Rocket motion fuel consumption

(1.1)

In absence of external forces

$$v(t) = v_0 - u \ln\left(\frac{m(t)}{m_0}\right)$$

After all the fuel is used up

$$v_f = v_0 - u \ln\left(\frac{m_f}{m_0}\right)$$

$$\rightarrow \quad \frac{v_f - v_0}{u} = -\ln\left(\frac{m_f}{m_0}\right) =$$

Thus,

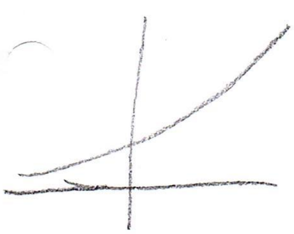
$$\frac{m_f}{m_0} = \exp\left[-\frac{(v_f - v_0)}{u}\right]$$

Fuel used : $M = m_0 - m_f$

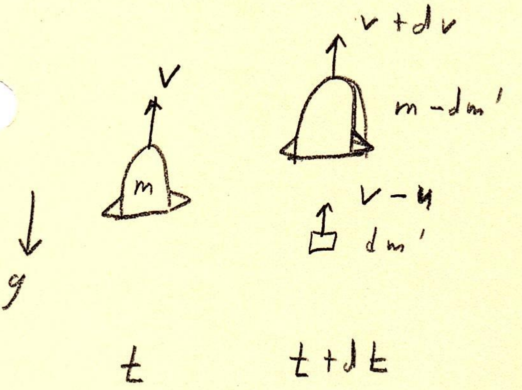
$$= m_0 - m_0 \exp\left[-\frac{(v_f - v_0)}{u}\right]$$

$$= m_0\left[1 - \exp\left[-\frac{(v_f - v_0)}{u}\right]\right]$$

Thus, $\boxed{M = m_0\left[1 - e^{-\frac{(v_f - v_0)}{u}}\right]}$



Rocket motion in a gravitational field (1.2)



$dm' = -dm > 0$

$u > 0$: exhaust speed w.r.t spaceship

$$F = \frac{dp}{dt}$$

$F = -mg$ (applied force)

$$dp = p(t+dt) - p(t)$$
$$= (m - dm')(v + dv) + dm'(v - u) - mv$$
$$= \cancel{mv} - \cancel{v\,dm'} + m\,dv - \underbrace{dm'\,dv}_{O(2)\ -\ \text{ignore}} + \cancel{dm'\,v} - u\,dm' - \cancel{mv}$$
$$= m\,dv - u\,dm'$$
$$= m\,dv + u\,dm$$

Thus, $\dfrac{dp}{dt} = m\dfrac{dv}{dt} + u\dfrac{dm}{dt}$

so $$\boxed{-mg = m\frac{dv}{dt} + u\frac{dm}{dt}}$$

NOTE: $$m\frac{dv}{dt} = -mg - u\frac{dm}{dt}$$
$$= -mg + \boxed{u\frac{dm'}{dt}}$$ — reaction force ~~[illegible]~~ from ejected mass dm'

Assume: $\alpha \equiv -\dfrac{dm}{dt}\ (>0)$ is constant

└ burn rate

Then $dt = -\dfrac{dm}{\alpha}$ $\rightarrow$

$$-mg\,dt = m\,dv + u\,dm$$
$$m\frac{g}{\alpha}\,dm = m\,dv + u\,dm$$
$$dv = \left(\frac{g}{\alpha} - \frac{u}{m}\right)dm$$

Integrate

$$\int_{v_0}^{v} dv = \int_{m_0}^{m} \left(\frac{g}{\alpha} - \frac{u}{m} \right) dm$$

$$\boxed{v - v_0 = \frac{g}{\alpha}(m - m_0) - u \ln\left(\frac{m}{m_0}\right)}$$

<u>Now</u>: $\alpha = -\frac{dm}{dt} \rightarrow dm = -\alpha\, dt$

$$\int_{m_0}^{m} dm = -\alpha \int_0^t dt$$

$$m - m_0 = -\alpha t$$

so $\boxed{m(t) = m_0 - \alpha t}$

$$\rightarrow \boxed{v - v_0 = -gt - u \ln\left(1 - \frac{\alpha t}{m_0}\right)}$$

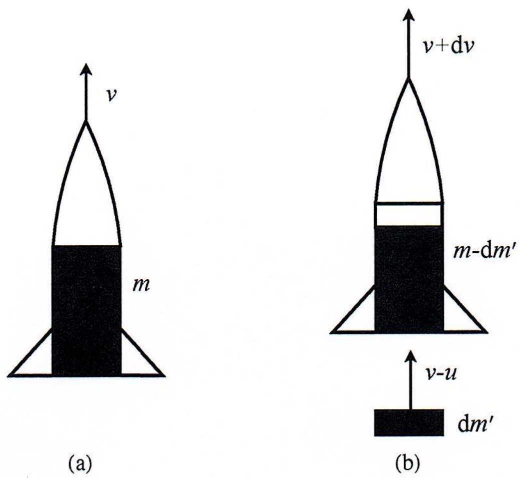


Fig. 1.1 A rocket moving in interstellar space, free of all external forces. Panel (a): Rocket at time t (mass m, velocity v). Panel (b): Rocket and exhaust at time $t+dt$ (mass $m-dm'$, velocity $v+dv$; mass dm', velocity $v-u$).

will have changed its velocity to $v+dv$. We will assume that the exhaust gases dm' exit the rocket with *constant* velocity, $-u$, with respect to the rocket, so that with respect to the fixed inertial frame, the exhaust gases are moving with velocity $v-u$. The change in the momentum of the system over the time interval t to $t+dt$ is then

$$\begin{aligned} dp &= p(t+dt) - p(t) \\ &= \left[(m-dm')(v+dv) + dm'(v-u)\right] - mv \\ &= m\,dv - u\,dm' \\ &= m\,dv + u\,dm\,, \end{aligned} \tag{1.3}$$

where we ignored the $-dm'\,dv$ term (since it is second-order small) to get the third line, and switched back to dm to get the last line. Since there are no external forces acting on the system, $dp/dt = F = 0$, which implies

$$0 = m\,dv + u\,dm\,, \tag{1.4}$$

or, equivalently,

$$dv = -u\frac{dm}{m}\,. \tag{1.5}$$

This is a separable differential equation, which can be immediately integrated, subject to the boundary (or initial) condition that $v = v_0$ when $m = m_0$:

$$v - v_0 = -u\ln(m/m_0)\,. \tag{1.6}$$

Galilean transformation: (1.3)

$$\vec{r}\,'(t) = \vec{r}(t) - u\,t$$

Follows that

$$m\ddot{\vec{r}}\,' = m\Big(\ddot{\vec{r}} - \underbrace{\frac{d^2}{dt^2}(ut)}_{=0}\Big)$$

$$= m\ddot{\vec{r}}$$

$$= \vec{F}$$

So Newton's 2nd law is invariant under such a transformation

Work done by velocity-dependent force

1.4

$$W_{12} = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{s} \quad , \quad \vec{F} = -b\vec{v}$$

a) For motion

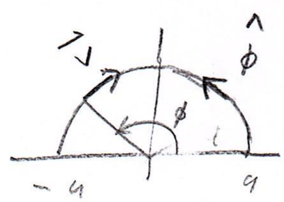
with const speed

$$W_{12} = \int_{-a}^{a} -bv\, dx \quad , \qquad v = \text{const}$$

$$= -bv \int_{-a}^{a} dx$$

$$= -2abv$$

b) For motion



with const speed

$$W_{12} = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{s} \quad , \qquad d\vec{s} = a\, d\phi\, \hat{\phi}$$

$$\vec{v} = -v\,\hat{\phi} \quad \text{where } v > 0$$

$$= \int_{\phi=\pi}^{0} +bv\,\hat{\phi}\cdot a\, d\phi\, \hat{\phi}$$

$$= +abv \int_{\pi}^{0} d\phi$$

$$= -\pi abv$$

NOTE: Force does more work along second path as distance is larger

Work done by quadratic drag force (1n5)

Find work done in moving the particle a distance a along a straight line.

→ 1-d motion: $F = -bv^2$, $d\vec{s} \to dx$ $\quad (\vec{F} = -bv^2\hat{v})$

$$W_{12} = \int_0^a F\,dx = -\int_0^a bv^2\,dx$$

From: $-bv^2 = F = ma = m\dfrac{dv}{dt}$

$$-bv\frac{dx}{\cancel{dt}} = m\frac{dv}{\cancel{dt}}$$

$$\int -\frac{b}{m}\,dx = \int \frac{dv}{v}$$

$$-\frac{b}{m}x = \ln v$$

$$v(x) = v_0 e^{-\frac{b}{m}x}$$

Thus,

$$W_{12} = -\int_0^a bv^2\,dx$$

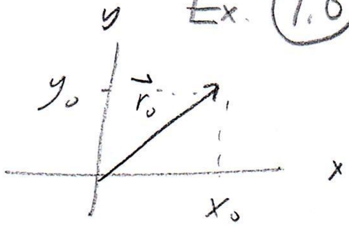
$$= -\int_0^a b v_0^2 e^{-\frac{2b}{m}x}\,dx$$

$$= -bv_0^2 \int_0^a e^{-\frac{2b}{m}x}\,dx$$

$$= -bv_0^2\left(\frac{m}{-2b}\right)\left(e^{-\frac{2ba}{m}} - 1\right)$$

$$= \frac{1}{2}mv_0^2\left(e^{-2ba/m} - 1\right)$$

Ex. 1.6



$$\vec{F} = F\hat{x} \qquad (\text{assume } F = \text{const})$$

$$\vec{F} = m\vec{a} = m\ddot{\vec{r}}$$

(a) $\ddot{x} = \frac{F}{m} \rightarrow x = x_0 + \cancel{v_{0x}}^{\,0}\, t + \frac{1}{2}\left(\frac{F}{m}\right)t^2$

$\ddot{y} = 0 \rightarrow y = y_0 + \cancel{v_{0y}}^{\,0}\, t$

Thus, $x(t) = x_0 + \frac{1}{2}\left(\frac{F}{m}\right)t^2$

$y(t) = y_0$

$$\rightarrow \vec{r}(t) = \left[x_0 + \frac{1}{2}\left(\frac{F}{m}\right)t^2\right]\hat{x} + y_0\hat{y}$$

$$= \vec{r}_0 + \frac{1}{2}\left(\frac{F}{m}\right)t^2\hat{x}$$

Also, $\vec{p}(t) = m\dot{\vec{r}}(t)$

$$= Ft\hat{x}$$

(b) $\vec{l} = \vec{r}\times\vec{p}$

$$= \left[\left(x_0 + \frac{1}{2}\left(\frac{F}{m}\right)t^2\right)\hat{x} + y_0\hat{y}\right]\times Ft\hat{x}$$

$$= -Fty_0\hat{z}$$

$$\frac{d\vec{l}}{dt} = -Fy_0\hat{z}$$

$$\vec{\tau} = \vec{r}\times\vec{F}$$

$$= \left[\left(x_0 + \frac{1}{2}\left(\frac{F}{m}\right)t^2\right)\hat{x} + y_0\hat{y}\right]\times F\hat{x}$$

$$= -Fy_0\hat{z} = \frac{d\vec{l}}{dt}$$

Since $\vec{\tau} \neq 0$, $\frac{d\vec{l}}{dt} \neq 0 \rightarrow$ Ang. momentum not conserved

1c) New coord system so that

$\vec{r}\,' = x_0 \hat{x}$

Solve EoMs for $\vec{F} = F\hat{x}$

$$\rightarrow \quad \vec{r}\,'(t) = \left(x_0 + \frac{1}{2}\left(\frac{F}{m}\right)t^2\right)\hat{x}$$

$$\vec{p}\,'(t) = Ft\,\hat{x}$$

$$\vec{l}\,' = \vec{r}\,' \times \vec{p}\,' = 0$$

$$\frac{d\vec{l}\,'}{dt} = 0$$

But $\vec{\tau}\,' = \vec{r}\,' \times \vec{F}$

$$= \left(x_0 + \frac{1}{2}\left(\frac{F}{m}\right)t^2\right)\hat{x} \times F\hat{x}$$

$$= 0$$

so $\vec{\tau}\,' = \dfrac{d\vec{l}\,'}{dt} = 0$

Angular momentum conserved in this system, but not in the other.

