta t}\right)$$

$$\rightarrow \quad \Delta \mu = -\vec{\nabla}\cdot(\mu \Delta \vec{s})$$

$$\Delta \vec{s} = \vec{\eta}$$

$$\mu = \mu_0(1+\sigma)$$

$$\mu_0 \sigma = -\vec{\nabla}\cdot(\mu_0 \vec{\eta})$$

$$\boxed{\sigma = -\vec{\nabla}\cdot\vec{\eta}}$$

10.6

Problem: Schrodinger equation

$$\mathcal{L}(\psi, \nabla\psi, \dot{\psi}, \vec{x}, t) = \frac{i\hbar}{2}(\psi^*\dot{\psi} - \psi\dot{\psi}^*) - \frac{\hbar^2}{2m}\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi - U(\vec{r},t)\,\psi^*\psi$$

ψ: complex.

a) Two equations (varying ψ, ψ^* treated independently)

$$0 = \frac{\partial\mathcal{L}}{\partial\psi} - \frac{d}{dx^i}\left(\frac{\partial\mathcal{L}}{\partial\psi_{,i}}\right) - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\right)$$

$$= -\frac{i\hbar}{2}\dot{\psi}^* - U\psi^* - \frac{d}{dx^i}\left(-\frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial x^i}\right) - \frac{d}{dt}\left(\frac{i\hbar\psi^*}{2}\right)$$

$$= -i\hbar\dot{\psi}^* - U\psi^* + \frac{\hbar^2}{2m}\nabla^2\psi^*$$

$$\boxed{\left(-\frac{\hbar^2}{2m}\nabla^2 + U\right)\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t}}$$

$$0 = \frac{\partial\mathcal{L}}{\partial\psi^*} - \frac{d}{dx^i}\left(\frac{\partial\mathcal{L}}{\partial\psi^*_{,i}}\right) - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*}\right)$$

$$= \frac{i\hbar}{2}\dot{\psi} - U\psi - \frac{\partial}{\partial x^i}\left(-\frac{\hbar^2}{2m}\frac{\partial\psi}{\partial x^i}\right) - \frac{d}{dt}\left(\frac{-i\hbar}{2}\psi\right)$$

$$= i\hbar\dot{\psi} - U\psi + \frac{\hbar^2}{2m}\nabla^2\psi$$

$$\boxed{\left(-\frac{\hbar^2}{2m}\nabla^2 + U\right)\psi = i\hbar\frac{\partial\psi}{\partial t}}$$

②

b) Hamiltonian density

$$\mathcal{H} = \sum_I \pi_I \dot{\varphi}_I - \mathcal{L}$$

$$\pi_1 = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{i\hbar}{2}\psi^*$$

$$\pi_2 = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} = -\frac{i\hbar}{2}\psi$$

$$\rightarrow \boxed{\mathcal{H}} = \underbrace{\frac{i\hbar}{2}\psi^*\dot{\psi} - \frac{i\hbar}{2}\psi\dot{\psi}^*}_{\cancel{\frac{i\hbar}{2}(\psi^*\dot{\psi} - \psi\dot{\psi}^*)}} - \Big[\cancel{\frac{i\hbar}{2}(\psi^*\dot{\psi} - \psi\dot{\psi}^*)} - \frac{\hbar^2}{2m}\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi - U(\vec{r},t)\psi^*\psi\Big]$$

$$\boxed{= \frac{\hbar^2}{2m}\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi + U(\vec{r},t)\psi^*\psi}$$

c) Hamiltonian

$$\boxed{H} = \int d^3x\, \mathcal{H}$$

$$= \int d^3x \Big[\frac{\hbar^2}{2m}\underbrace{\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi}_{\vec{\nabla}\cdot(\psi^*\vec{\nabla}\psi) - \psi^*\nabla^2\psi} + U\psi^*\psi\Big]$$

$$= \underbrace{\int_{\partial S} da\, \hat{n}\cdot\vec{\nabla}(\psi^*\vec{\nabla}\psi)}_{\text{assume} = 0} + \int d^3x\, \psi^*\Big[-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi\Big]$$

$$\boxed{= \int d^3x\, \psi^*\hat{H}\psi} = \langle \psi^* | H\psi \rangle$$

d) Conserved current for ~~=~~ wave function given symmetry of $\mathcal{L}$ under $\psi \to \psi' = e^{i\lambda}\psi$

$$\delta\psi = \left(\frac{\partial \psi'}{\partial \lambda}\right)_{\lambda=0}$$

$$= \left(i e^{i\lambda}\psi\right)\Big|_{\lambda=0}$$

$$= i\psi$$

Thus,

$$J_a^{\alpha} = \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\, i\psi + \frac{\partial \mathcal{L}}{\partial \psi^*_{,\alpha}}(-i\psi^*)$$

Now:

$$\frac{\partial \mathcal{L}}{\partial \psi_{,t}} = \frac{i\hbar}{2}\psi^*$$

$$\frac{\partial \mathcal{L}}{\partial \psi_{,i}} = -\frac{\hbar^2}{2m}\psi^*_{,i}$$

Thus,

$$\boxed{J_a^{t}} = \frac{i\hbar}{2}\psi^* i\psi - \frac{i\hbar}{2}\psi(-i\psi^*)$$

$$= -\frac{\hbar}{2}\left(\psi^*\psi + \psi^*\psi\right)$$

$$= -\hbar|\psi|^2$$

and

$$\boxed{J_a^{i}} = -\frac{\hbar^2}{2m}\psi^*_{,i}\, i\psi - \frac{\hbar^2}{2m}\psi_{,i}(-i\psi^*)$$

$$= -\frac{i\hbar^2}{2m}\left(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right)$$

e) Check continuity equation

1^{st} divide everything by $-\hbar$

$$j^t = |\psi|^2 = \psi^* \psi$$

$$\vec{J} = \frac{i\hbar}{2m}\left(\psi \vec{\nabla}\psi^* - \psi^* \vec{\nabla}\psi\right)$$

$$\partial_t j^t + \vec{\nabla}\cdot\vec{J} = \frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t}$$

$$+\frac{i\hbar}{2m}\left(\cancel{\vec{\nabla}\psi\cdot\vec{\nabla}\psi^*} + \psi\nabla^2\psi^* - \cancel{\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi} - \psi^*\nabla^2\psi\right)$$

$$= \psi\left(\frac{i\hbar}{2m}\nabla^2\psi^* + \frac{\partial\psi^*}{\partial t}\right)$$

$$-\psi^*\left(\frac{i\hbar}{2m}\nabla^2\psi - \frac{\partial\psi}{\partial t}\right)$$

multiply by $\frac{-\hbar}{i}$

$$RHS = \psi\left(\frac{-\hbar^2}{2m}\nabla^2\psi^* + i\hbar\frac{\partial\psi^*}{\partial t}\right)$$

$$-\psi^*\left(\frac{-\hbar^2}{2m}\nabla^2\psi - i\hbar\frac{\partial\psi}{\partial t}\right)$$

$$= \psi(-U\psi^*) - \psi^*(-U\psi^*)$$

$$= -U|\psi|^2 + U|\psi|^2$$

$$= 0$$

f) conserved charge

$$Q = \int d^3x \ J_a^{\,t}$$

$$= -t \int_{\text{all space}} d^3x \ |\psi|^2$$

consistent with $|\psi|^2$ being probability density