Expression for total linear momentum (1.7)

Show: $\vec{P} = M\dot{\vec{R}}$

$$M\dot{\vec{R}} = \cancel{M} \frac{d}{dt}\left(\frac{1}{\cancel{M}} \sum_I m_I \vec{r}_I\right)$$

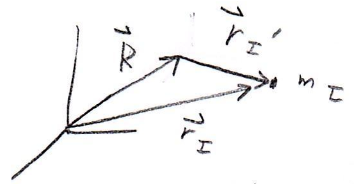
$$= \sum_I m_I \dot{\vec{r}}_I$$

$$= \sum_I m_I \vec{v}_I$$

$$= \sum_I p_I$$

$$= \vec{P}$$

Expression for total angular momentum:



$$\vec{r}_I = \vec{R} + \vec{r}_I{}'$$

Show: $\vec{L} = \vec{R} \times \vec{P} + \sum_I \vec{r}_I{}' \times \vec{p}_I{}'$

Proof:
$$\vec{L} = \sum_I \vec{l}_I$$
$$= \sum_I \vec{r}_I \times \vec{p}_I$$
$$= \sum_I \vec{r}_I \times m_I \frac{d\vec{r}_I}{dt}$$
$$= \sum_I m_I \left(\vec{R} + \vec{r}_I{}'\right) \times \left[\frac{d\vec{R}}{dt} + \frac{d\vec{r}_I{}'}{dt}\right]$$
$$= \sum_I m_I \left[\vec{R} \times \frac{d\vec{R}}{dt} + \vec{r}_I{}' \times \frac{d\vec{r}_I{}'}{dt} + \vec{R} \times \frac{d\vec{r}_I{}'}{dt} + \vec{r}_I{}' \times \frac{d\vec{R}}{dt}\right]$$
$$= M\vec{R} \times \frac{d\vec{R}}{dt} + \sum_I \vec{r}_I{}' \times \left(m_I \frac{d\vec{r}_I{}'}{dt}\right) + \vec{R} \times \sum_I m_I \frac{d\vec{r}_I{}'}{dt} + \left(\sum_I m_I \vec{r}_I{}'\right) \times \frac{d\vec{R}}{dt}$$
$$= \vec{R} \times \vec{P} + \sum_I \vec{r}_I{}' \times \vec{p}_I{}' + \vec{R} \times \underbrace{\frac{d}{dt}\left(\sum_I m_I \vec{r}_I{}'\right)}_{=0} + \underbrace{\left(\sum_I m_I \vec{r}_I{}'\right)}_{=0} \times \frac{d\vec{R}}{dt}$$
(see below)
$$= \vec{R} \times \vec{P} + \sum_I \vec{r}_I{}' \times \vec{p}_I{}'$$

NOTE: $\left.\begin{array}{l} \vec{R} = \frac{1}{M} \sum_I m_I \vec{r}_I \\ \vec{r}_I{}' = \vec{r}_I - \vec{R} \end{array}\right\} \Rightarrow \sum_I m_I \vec{r}_I{}' = \sum_I m_I \left(\vec{r}_I - \vec{R}\right)$

$$= \sum_I m_I \vec{r}_I - \left(\sum_I m_I\right)\vec{R}$$
$$= M\vec{R} - M\vec{R} = 0$$

Exer (1.8) Verify $\frac{d\vec{L}}{dt} = \vec{\tau}^{(e)}$

$$\vec{L} = \sum_I \vec{l}_I$$

$$\frac{d\vec{L}}{dt} = \sum_I \frac{d\vec{l}_I}{dt}$$

$$= \sum_I \frac{d}{dt}\left(\vec{r}_I \times \vec{p}_I\right)$$

$$= \sum_I \left(\underbrace{\cancel{\vec{v}_I \times \vec{p}_I}}_{0} + \vec{r}_I \times \dot{\vec{p}}_I\right)$$

since $\vec{p}_I = m\vec{r}_I$

$$= \sum_I \vec{r}_I \times \vec{F}_I$$

$$= \sum_I \vec{r}_I \times \left(\vec{F}_I^{(e)} + \sum_{J \neq I} \vec{F}_{JI}\right)$$

$$= \sum_I \vec{r}_I \times \vec{F}_I^{(e)} + \sum_I \sum_{J \neq I} \vec{r}_I \times \vec{F}_{JI}$$

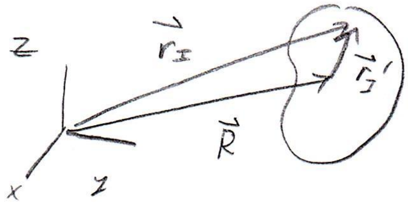
$$= \sum_I \vec{\tau}_I^{(e)} + \frac{1}{2} \sum_{\substack{I,J \\ I \neq J}} \left(\vec{r}_I \times \vec{F}_{JI} + \vec{r}_J \times \underbrace{\vec{F}_{IJ}}_{= -\vec{F}_{JI}}\right)$$

$$= \sum_I \vec{\tau}_I^{(e)} + \frac{1}{2} \sum_{\substack{I,J \\ I \neq J}} \cancel{\left(\vec{r}_I - \vec{r}_J\right) \times \underbrace{\vec{F}_{JI}}_{\propto \left(\vec{r}_I - \vec{r}_J\right)}} \rightarrow 0$$

$$= \sum_I \vec{\tau}_I^{(e)}$$

$$= \vec{\tau}^{(e)}$$

Exer 1.9 :



$$\vec L = \sum_I \vec r_I \times \vec p_I$$
$$= \sum_I (\vec R + \vec r_I') \times \vec p_I$$
$$= \vec R \times \left(\sum_I \vec p_I\right) + \sum_I \vec r_I' \times \vec p_I$$
$$= \vec R \times \vec P + \sum_I \vec r_I' \times \vec p_I$$

Now: $\vec p_I = m_I \vec v_I = m_I \dot{\vec r}_I$

$$= m_I (\vec R + \vec r_I')^{\cdot}$$
$$= m_I \dot{\vec R} + m_I \dot{\vec r}_I'$$

Thus, $\vec L = \vec R \times \vec P + \sum_I \vec r_I' \times (m_I \dot{\vec R} + \underbrace{m_I \dot{\vec r}_I'}_{\vec p_I'})$

$$= \vec R \times \vec P + \left(\sum_I \cancel{m_I \vec r_I'}\right) \times \dot{\vec R} + \sum_I \vec r_I' \times \vec p_I'$$

0

since $\vec R$ is com

$$= \vec R \times \vec P + \sum_I \vec r_I' \times \vec p_I'$$

Torque equation for example problem: 1.10

$$\vec{\tau} = \sum_I \vec{\tau}_I = \vec{r}_1 \times (f\,\hat{z} \times \vec{r}_{12}) + \underset{= -\vec{r}_1}{\vec{r}_2} \times (f\,\hat{z} \times \underset{= -\vec{r}_{12}}{\vec{r}_{21}})$$

$$= 2\vec{r}_1 \times (f\,\hat{z} \times \vec{r}_{12})$$

$$= 4f\,\vec{r} \times (\hat{z} \times \vec{r}) \quad \text{using } \vec{r}_{12} = 2\vec{r}_1 \equiv 2\vec{r}$$

$$= 4f\,[\hat{z}(\vec{r}\cdot\vec{r}) - \vec{r}(\underbrace{\vec{r}\cdot\hat{z}}_{0})]$$

$$= 4f r^2 \hat{z} \qquad \text{since } \vec{r} \text{ in } (x,y) \text{ plane}$$

Now: $\omega = \sqrt{\frac{f}{m}} \rightarrow f = m\omega^2$

Also: $r^2 = x^2 + y^2$

$$= ch^2 c^2 + sh^2 s^2$$

$$= ch^2(1 - s^2) + sh^2 s^2$$

$$= ch^2 - s^2 \underbrace{(ch^2 - sh^2)}_{=1}$$

$$= ch^2 - s^2$$

$$= \left(\frac{1 + \cosh(2\omega t)}{2}\right) - \left(\frac{1 - \cos(2\omega t)}{2}\right)$$

$$= \frac{1}{2}(\cosh(2\omega t) + \cos(2\omega t))$$

$$\rightarrow \vec{\tau} = 4m\omega^2 \frac{1}{2}(ch(2\omega t) + c(2\omega t))\hat{z}$$

$$= 2m\omega^2(ch(2\omega t) + c(2\omega t))\hat{z}$$

Recall:

$$\vec{L} = ma^2\omega\,[\sin(2\omega t) + \sinh(2\omega t)]\,\hat{z}$$

$$\rightarrow \frac{d\vec{L}}{dt} = 2ma^2\omega^2\,[\cos(2\omega t) + \cosh(2\omega t)]\,\hat{z}$$

$$= \vec{\tau}$$

Time rate of change of work for example problem: 1.11

$$\frac{dW}{dt} = 4f\,(\hat{z} \times \vec{r}_1)\cdot\vec{v}_1$$
$$= 4f\,(\vec{r}_1 \times \vec{v}_1)\cdot\hat{z}$$
$$= 4m\omega^2(\vec{r}_1 \times \vec{v}_1)\cdot\hat{z}$$
$$= 4\omega^2(\vec{r}_1 \times \vec{p}_1)\cdot\hat{z}$$
$$= 4\omega^2\,\vec{L}_1\cdot\hat{z}$$
$$= 2\omega^2\,\vec{L}\cdot\hat{z}$$
$$= 2\omega^2 L \quad (\text{since } \vec{L} \text{ points along } \hat{z})$$

Integrate this:

$$dW = 2\omega^2 L\,dt$$
$$= 2\omega^2 (ma^2\omega)[\sin(2\omega t) + \sinh(2\omega t)]\,dt$$

$$\rightarrow \quad W = \cancel{2}ma^2\omega^3\left[\frac{1}{\cancel{2}\omega}-\cos(2\omega t) + \frac{\cosh(2\omega t)}{\cancel{2}\omega}\right]$$
$$= ma^2\omega^2[\cosh(2\omega t) - \cos(2\omega t)]$$
$$= fa^2[\cosh(2\omega t) - \cos(2\omega t)]$$

$$T = \sum_I \tfrac{1}{2} m_I v_I^2$$
$$= \cancel{2}\cdot\left(\tfrac{1}{\cancel{2}} m v_1^2\right)$$
$$= m(\dot{x}^2 + \dot{y}^2)$$
$$= ma^2\omega^2[(-s\,ch + c\,sh)^2 + (c\,sh + s\,ch)^2]$$
$$= ma^2\omega^2[s^2ch^2 + c^2sh^2 \cancel{- 2sc\,ch\,sh} + c^2sh^2 + s^2ch^2 + \cancel{2sc\,ch\,sh}]$$
$$= 2ma^2\omega^2[s^2ch^2 + c^2sh^2]$$
$$= 2ma^2\omega^2[(1-c^2)ch^2 + c^2sh^2]$$
$$= 2ma^2\omega^2[ch^2 - c^2\underbrace{(ch^2 - sh^2)}_{1}]$$

$= 2ma^2\omega^2[ch^2 - c^2]$

$= 2ma^2\omega^2\left[\left(\frac{1+\cosh(2\omega t)}{2}\right) - \left(\frac{1+\cos(2\omega t)}{2}\right)\right]$

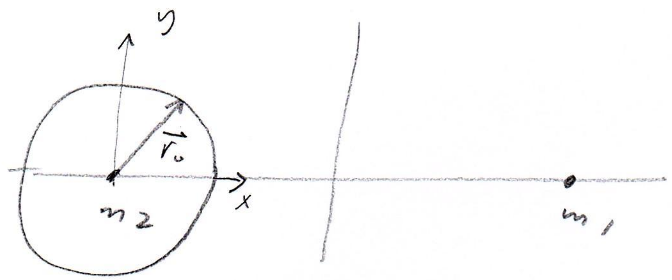
$= ma^2\omega^2[\cosh(2\omega t) - \cos(2\omega t)]$

$= fa^2[\cosh(2\omega t) - \cos(2\omega t)]$

$= W$

Non-conservative force for example problem: 1,12

Put the origin centered on m_2



$$\vec{F} = f\hat{z} \times \vec{r}_0$$
$$= f r_0 \hat{\phi}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{\phi=0}^{2\pi} \vec{F} \cdot r_0 \, d\phi \, \hat{\phi}$$
$$= \int_{\phi=0}^{2\pi} \left(f r_0 \hat{\phi} \right) \cdot r_0 \, d\phi \, \hat{\phi}$$
$$= f r_0^2 \int_{\phi=0}^{2\pi} d\phi$$
$$= 2\pi f r_0^2$$
$$\neq 0$$

so $\vec{F}$ is not conservative

Instantaneous angular velocity vector: (1.13)

Let

$$\Lambda'_{ij} = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\Lambda}'_{ij} = \omega \begin{bmatrix} -\sin & \cos & 0 \\ -\cos & -\sin & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now $M_{jk} = \sum_{i'} \Lambda'_{ij} \dot{\Lambda}'_{ik} = \Lambda^T \dot{\Lambda}$

$$= \omega \begin{bmatrix} c & -s & 0 \\ +s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s & c & 0 \\ -c & -s & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \omega \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{So } M_{12} = \omega, \quad M_{21} = -\omega, \quad \text{all other} = 0}$

$$\omega_i = \frac{1}{2} \sum_{j,k} \epsilon_{ijk} M_{jk}$$

$$\rightarrow \omega_1 = \frac{1}{2} \sum_{j,k} \epsilon_{1jk} M_{jk} = \frac{1}{2}\left(\epsilon_{123} \cancel{M_{23}}^{0} + \epsilon_{132} \cancel{M_{32}}^{0}\right) = 0$$

$$\omega_2 = \frac{1}{2} \sum_{j,k} \epsilon_{2jk} M_{jk} = \frac{1}{2}\left(\epsilon_{231} \cancel{M_{31}}^{0} + \epsilon_{213} \cancel{M_{13}}^{0}\right) = 0$$

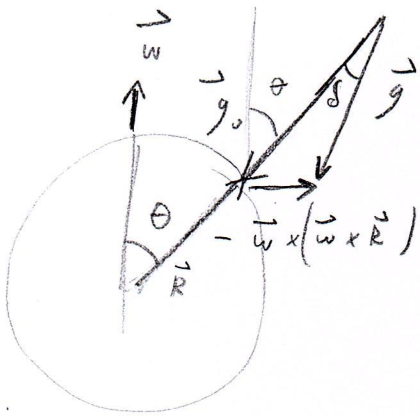
$$\omega_3 = \frac{1}{2} \sum_{j,k} \epsilon_{3jk} M_{jk} = \frac{1}{2}\left(\epsilon_{312} M_{12} + \epsilon_{321} M_{21}\right)$$

$$= \frac{1}{2}\left(1 \cdot \omega + (-1)(-\omega)\right)$$

$$= \omega$$

So $\boxed{\vec{\omega} = \omega \hat{z}}$

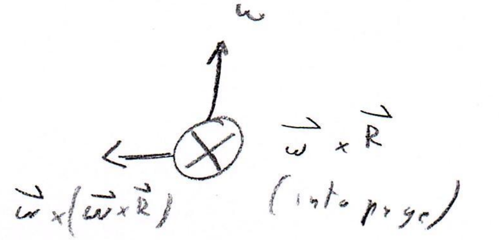
Deflection of plumb line away from $\vec{g}_0$ (1.14) (1)



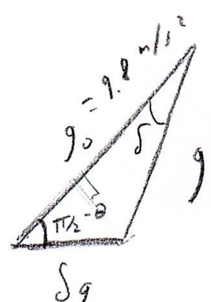
$R = 6400$ km $= 6.4\times10^{7}$ m

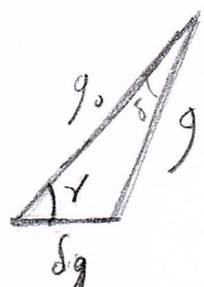
$\omega = \dfrac{2\pi}{1\,\text{day}} = \dfrac{2\pi}{86400\,s} = 7.27\times10^{-5}\,\dfrac{\text{rad}}{\text{sec}}$

$g = 9.8$ m/s^2



$$|\vec{\omega}\times(\vec{\omega}\times\vec{R})| = \omega\sin\theta\,\omega R$$
$$= \omega^2\sin\theta\, R$$
$$= 0.3385\sin\theta\,\frac{m}{s^2} \equiv |\delta\vec{g}| \quad \text{(about 3\% of } g_0\text{)}$$





$\cos\gamma = \cos(\pi/2 - \theta)$

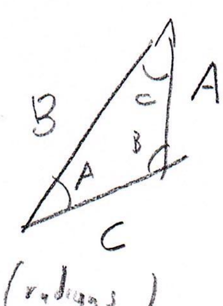
$= \cos(\pi/2)\cos\theta + \sin(\pi/2)\sin\theta$

$g^2 = (\delta g)^2 + g_0^2 - 2\,\delta g\, g_0 \cos\gamma$

$g = (\delta g)^2 + g_0^2 - 2\,\delta g\, g_0 \sin\theta$

law of sines

$$\frac{\sin A}{A} = \frac{\sin B}{B} = \frac{\sin C}{C}$$

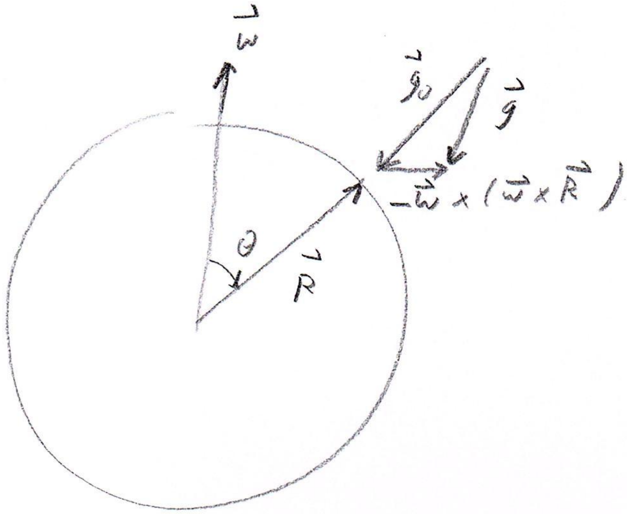


$\delta g \approx g_0\cdot\delta$

Thus $\delta \approx \dfrac{\delta g}{g_0} = \dfrac{0.3385\sin\theta}{9.8} = 0.0345\sin\theta$ (radians)

$\dfrac{.3}{9.8} \approx \dfrac{.3}{10} = .03$

Deflection of plumb line away from $\vec{g}_0$ (1.14)



$R = 6400\ \text{km}$
$= 6.4\times 10^{6}\ \text{m}$

$\omega = \dfrac{2\pi}{\text{day}} = \dfrac{2\pi}{86400\,\text{s}}$

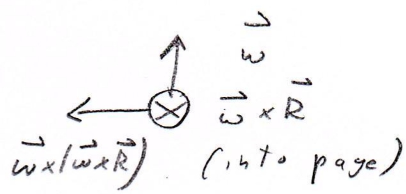
$= 7.27\times 10^{-5}\ \dfrac{\text{rad}}{\text{sec}}$

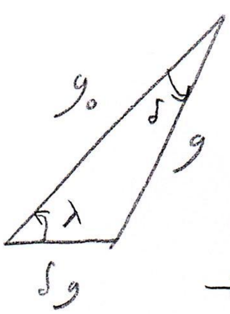
$g_0 = 9.8\ \text{m/s}^2$

$|-\vec{\omega}\times(\vec{\omega}\times\vec{R})| = \omega\sin\theta\,\omega R$
$= \omega^2\sin\theta R$

$= 0.03385\sin\theta\ \dfrac{\text{m}}{\text{s}^2}$

$= |\delta\vec{g}|$ (about 0.34% of g_0)





$\lambda = \pi/2 - \theta$ (known)

$\delta g = \omega^2 R\sin\theta$ (known)

$g_0 = 9.8\ \text{m/s}^2$ (known)

$\rightarrow\ g^2 = g_0^2 + \delta g^2 - 2g_0\,\delta g\cos\lambda$ (law of cosines)

$= g_0^2 + \delta g^2 - 2g_0\,\delta g\sin\theta$

Thus, $g = \sqrt{g_0^2 + \delta g^2 - 2g_0\delta g\sin\theta}$

$= g_0\sqrt{1 - 2\left(\dfrac{\delta g}{g_0}\right)\sin\theta + \left(\dfrac{\delta g}{g_0}\right)^2}$

$\approx g_0\left[1 - \left(\dfrac{\delta g}{g_0}\right)\sin\theta\right]$

$\cos\lambda = \cos\left(\frac{\pi}{2} - \theta\right)$

$= \underbrace{\cancel{\cos\left(\frac{\pi}{2}\right)}}_{0}\cos\theta + \sin\left(\frac{\pi}{2}\right)\sin\theta$

$= \sin\theta$

$$\cancel{\delta g} \quad \frac{\sin \delta}{\delta g} = \frac{\sin \lambda}{g} \qquad \text{(law of sines)}$$

small angle approx: $\sin \delta \approx \delta$

$$\sin \lambda = \sin\left(\frac{\pi}{2} - \theta\right) = \sin\frac{\pi}{2}\cos\theta - \cancel{\cos\frac{\pi}{2}}\sin\theta = \cos\theta$$

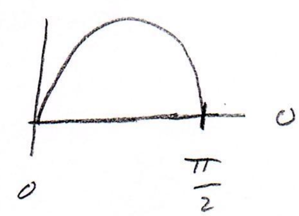
Thus

$$\delta \approx \delta g \cdot \frac{\sin \lambda}{g}$$

$$\approx \frac{\delta g}{g_0} \cos\theta \qquad \text{(to leading order)}$$

$$= \frac{\omega^2 R \sin\theta \cos\theta}{g_0}$$

$$= \boxed{\frac{1}{2} \frac{\omega^2 R \sin 2\theta}{g_0}}$$



δ max at $\theta = \pi/4$

$$\delta\Big|_{max} = \frac{1}{2} \frac{\omega^2 R \sin(\pi/2)}{g_0}$$

$$= \frac{1}{2} \frac{\omega^2 R}{g_0} = \boxed{0.0017} \left(\frac{180°}{\pi}\right) = 0.099° \simeq \boxed{0.1°}$$

$$\delta g = \omega^2 R \sin\theta$$

$$= 0.03385 \sin\theta \ m/s^2$$

max at equator ($\pi/2$)

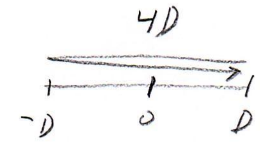
$$\left(\frac{\delta g}{g_0}\right)_{max} = \frac{0.03385}{9.8} = \boxed{0.00345} \simeq 0.3\%$$

Verify that $|\omega_x \dot{y}| \ll g$ (1.15)

$$\omega_x \approx -\omega \sin\theta$$

$$|\omega_x| \le \omega$$

$$\dot{y} \sim \frac{4D}{P}$$



where $P = \dfrac{2\pi}{\Omega} = \dfrac{2\pi}{\sqrt{g/L}} \rightarrow \dfrac{L}{g} = \dfrac{P^2}{4\pi^2}$

Thus, $$\frac{|\omega_x \dot{y}|}{g} \lesssim \omega\left(\frac{4D}{P}\right)\frac{1}{g}$$

$$= 4\left(\frac{\omega}{P}\right)\left(\frac{D}{L}\right)\left(\frac{L}{g}\right)$$

$$= 4\left(\frac{D}{L}\right)\left(\frac{2\pi}{1\,\text{day}}\right)\frac{P}{4\pi^2}$$

$$= 4\left(\frac{D}{L}\right)\left(\frac{1}{2\pi}\right)\left(\frac{P}{1\,\text{day}}\right)$$

$$\simeq \frac{2}{\pi}\underbrace{\left(\frac{D}{L}\right)}_{\ll 1}\underbrace{\left(\frac{P}{1\,\text{day}}\right)}_{\ll 1}$$

$$P = \frac{2\pi}{\sqrt{g/L}} = \frac{2\pi}{\sqrt{10/30}} \simeq 11\,s$$

Foucault pendulum solution: 1.16

$$\xi(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}$$

where $\lambda_\pm = -i(\omega_z \mp \Omega)$

$$\rightarrow \boxed{\xi(t) = Ae^{-i\omega_z t}e^{i\Omega t} + Be^{-i\omega_z t}e^{-i\Omega t} = [Ae^{i\Omega t} + Be^{-i\Omega t}]e^{-i\omega_z t}}$$

Initial conditions:

$\xi(0) = D$, $\dot{\xi}(0) = 0$

$$\boxed{D = \xi(0) = A + B} \quad (1)$$

$$\dot{\xi}(t) = \lambda_+ Ae^{\lambda_+ t} + \lambda_- Be^{\lambda_- t}$$

$$\rightarrow \dot{\xi}(0) = \lambda_+ A + \lambda_- B$$

$$= -i(\omega_z - \Omega)A - i(\omega_z + \Omega)B$$

$$= -i\omega_z(A+B) + i\Omega(A-B)$$

Thus, $\boxed{0 = -i\omega_z(A+B) + i\Omega(A-B)}$ (2)

Substituting (1) into (2) gives

$$0 = -i\omega_z D + i\Omega(A-B)$$

$$\cancel{i}\omega_z D = \cancel{i}\Omega(A-B)$$

$$A - B = \left(\frac{\omega_z}{\Omega}\right)D$$

and $A + B = D$

$$\Longrightarrow \boxed{\begin{aligned} A &= \frac{1}{2}\left(1+\frac{\omega_z}{\Omega}\right)D \\ B &= \frac{1}{2}\left(1-\frac{\omega_z}{\Omega}\right)D \end{aligned}}$$

Therefore,

$$\zeta(t) = \left[\tfrac{1}{2}\left(1+\frac{\omega_z}{\Omega}\right) D\, e^{i\Omega t} + \tfrac{1}{2}\left(1-\frac{\omega_z}{\Omega}\right) D\, e^{-i\Omega t}\right] e^{-i\omega_z t}$$

$$= D\left[\underbrace{\tfrac{1}{2}\left(e^{i\Omega t}+e^{-i\Omega t}\right)}_{\cos\Omega t} + \underbrace{\tfrac{1}{2}\left(\frac{\omega_z}{\Omega}\right)\left(e^{i\Omega t}-e^{-i\Omega t}\right)}_{\left(\frac{\omega_z}{\Omega}\right) i\sin\Omega t}\right] e^{-i\omega_z t}$$

$$= D\left[\cos\Omega t + i\left(\frac{\omega_z}{\Omega}\right)\sin\Omega t\right] e^{-i\omega_z t}$$

$$= D\left[\cos\Omega t + i\left(\frac{\omega_z}{\Omega}\right)\sin\Omega t\right]\left(\cos(\omega_z t) - i\sin(\omega_z t)\right)$$

$$= D\left[\cos(\Omega t)\cos(\omega_z t) + \left(\frac{\omega_z}{\Omega}\right)\sin(\Omega t)\sin(\omega_z t)\right]$$

$$+ i D\left[-\cos(\Omega t)\sin(\omega_z t) + \left(\frac{\omega_z}{\Omega}\right)\sin(\Omega t)\cos(\omega_z t)\right]$$

$$\boxed{\begin{aligned} x(t) &= D\left[\cos(\omega_z t)\cos(\Omega t) + \frac{\omega_z}{\Omega}\sin(\omega_z t)\sin(\Omega t)\right] \\ y(t) &= D\left[-\sin(\omega_z t)\cos(\Omega t) + \frac{\omega_z}{\Omega}\cos(\omega_z t)\sin(\Omega t)\right] \end{aligned}}$$

Time derivatives of basis vectors for spherical pendulum example:

1.17

Use $$\frac{d\hat{e}}{dt} = \frac{\partial \hat{e}}{\partial r}\dot{r} + \frac{\partial \hat{e}}{\partial \theta}\dot{\theta} + \frac{\partial \hat{e}}{\partial \phi}\dot{\phi}$$

From App A:

$$\frac{\partial \hat{r}}{\partial r} = 0 \,, \quad \frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} \,, \quad \frac{\partial \hat{r}}{\partial \phi} = \sin\theta\,\hat{\phi}$$

$$\rightarrow \boxed{\frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta} + \sin\theta\,\dot{\phi}\,\hat{\phi}}$$

From App A:

$$\frac{\partial \hat{\theta}}{\partial r} = 0 \,, \quad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r} \,, \quad \frac{\partial \hat{\theta}}{\partial \phi} = \cos\theta\,\hat{\phi}$$

$$\rightarrow \boxed{\frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r} + \cos\theta\,\dot{\phi}\,\hat{\phi}}$$

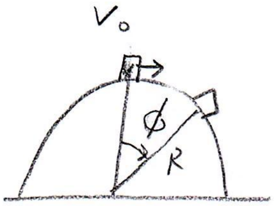
From App A:

$$\frac{\partial \hat{\phi}}{\partial r} = 0 \,, \quad \frac{\partial \hat{\phi}}{\partial \theta} = 0 \,, \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$$

$$\rightarrow \boxed{\frac{d\hat{\phi}}{dt} = -\sin\theta\,\dot{\phi}\,\hat{r} - \cos\theta\,\dot{\phi}\,\hat{\theta}}$$

Redo sticker problem allowing for initial velocity v_0:

1.18



Same analysis as before:

$$\frac{1}{2}\dot{\phi}^2 = -\frac{g}{R}\cos\phi + c$$

c determined from the requirement that

$v_0 = R\dot{\phi}$ at $\phi = 0$.

$$\rightarrow \quad \frac{1}{2}\left(\frac{v_0}{R}\right)^2 = -\frac{g}{R}\underbrace{\cos 0}_{1} + c$$

$$c = \frac{1}{2}\left(\frac{v_0}{R}\right)^2 + \frac{g}{R}$$

Thus, $\frac{1}{2}\dot{\phi}^2 = \frac{g}{R}(1-\cos\phi) + \frac{1}{2}\left(\frac{v_0}{R}\right)^2$

Equation for normal force:

$$\begin{aligned} F_n &= mg\cos\phi - mR\dot{\phi}^2 \\ &= mg\cos\phi - mR\left[\frac{2g}{R}(1-\cos\phi) + \left(\frac{v_0}{R}\right)^2\right] \\ &= mg\cos\phi - 2mg(1-\cos\phi) - \frac{mv_0^2}{R} \\ &= mg\left[3\cos\phi - 2\right] - \frac{mv_0^2}{R} \end{aligned}$$

$$F_n = 0 \iff mg\left[3\cos\phi - 2\right] = \frac{mv_0^2}{R}$$

$$3\cos\phi - 2 = \frac{v_0^2}{Rg}$$

$$\boxed{\cos\phi = \frac{2}{3} + \frac{1}{3}\frac{v_0^2}{Rg}}$$

Minimum velocity needed for skier to leave surface at $\phi = 0$:

$$\cos 0 = \frac{2}{3} + \frac{1}{3}\frac{v_0^2}{Rg}$$

$$1 = \frac{2}{3} + \frac{1}{3}\frac{v_0^2}{Rg}$$

$$\frac{1}{3} = \frac{1}{3}\frac{v_0^2}{Rg}$$

$$\boxed{v_0 = \sqrt{Rg}}$$

Acceleration Vector in sph. polar coords: (Prob 1.1)

$$\vec{r} = r\hat{r}$$

Velocity:

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

Now: $$\frac{d\hat{r}}{dt} = \frac{\partial\hat{r}}{\partial r}\dot{r} + \frac{\partial\hat{r}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{r}}{\partial\phi}\dot{\phi}$$

Use results from App A:

$$\frac{\partial\hat{r}}{\partial r} = 0, \quad \frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}, \quad \frac{\partial\hat{r}}{\partial\phi} = \sin\theta\,\hat{\phi}$$

$$\rightarrow \quad \frac{d\hat{r}}{dt} = \hat{\theta}\dot{\theta} + \sin\theta\,\dot{\phi}\,\hat{\phi}$$

Thus, $$\boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}}$$

Acceleration:

$$\vec{a} \equiv \frac{d\vec{v}}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$+ \dot{r}\sin\theta\,\dot{\phi}\hat{\phi} + r\cos\theta\,\dot{\theta}\dot{\phi}\hat{\phi} + r\sin\theta\,\ddot{\phi}\hat{\phi}$$

$$+ r\sin\theta\,\dot{\phi}\frac{d\hat{\phi}}{dt}$$

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi} \quad \text{(as before)}$$

$$\frac{d\hat{\theta}}{dt} = \frac{\partial\hat{\theta}}{\partial r}\dot{r} + \frac{\partial\hat{\theta}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\theta}}{\partial\phi}\dot{\phi}$$

From App A:

$$\frac{\partial\hat{\theta}}{\partial r} = 0, \quad \frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}, \quad \frac{\partial\hat{\theta}}{\partial\phi} = \cos\theta\,\hat{\phi}$$

$$\rightarrow \quad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r} + \cos\theta\,\dot{\phi}\,\hat{\phi}$$

Also,

$$\frac{d\hat{\phi}}{dt} = \frac{\partial\hat{\phi}}{\partial r}\dot{r} + \frac{\partial\hat{\phi}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\phi}}{\partial\phi}\dot{\phi}$$

From App A:

$$\frac{\partial\hat{\phi}}{\partial r} = 0, \quad \frac{\partial\hat{\phi}}{\partial\theta} = 0, \quad \frac{\partial\hat{\phi}}{\partial\phi} = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$$

$$\rightarrow \quad \frac{d\hat{\phi}}{dt} = -\sin\theta\,\dot{\phi}\,\hat{r} - \cos\theta\,\dot{\phi}\,\hat{\theta}$$

Thus,

$$\begin{aligned}\vec{a} = {} & \ddot{r}\hat{r} + \dot{r}\left(\dot{\theta}\hat{\theta} + \sin\theta\,\dot{\phi}\,\hat{\phi}\right) + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} \\ & + r\dot{\theta}\left(-\dot{\theta}\hat{r} + \cos\theta\,\dot{\phi}\,\hat{\phi}\right) + \dot{r}\sin\theta\,\dot{\phi}\,\hat{\phi} \\ & + r\cos\theta\,\dot{\theta}\dot{\phi}\hat{\phi} + r\sin\theta\,\ddot{\phi}\,\hat{\phi} \\ & + r\sin\theta\,\dot{\phi}\left(-\sin\theta\,\dot{\phi}\,\hat{r} - \cos\theta\,\dot{\phi}\,\hat{\theta}\right)\end{aligned}$$

$$\boxed{\begin{aligned} = {} & \hat{r}\left[\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2\right] \\ & + \hat{\theta}\left[2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2\right] \\ & + \hat{\phi}\left[2\sin\theta\,\dot{r}\dot{\phi} + 2r\cos\theta\,\dot{\theta}\dot{\phi} + r\sin\theta\,\ddot{\phi}\right]\end{aligned}}$$

Acceleration vector in cylindrical coords (ρ, ϕ, z) Prob 1.2 ①

$$\vec{r} = \rho\hat{\rho} + z\hat{z}$$

velocity:

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= \dot{\rho}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} + \dot{z}\hat{z} + \cancel{z\frac{d\hat{z}}{dt}} \to 0$$

$$= \dot{\rho}\hat{\rho} + \dot{z}\hat{z} + \rho\frac{d\hat{\rho}}{dt}$$

Now: $$\frac{d\hat{\rho}}{dt} = \frac{\partial\hat{\rho}}{\partial\rho}\dot{\rho} + \frac{\partial\hat{\rho}}{\partial\phi}\dot{\phi} + \frac{\partial\hat{\rho}}{\partial z}\dot{z}$$

From App A:

$$\frac{\partial\hat{\rho}}{\partial\rho} = 0, \quad \frac{\partial\hat{\rho}}{\partial\phi} = \hat{\phi}, \quad \frac{\partial\hat{\rho}}{\partial z} = 0$$

$$\rightarrow \quad \frac{d\hat{\rho}}{dt} = \dot{\phi}\hat{\phi}$$

so $$\boxed{\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}}$$

Acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \ddot{\rho}\hat{\rho} + \dot{\rho}\frac{d\hat{\rho}}{dt} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}\frac{d\hat{\phi}}{dt} + \ddot{z}\hat{z}$$

Now: $$\frac{d\hat{\phi}}{dt} = \frac{\partial\hat{\phi}}{\partial\rho}\dot{\rho} + \frac{\partial\hat{\phi}}{\partial\phi}\dot{\phi} + \frac{\partial\hat{\phi}}{\partial z}\dot{z}$$

From App. A:

$$\frac{\partial \hat{\phi}}{\partial \rho} = 0 \quad , \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{\rho} \quad , \quad \frac{\partial \hat{\phi}}{\partial z} = 0$$

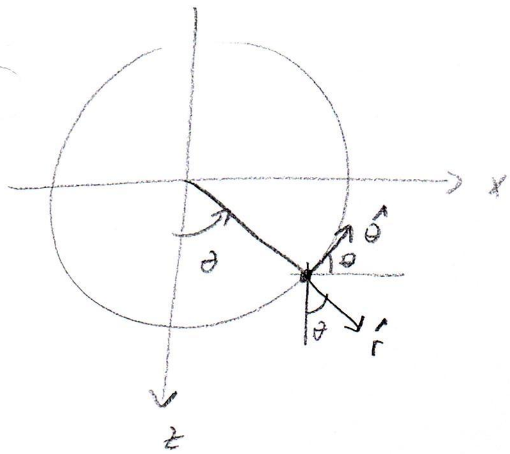
Thus, $$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{\rho}$$

$$\rightarrow \vec{a} = \ddot{\rho}\hat{\rho} + \dot{\rho}\dot{\phi}\hat{\phi} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi}$$

$$- \rho\dot{\phi}^2\hat{\rho} + \ddot{z}\hat{z}$$

$$= \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\rho} + \left(2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}\right)\hat{\phi} + \ddot{z}\hat{z}$$

Planar pendulum: Prob 1.3

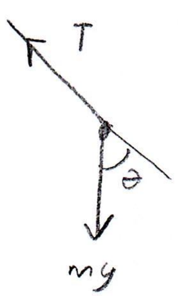


$$\hat{r} = \cos\theta\,\hat{z} + \sin\theta\,\hat{x}$$
$$\hat{\theta} = -\sin\theta\,\hat{z} + \cos\theta\,\hat{x}$$

$$\frac{d\hat{r}}{dt} = \dot{\theta}\left[-\sin\theta\,\hat{z} + \cos\theta\,\hat{x}\right] = \dot{\theta}\,\hat{\theta}$$
$$\frac{d\hat{\theta}}{dt} = \dot{\theta}\left[-\cos\theta\,\hat{z} - \sin\theta\,\hat{x}\right] = -\dot{\theta}\,\hat{r}$$

$$\vec{r} = l\,\hat{r}$$
$$\vec{v} = \frac{d\vec{r}}{dt} = l\,\frac{d\hat{r}}{dt} = l\,\dot{\theta}\,\hat{\theta} \qquad \text{(tgt to circle)}$$
$$\vec{a} = \frac{d\vec{v}}{dt} = l\,\ddot{\theta}\,\hat{\theta} + l\,\dot{\theta}\,\frac{d\hat{\theta}}{dt}$$
$$= \underbrace{l\,\ddot{\theta}\,\hat{\theta}}_{\text{tgt}} \underbrace{- l\,\dot{\theta}^2\,\hat{r}}_{\text{centripetal}}$$

Net force:



$$\vec{T} = -T\,\hat{r}$$
$$m\vec{g} = mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{\theta}$$
$$\vec{F} = (mg\cos\theta - T)\,\hat{r} - mg\sin\theta\,\hat{\theta}$$
$$m\vec{a} = ml\,\ddot{\theta}\,\hat{\theta} - ml\,\dot{\theta}^2\,\hat{r}$$

Thus, $\vec{F} = m\vec{a} \iff$

$$\boxed{\begin{aligned} mg\cos\theta - T &= -ml\,\dot{\theta}^2 \\ -mg\sin\theta &= ml\,\ddot{\theta} \end{aligned}}$$

(a) $T = mg\cos\theta + ml\dot{\theta}^2$

(b) $\cancel{m} l\ddot{\theta} = -\cancel{m} g\sin\theta$

$\ddot{\theta} = -\frac{g}{l}\sin\theta$ [oscillator equation]

multiply by $\dot{\theta}$

$\dot{\theta}\ddot{\theta} = -\frac{g}{l}\dot{\theta}\sin\theta$

$\frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2\right) = \frac{d}{dt}\left(\frac{g}{l}\cos\theta\right)$

$\rightarrow \boxed{\frac{1}{2}\dot{\theta}^2 = \frac{g}{l}\cos\theta + C}$

rearrange:

$\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta = C$

$\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = C\cdot ml^2$

$T + U' = E$ (so C is related to E)

$U = -mgl\cos\theta + mgl$

$= mgl(1-\cos\theta)$

so $U=0$ at bottom.

(c) $T = mg\cos\theta + ml\dot{\theta}^2$

max value of θ having $T \geq 0$ and $\dot{\theta}=0$

$\longrightarrow 0 = mg\cos\theta_0 \rightarrow \boxed{\theta_0 = \pi/2}$

(d) minimum initial velocity for loop-the-loop?

Determine C in terms of initial velocity $V_0 = l\dot{\theta}\Big|_{\text{at }\theta=0}$

$\frac{1}{2}\frac{V_0^2}{l^2} = \frac{g}{l}\cos\theta\Big|_{\theta=0} + C$

$$\frac{1}{2}\frac{v_0^2}{\ell^2} = \frac{g}{\ell} + C$$

$$\rightarrow \quad C = \frac{1}{2}\frac{v_0^2}{\ell^2} - \frac{g}{\ell}$$

Thus, $\frac{1}{2}\dot{\theta}^2 - \frac{g}{\ell}\cos\theta = \frac{1}{2}\frac{v_0^2}{\ell^2} - \frac{g}{\ell}$

$$\boxed{\frac{1}{2}\dot{\theta}^2 + \frac{g}{\ell}(1-\cos\theta) = \frac{1}{2}\left(\frac{v_0}{\ell}\right)^2}$$

At top of loop-the-loop, $\theta = \pi \rightarrow \cos\theta = -1$

$$T = mg\cos\theta + m\ell\dot{\theta}^2$$

$$\rightarrow \quad 0 = -mg + m\ell\dot{\theta}^2$$

so $\boxed{\dot{\theta} = \sqrt{\frac{g}{\ell}}}$ at top with $T=0$.

<u>Minimum v_0:</u>

At top $\frac{1}{2}\left(\sqrt{\frac{g}{\ell}}\right)^2 + \frac{g}{\ell}(1+1) = \frac{1}{2}\left(\frac{v_0}{\ell}\right)^2$

$$\frac{1}{2}\left(\frac{g}{\ell}\right) + 2\left(\frac{g}{\ell}\right) = \frac{1}{2}\left(\frac{v_0}{\ell}\right)^2$$

$$\frac{5}{\cancel{2}}\frac{g}{\ell} = \frac{1}{\cancel{2}}\left(\frac{v_0}{\ell}\right)^2$$

$$\rightarrow \quad \boxed{v_0 = \sqrt{5g\ell}}$$

Simple conservation of energy argument:

At bottom $E = KE = \frac{1}{2} m v_0^2$

At top $E = 2mgl + \frac{1}{2} m v^2$

centripetal force given only by $mg \rightarrow$

$\cancel{m} g = \frac{\cancel{m} v^2}{\cancel{l}}$

$\rightarrow v^2 = gl$

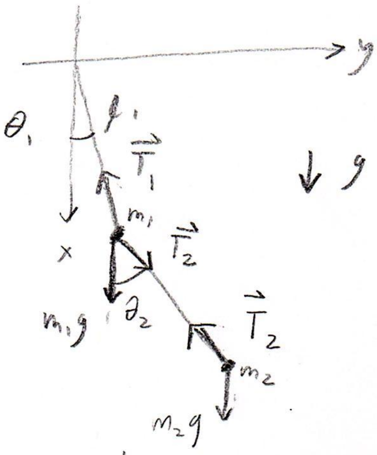
$= 2mgl + \frac{1}{2} mgl$

$= \frac{5}{2} mgl$

Thus, $\frac{1}{\cancel{2}} \cancel{m} v_0^2 = \frac{5}{\cancel{2}} \cancel{m} g l$

$v_0 = \sqrt{5gl}$

Double pendulum: Prob 1.4



$$\vec{r}_1 = l_1 \cos\theta_1 \hat{x} + l_1 \sin\theta_1 \hat{y}$$

$$\vec{r}_2 = \vec{r}_1 + l_2 \cos\theta_2 \hat{x} + l_2 \sin\theta_2 \hat{y}$$

$$= (l_1 \cos\theta_1 + l_2 \cos\theta_2)\hat{x}$$

$$+ (l_1 \sin\theta_1 + l_2 \sin\theta_2)\hat{y}$$

Calculate velocity and acceleration vectors

$$\vec{v}_1 = -l_1 \sin\theta_1 \dot{\theta}_1 \hat{x} + l_1 \cos\theta_1 \dot{\theta}_1 \hat{y}$$

$$= l_1 \dot{\theta}_1 [-\sin\theta_1 \hat{x} + \cos\theta_1 \hat{y}]$$

$$\vec{v}_2 = (-l_1 \sin\theta_1 \dot{\theta}_1 - l_2 \sin\theta_2 \dot{\theta}_2)\hat{x}$$

$$+ (l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2)\hat{y}$$

$$= l_1 \dot{\theta} [-\sin\theta_1 \hat{x} + \cos\theta_1 \hat{y}]$$

$$+ l_2 \dot{\theta}_2 [-\sin\theta_2 \hat{x} + \cos\theta_2 \hat{y}]$$

$$\vec{a}_1 = l_1 \dot{\theta}_1 [-\sin\theta_1 \hat{x} + \cos\theta_1 \hat{y}]$$

$$+ l_1 \dot{\theta}_1 [-\cos\theta_1 \dot{\theta}_1 \hat{x} - \sin\theta_1 \dot{\theta}_1 \hat{y}]$$

$$= l_1 \ddot{\theta}_1 [-\sin\theta_1 \hat{x} + \cos\theta_1 \hat{y}] - l_1 \dot{\theta}_1^2 [\cos\theta_1 \hat{x} + \sin\theta_1 \hat{y}]$$

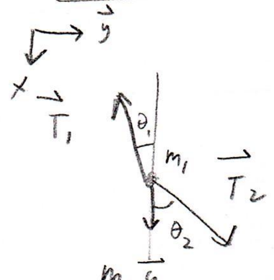
$$\vec{a}_2 = l_1 \ddot{\theta}_1 [-\sin\theta_1 \hat{x} + \cos\theta_1 \hat{y}] - l_1 \dot{\theta}_1^2 [\cos\theta_1 \hat{x} + \sin\theta_1 \hat{y}]$$

$$+ l_2 \ddot{\theta}_2 [-\sin\theta_2 \hat{x} + \cos\theta_2 \hat{y}] - l_2 \dot{\theta}_2^2 [\cos\theta_2 \hat{x} + \sin\theta_2 \hat{y}]$$

$$\vec{a}_1 = -l_1\left(\ddot{\theta}_1 \sin\theta_1 + \dot{\theta}_1^2 \cos\theta_1\right)\hat{x} + l_1\left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1\right)\hat{y}$$

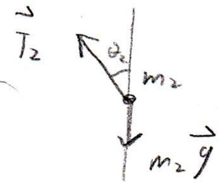
$$\vec{a}_2 = \left[-l_1\left(\ddot{\theta}_1 \sin\theta_1 + \dot{\theta}_1^2 \cos\theta_1\right) - l_2\left(\ddot{\theta}_2 \sin\theta_2 + \dot{\theta}_2^2 \cos\theta_2\right)\right]\hat{x}$$

$$+ \left[l_1\left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1\right) + l_2\left(\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2\right)\right]\hat{y}$$



$$\vec{F}_1 = -T_1 \cos\theta_1 \hat{x} - T_1 \sin\theta_1 \hat{y} + T_2 \cos\theta_2 \hat{x} + T_2 \sin\theta_2 \hat{y} + m_1 g \hat{x}$$

$$= \left(m_1 g - T_1 \cos\theta_1 + T_2 \cos\theta_2\right)\hat{x} + \left(-T_1 \sin\theta_1 + T_2 \sin\theta_2\right)\hat{y}$$



$$\vec{F}_2 = -T_2 \cos\theta_2 \hat{x} - T_2 \sin\theta_2 \hat{y} + m_2 g \hat{x}$$

$$= \left(m_2 g - T_2 \cos\theta_2\right)\hat{x} - T_2 \sin\theta_2 \hat{y}$$

2nd law:

$$m_1 g - T_1 \cos\theta_1 + T_2 \cos\theta_2 = -m_1 l_1\left(\ddot{\theta}_1 \sin\theta_1 + \dot{\theta}_1^2 \cos\theta_1\right) \quad (1)$$

$$-T_1 \sin\theta_1 + T_2 \sin\theta_2 = m_1 l_1\left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1\right) \quad (2)$$

$$m_2 g - T_2 \cos\theta_2 = m_2\left[-l_1\left(\ddot{\theta}_1 \sin\theta_1 + \dot{\theta}_1^2 \cos\theta_1\right) - l_2\left(\ddot{\theta}_2 \sin\theta_2 + \dot{\theta}_2^2 \cos\theta_2\right)\right] \quad (3)$$

$$-T_2 \sin\theta_2 = m_2\left[l_1\left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1\right) + l_2\left(\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2\right)\right] \quad (4)$$

Thus, from (4)

$$\boxed{T_2 \sin\theta_2 = -m_2 \left[l_1 \left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1 \right) + l_2 \left(\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2 \right) \right]}$$

<u>Add (2) and (4):</u>

$$-T_1 \sin\theta_1 = (m_1 + m_2)\, l_1 \left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1 \right) + m_2 l_2 \left(\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2 \right)$$

so

$$\boxed{T_1 \sin\theta_1 = -(m_1 + m_2)\, l_1 \left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1 \right) - m_2 l_2 \left(\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2 \right)}$$

<u>(3) $\sin\theta_2$ − (4) $\cos\theta_2$:</u>

$$\begin{aligned} m_2 g \sin\theta_2 &= -m_2 \sin\theta_2 \left[l_1 \left(\ddot{\theta}_1 \sin\theta_1 + \dot{\theta}_1^2 \cos\theta_1 \right) \right. \\ &\qquad \left. l_2 \left(\ddot{\theta}_2 \sin\theta_2 + \cancel{\dot{\theta}_2^2 \cos\theta_2} \right) \right] \\ &\quad - m_2 \cos\theta_2 \left[l_1 \left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1 \right) \right. \\ &\qquad \left. + l_2 \left(\ddot{\theta}_2 \cos\theta_2 - \cancel{\dot{\theta}_2^2 \sin\theta_2} \right) \right] \\ &= -m_2 \left[l_1 \ddot{\theta}_1 \left(\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \right) \right. \\ &\qquad + l_2 \ddot{\theta}_2 \left(\sin^2\theta_2 + \cos^2\theta_2 \right) \\ &\qquad \left. + l_1 \dot{\theta}_1^2 \left(\cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 \right) \right] \end{aligned}$$

Now:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cancel{m_2} g \sin\theta_2 = -\cancel{m_2}\left[l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + l_2 \ddot{\theta}_2 - l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) \right]$$

$$\boxed{g \sin\theta_2 = -l_1 \ddot{\theta}_1 \cos\Delta + l_1 \dot{\theta}_1^2 \sin\Delta - l_2 \ddot{\theta}_2}$$

---

(1) $\sin\theta_1$ $-$(2) $\cos\theta_1$ $+$(3) $\sin\theta_1$ $-$(4) $\cos\theta_1$:

$$LHS = m_1 g \sin\theta_1 \cancel{- T_1 \cos\theta_1 \sin\theta_1} \cancel{+ T_2 \cos\theta_2 \sin\theta_1} \cancel{+ T_1 \sin\theta_1 \cos\theta_1} \cancel{- T_2 \sin\theta_2 \cos\theta_1}$$
$$+ m_2 g \sin\theta_1 \cancel{- T_2 \cos\theta_2 \sin\theta_1} \cancel{+ T_2 \sin\theta_2 \cos\theta_1}$$

$$= (m_1 + m_2) g \sin\theta_1$$

$$RHS = -m_1 l_1 \left[\sin\theta_1 (\ddot{\theta}_1 \sin\theta_1 \cancel{+ \dot{\theta}_1^2 \cos\theta_1}) + \cos\theta_1 (\ddot{\theta}_1 \cos\theta_1 \cancel{- \dot{\theta}_1^2 \sin\theta_1}) \right]$$

$$+ m_2 \Big[-l_1 \sin\theta_1 (\ddot{\theta}_1 \sin\theta_1 \cancel{+ \dot{\theta}_1^2 \cos\theta_1})$$
$$- l_1 \cos\theta_1 (\ddot{\theta}_1 \cos\theta_1 \cancel{- \dot{\theta}_1^2 \sin\theta_1})$$
$$- l_2 \sin\theta_1 (\ddot{\theta}_2 \sin\theta_2 + \dot{\theta}_2^2 \cos\theta_2)$$
$$- l_2 \cos\theta_1 (\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2) \Big]$$

~~RHS = $(m_1+m_2) l_1 \ddot{\theta}_1 \cos\Delta$ $m_2 l_2 \ddot{\theta}_2$~~

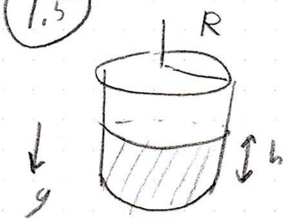
$$
\begin{aligned}
RHS = &- m_1 l_1 \ddot{\theta}_1 (\sin^2\theta_1 + \cos^2\theta_1) \\
&- m_2 l_1 \ddot{\theta}_1 (\sin^2\theta_1 + \cos^2\theta_1) \\
&- m_2 l_2 \ddot{\theta}_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) \\
&- m_2 l_2 \dot{\theta}_2^2 (\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2) \\
= &- (m_1+m_2) l_1 \ddot{\theta}_1 - m_2 l_2 \ddot{\theta}_2 \cos\Delta - m_2 l_2 \dot{\theta}_2^2 \sin\Delta
\end{aligned}
$$

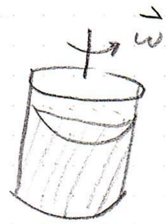
thus,

$$\boxed{(m_1+m_2) g \sin\theta_1 = -(m_1+m_2) l_1 \ddot{\theta}_1 - m_2 l_2 \left(\ddot{\theta}_2 \cos\Delta + \dot{\theta}_2^2 \sin\Delta\right)}$$

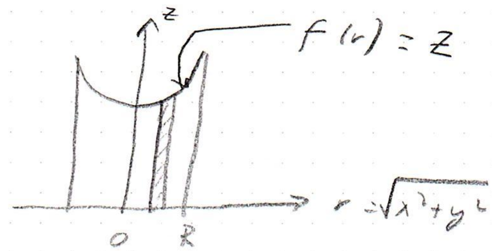
Problem: Shape of surface of water in a rotating bucket

1.5





Azimuthally symmetric:



cylindrical shell

$dm = \rho\, 2\pi r\, dr\, z(r)$

~~U_{tot} = KE + PE~~

$U_{tot} =$ rotational KE + $U_{gravity}$

rotational KE $\equiv U_{centrifugal}$

$$\vec{F}_{centrifugal} = -m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}) = m\omega^2 r\,\hat{r}$$

$$\rightarrow U_{centrifugal} = -\frac{1}{2}m\omega^2 r^2$$

Thus,

$$dU_{tot} = -\frac{1}{2}dm\,\omega^2 r^2 + dm\cdot g\cdot \underbrace{\frac{1}{2}z(r)}_{\text{avg height}}$$

$$= \frac{1}{\cancel{2}}\left[-\rho\,\cancel{2}\pi r\,z\,dr\,\omega^2 r^2 + \rho\,\cancel{2}\pi r\,z\,dr\,g\,z\right]$$

$$= \pi\rho\left[-\omega^2 r^3 z + g r z^2\right]dr$$

Constraint: volume $= \pi R^2 h$

$$\int_0^R 2\pi r\,dr\,z = \pi R^2 h$$

so
$$\boxed{\int_0^R 2\pi r z\,dr - \pi R^2 h = 0}$$

Extremize: $I[z] = \int_0^R \pi\rho\left[-\omega^2 r^3 z + g r z^2\right]dr$

subject to $J[z] = 2\pi\int_0^R r z\,dr - \pi R^2 h = 0$

$$F(z, z', r) = \pi\rho\left[-\omega^2 r^3 z + gr\, z^2\right]$$

$$G(z, z', r) = 2\pi r z$$

i) $$\left(\cancel{\frac{d}{dr}\left(\frac{\partial F}{\partial z'}\right)}^{\,0} - \frac{\partial F}{\partial z}\right) + \lambda\left[\frac{d}{dr}\left(\cancel{\frac{\partial G}{\partial z'}}^{\,0}\right) - \frac{\partial G}{\partial z}\right] = 0$$

ii) $$\int_0^R 2\pi r z\, dr = \pi R^2 h$$

$$0 = \frac{\partial F}{\partial z} + \lambda \frac{\partial G}{\partial z}$$

$$= \pi\rho\left[-\omega^2 r^3 + 2grz\right] + \lambda 2\pi r$$

$$= -\cancel{\pi}\rho\omega^2 r^3 + 2\cancel{\pi}\rho g r z + \lambda 2\cancel{\pi} r$$

$$= -\rho\omega^2 r^3 + 2\rho g r z + 2\lambda r$$

$$\rightarrow \quad z = \frac{-2\lambda r + \rho\omega^2 r^3}{2\rho g r}$$

$$= \frac{1}{2\rho g}\left[-2\lambda + \rho\omega^2 r^2\right]$$

$$= \frac{-\lambda}{\rho g} + \frac{1}{2g}\omega^2 r^2 \qquad = A + Br^2$$

Substitute: $$\pi R^2 h = \int_0^R 2\pi r\left[\frac{-\lambda}{\rho g} + \frac{1}{2g}\omega^2 r^2\right] dr$$

$$= -\frac{2\pi\lambda}{\rho g}\frac{r^2}{2}\Big|_0^R + \frac{\pi\omega^2}{g}\frac{r^4}{4}\Big|_0^R$$

$$= -\frac{\pi\lambda}{\rho g}R^2 + \frac{\pi}{4g}\omega^2 R^4$$

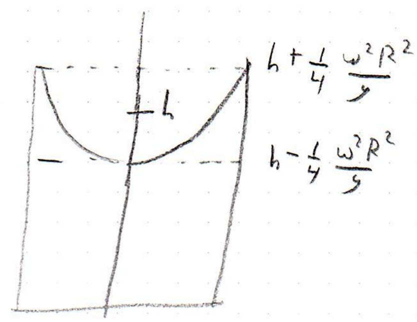
$$\cancel{\pi} R^2 h = \cancel{\pi R^2}\left[\frac{-\lambda}{\rho g} + \frac{R^2\omega^2}{4g}\right]$$

$$h = \frac{-\lambda}{\rho g} + \frac{R^2\omega^2}{4g}$$

$$\rightarrow \quad \boxed{\lambda = \rho g\left(-h + \frac{R^2\omega^2}{4g}\right) = -\rho g h + \frac{1}{4}\rho \omega^2 R^2}$$

Thus,

$$\boxed{z = \frac{-\lambda}{\rho g} + \frac{1}{2g}\omega^2 r^2 = h - \frac{R^2\omega^2}{4g} + \frac{1}{2g}\omega^2 r^2 = h + \frac{\omega^2}{2g}\left[r^2 - \frac{R^2}{2}\right]}$$



NOTE: $r = 0 \rightarrow z(0) = h - \frac{1}{4}\frac{\omega^2 R^2}{g}$

$r = R \quad z(0) = h + \frac{1}{4}\omega^2\frac{R^2}{g}$

Check volume:

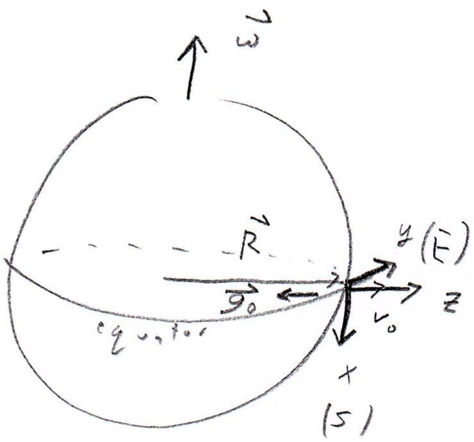
$$2\pi\int_0^R r z \, dr = 2\pi\int_0^R r\left(h + \frac{\omega^2}{2g}\left[r^2 - \frac{R^2}{2}\right]\right) dr$$

$$= 2\pi\left[\int_0^R h r \, dr + \frac{\omega^2}{2g}\int_0^R \left(r^3 - \frac{R^2 r}{2}\right) dr\right]$$

$$= 2\pi\left[h\frac{R^2}{2} + \frac{\omega^2}{2g}\cancel{\left(\frac{R^4}{4} - \frac{R^2}{2}\frac{R^2}{2}\right)}\right]$$

$$= \pi R^2 h \quad \checkmark$$

Prob: 1.6 Launch from equator



$$m\vec{a} = \vec{F} - m\dot{\vec{\omega}}\times\vec{r} - 2m\vec{\omega}\times\vec{v} - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\ddot{\vec{R}}_f$$

$$\ddot{\vec{R}}_f = \vec{\omega}\times(\vec{\omega}\times\vec{R})$$

$$\dot{\vec{\omega}} = 0$$

$$\vec{F} = m\vec{g}_0$$

$\vec{F} = m\vec{g}_0$

$\vec{\omega} = \text{const}$

a) Thus,

$$m\vec{a} = m\left[\vec{g}_0 - \vec{\omega}\times(\vec{\omega}\times(\vec{r}+\vec{R}))\right] - 2m\vec{\omega}\times\vec{v}$$

$$\boxed{\vec{a} = \vec{g}_0 - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - \vec{\omega}\times(\vec{\omega}\times\vec{R}) - 2\vec{\omega}\times\vec{v}}$$

Let

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ (wrt non-inertial frame)

$\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$ time derivative wrt non-inertial frame

$\vec{a} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$

Now:

$\vec{g}_0 = -g_0\hat{z}$

$\vec{R} = R\hat{z}$

$\vec{\omega} = -\omega\hat{x}$

Thus,

$$\vec{\omega}\times\vec{r} = -\omega\hat{x}\times(x\hat{x} + y\hat{y} + z\hat{z}) = -\omega y\hat{z} + \omega z\hat{y}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\omega\hat{x}\times(-\omega y\hat{z} + \omega z\hat{y}) = -\omega^2 y\hat{y} - \omega^2 z\hat{z}$$

$$\vec{\omega} \times \vec{R} = -\omega\hat{x} \times R\hat{z}$$
$$= \omega R \hat{y}$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{R}) = -\omega\hat{x} \times \omega R \hat{y}$$
$$= -\omega^2 R \hat{z}$$

$$\vec{\omega} \times \vec{v} = -\omega\hat{x} \times (\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z})$$
$$= -\omega\dot{y}\hat{z} + \omega\dot{z}\hat{y}$$

$$\vec{a} = \vec{g_0} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - \vec{\omega} \times (\vec{\omega} \times \vec{R}) - 2\vec{\omega} \times \vec{v}$$

$$\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z} = -g_0\hat{z} + \omega^2 y\hat{y} + \omega^2 z\hat{z} + \omega^2 R\hat{z} + 2\omega\dot{y}\hat{z} - 2\omega\dot{z}\hat{y}$$

$$= \hat{y}(\omega^2 y - 2\omega\dot{z}) + \hat{z}(-g_0 + \omega^2 z + \omega^2 R + 2\omega\dot{y})$$

Thus,

$$\boxed{\begin{aligned} \ddot{x} &= 0 \\ \ddot{y} &= \omega^2 y - 2\omega\dot{z} \\ \ddot{z} &= (-g_0 + \omega^2 R) + \omega^2 z + 2\omega\dot{y} \end{aligned}}$$

b) If $\omega = 0$

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} = -g_0$$

$$\rightarrow \quad y_0(t) = 0, \quad z_0(t) = -\tfrac{1}{2} g_0 t^2 + v_0 t \qquad (z_0(0) = 0)$$

Perturbed trajectory

$$y(t) = \psi(t), \quad z(t) = z_0(t) + \zeta(t)$$

To 1st order:

$$\boxed{\begin{aligned} \ddot{\psi} &= \cancel{\omega^2 y} - 2\omega\big(\dot{z}_0(t) + \overbrace{\dot{\zeta}(t)}^{2^{nd}\text{ order}}\big) \\ &= -2\omega \dot{z}_0 \\ &= -2\omega(v_0 - g_0 t) \end{aligned}}$$

$$\ddot{z}_0 + \ddot{\zeta} = (-g_0 + \underbrace{\omega^2 R}_{2^{nd}\text{ order}}) + \underbrace{\omega^2 z}_{2^{nd}\text{ order}} + \underbrace{2\omega\dot{\psi}}_{2^{nd}\text{ order}}$$

$$\rightarrow \quad -g_0 + \ddot{\zeta} = -g_0$$

or $\boxed{\ddot{\zeta} = 0}$

c) Solve these equations

$$\ddot{\zeta} = 0 \quad \rightarrow \quad \zeta(t) = At + B$$

But $A = 0$, $B = 0$ in order for $z(t) = z_0(t) + \zeta(t)$ to satisfy $z(0) = 0$, $\dot{z}(0) = v_0$. Thus $\boxed{\zeta(t) = 0}$

$$\begin{aligned} \ddot{\psi} &= -2\omega(v_0 - g_0 t) \\ &= -2\omega v_0 + 2\omega g_0 t \end{aligned}$$

$$\rightarrow \quad \dot{\psi} = -2\omega v_0 t + \omega g_0 t^2 + A$$

$$\rightarrow \quad \psi = -\omega v_0 t^2 + \frac{1}{3}\omega g_0 t^3 + At + B$$

Again, in order for $y(t) = \psi(t)$ to satisfy $y(0)=0$, $\dot{y}(0)=0$ $\rightarrow$ $A=0$, $B=0$.

Thus, $\boxed{\psi(t) = \frac{1}{3}\omega g_0 t^3 - \omega v_0 t^2}$

d) Suppose the project is launched to a height $h = 100$ km. How far from the launch site does it land?

<u>vertical motion:</u> At top of trajectory $\dot{z}(T_{\frac{1}{2}}) = 0$

$$0 = -g_0 T_{\frac{1}{2}} + v_0$$

$$\boxed{T_{\frac{1}{2}} = \frac{v_0}{g_0}}$$

Time back to Earth: $T = 2T_{\frac{1}{2}} = \boxed{\frac{2v_0}{g_0}}$

At top of trajectory

$$h = z_0(T_{\frac{1}{2}}) = v_0 T_{\frac{1}{2}} - \frac{1}{2}g_0 (T_{\frac{1}{2}})^2$$

$$= \frac{v_0^2}{g_0} - \frac{1}{2}g_0\left(\frac{v_0^2}{g_0^2}\right)$$

$$= \frac{1}{2}\frac{v_0^2}{g_0}$$

Thus, $\boxed{v_0 = \sqrt{2g_0 h}}$ — initial velocity required to reach height h

Distance from landing site:

$$\psi(T) = \frac{1}{3}\omega g_0 T^3 - \omega v_0 T^2$$

$$= \frac{1}{3}\omega g_0 \left(\frac{2v_0}{g_0}\right)^3 - \omega v_0 \left(\frac{2v_0}{g_0}\right)^2$$

$$= \frac{8}{3}\frac{\omega v_0^3}{g_0^2} - \underbrace{4\frac{\omega v_0^3}{g_0^2}}_{\frac{12}{3}}$$

$$= -\frac{4}{3}\frac{\omega v_0^3}{g_0^2}$$

lands to the $\boxed{\text{west}}$

$$\psi(T) = -\frac{4}{3}\frac{\omega}{g_0^2}(2g_0 h)^{3/2}$$

$$= -\frac{4}{3}\frac{\omega}{g_0^{1/2}} 2\sqrt{2}\, h^{3/2}$$

$$= -\frac{8\sqrt{2}}{3}\underbrace{\underbrace{\omega h}_{\text{velocity}} \underbrace{\sqrt{\frac{h}{g_0}}}_{\text{time}}}_{\text{distance}}$$

$$\frac{\text{velocity}^3 \cdot \frac{1}{T}}{\frac{L^2}{T^4}} = \frac{\left(\frac{L}{T}\right)^3 \cdot \frac{1}{T}}{\frac{L^2}{T^4}} = L$$

substitute numbers:

$$\psi(T) = \frac{-8\sqrt{2}}{3}\left(\frac{2\pi}{24 \cdot 3600\,s}\right)(100\times10^3\,m)\sqrt{\frac{100\times10^3\,m}{9.8\,m/s^2}}$$

$$= \boxed{-2.77\ km}$$

Prob 1.7 Dropped from equator

Following the analysis in the previous problem:

$y_0(t) = 0, \qquad z_0(t) = h - \frac{1}{2} g_0 t^2$

$y(t) = \psi(t), \qquad z(t) = z_0(t) + \zeta(t)$

$\ddot{\zeta} = 0 \rightarrow \zeta(t) = At + B$

In order for $z(t) = z_0(t) + \zeta(t)$ to satisfy

$z(0) = h, \ \dot{z}(0) = 0 \rightarrow A = 0, \ B = 0$

$\ddot{\psi} = -2\omega \dot{z}_0 = 2\omega g_0 t$

$\rightarrow \psi(t) = \frac{1}{3}\omega g_0 t^3 + At + B$

In order for $y(t) = \psi(t)$ to satisfy

$y(0) = 0, \ \dot{y}(0) = 0 \rightarrow A = 0, \ B = 0$

Thus, $$\boxed{\begin{aligned} z(t) &= h - \tfrac{1}{2} g_0 t^2 \\ y(t) &= \tfrac{1}{3}\omega g_0 t^3 \end{aligned}}$$

object hits the ground at time T:

$0 = h - \frac{1}{2} g_0 T^2 \rightarrow T = \sqrt{\frac{2h}{g_0}}$

Distance traveled in y-direction

$$y(T) = \frac{1}{3}\omega g_0 \left(\frac{2h}{g_0}\right)^{3/2} = \frac{1}{3} 2\sqrt{2}\, \omega g_0 \frac{h^{3/2}}{g_0^{3/2}}$$

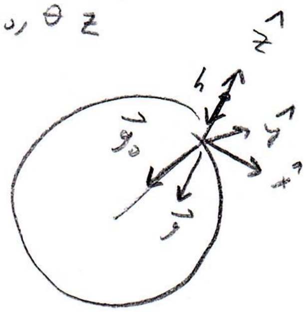
$$= \boxed{\frac{2\sqrt{2}}{3}\omega h \sqrt{\frac{h}{g_0}}} = \boxed{0.69 \text{ km}} \ \text{(to the east)}$$

Prob: Drop from arbitrary latitude

Again define $\hat{z}$ wrt $\vec{g} \equiv \vec{g}_0 - \vec{\omega}\times(\vec{\omega}\times\vec{R})$

$$\vec{\omega} \approx -\omega\sin\theta\,\hat{x} + \omega\cos\theta\,\hat{z}$$

Equations:



$$\ddot{x} = \omega\cos\theta\,(\omega\sin\theta\, z + \omega\cos\theta\, x) + 2\omega\cos\theta\,\dot{y}$$

$$\ddot{y} = \omega^2 y - 2(\omega\sin\theta\,\dot{z} + \omega\cos\theta\,\dot{x})$$

$$\ddot{z} = -g + 2\omega\sin\theta\,\dot{y} + \omega\sin\theta\,(\omega\sin\theta\, z + \omega\cos\theta\, x)$$

0th order:

$$\ddot{z} = -g$$
$$z(0) = h, \quad \dot{z}(0) = 0$$
$$\boxed{z(t) = h - \tfrac{1}{2} g t^2} \rightarrow z(0) = h$$
$$\dot{z}(t) = -gt \rightarrow \dot{z}(0) = 0$$
$$\ddot{z}(t) = -g$$

$\ddot{y} = 0$

$\ddot{z} = 0$

1st order:

$$\ddot{y} = -2\omega\sin\theta\,\dot{z}$$
$$= +2\omega\sin\theta\, g t$$
$$\rightarrow \dot{y} = \omega\sin\theta\, g t^2$$
$$\rightarrow \boxed{y = \tfrac{1}{3}\omega\sin\theta\, g\, t^3}$$

2nd order:

$$\ddot{x} = \omega^2\sin\theta\cos\theta\,\underbrace{\left[h - \tfrac{1}{2} g t^2\right]}_{z} + \overset{\text{2nd order, ignore}}{\cancel{\omega^2\cos^2\theta\, x}}$$
$$+ 2\omega\cos\theta\,\underbrace{\left[\omega\sin\theta\, g t^2\right]}_{\dot{y}}$$

$$\ddot{x} = \omega^2 \sin\theta \cos\theta \left[h - \tfrac{1}{2} g t^2 \right] + 2\omega^2 \sin\theta \cos\theta \, g t^2$$

$$= \omega^2 \sin\theta \cos\theta \left[h - \tfrac{1}{2} g t^2 + 2 g t^2 \right]$$

$$= \omega^2 \sin\theta \cos\theta \left[h + \tfrac{3}{2} g t^2 \right]$$

$$\to \quad \dot{x} = \omega^2 \sin\theta \cos\theta \left[h t + \tfrac{1}{2} g t^3 \right]$$

$$\to \quad \boxed{x = \omega^2 \sin\theta \cos\theta \left[\tfrac{1}{2} h t^2 + \tfrac{1}{8} g t^4 \right]}$$

Displacement: T given by $z(T) = 0 = h - \frac{1}{2} g T^2 \to T = \sqrt{\frac{2h}{g}}$

$$\Delta y = y(T)$$

$$= \frac{1}{3} \omega \sin\theta \, g \left(\frac{2h}{g} \right)^{3/2}$$

$$= \frac{2^{3/2}}{3} \omega \sin\theta \frac{h^{3/2}}{g^{1/2}}$$

$$= \frac{2\sqrt{2}}{3} \omega \sin\theta \sqrt{\frac{h^3}{g}}$$

$$= \boxed{\frac{2}{3} \omega \sin\theta \sqrt{\frac{2h^3}{g}}} \quad \text{vs.} \quad \overbrace{-\frac{8}{3} \omega \sin\theta \sqrt{\frac{2h^3}{g}}}^{\text{for Launch}} \quad \left(\text{differ by } -4\times \right)$$

$$\Delta x = x(T)$$

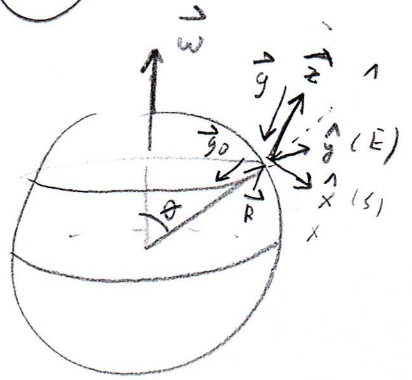
$$= \omega^2 \sin\theta \cos\theta \left[\frac{1}{\cancel{2}} h \left(\frac{\cancel{2}h}{g} \right) + \frac{1}{8} g \frac{4h^2}{g^2} \right]$$

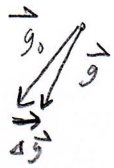
$$= \omega^2 \sin\theta \cos\theta \left[\frac{3}{2} \frac{h^2}{g} \right]$$

$$= \boxed{\frac{3}{2} \omega^2 \sin\theta \cos\theta \frac{h^2}{g}} \quad \text{vs.} \quad \overbrace{-8 \omega^2 \sin\theta \cos\theta \frac{h^2}{g}}^{\text{for launch}}$$

$$\left(\text{differ by } -\frac{16}{3} \times \right)$$

prob (1.8) Launch from arbitrary latitude





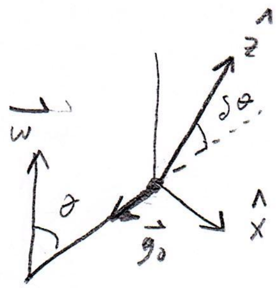
NOTE: $\hat{z}$ is opposite $\vec{g} \equiv \vec{g}_0 - \vec{\omega}\times(\vec{\omega}\times\vec{R})$

$$\vec{g} = -g\hat{z}$$

$$\vec{V}_0 = V_0\hat{z}$$

$$\vec{a} = \left(\vec{g}_0 - \vec{\omega}\times(\vec{\omega}\times\vec{R})\right) - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2\vec{\omega}\times\vec{v}$$

$$= \vec{g} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2\vec{\omega}\times\vec{v}$$



$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$$

$$\vec{a} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$

$$\vec{\omega} = -\omega\sin(\theta-\delta\theta)\hat{x} + \omega\cos(\theta-\delta\theta)\hat{z}$$

$$\approx -\omega\sin\theta\,\hat{x} + \omega\cos\theta\,\hat{z}$$

$\delta\theta = O(\omega^2)$
[2nd-order in ω^2]

Thus, $\cancel{\vec{\omega}\times\vec{R} = (\omega\sin\theta\,\hat{x} + \omega\cos\theta\,\hat{z})\times R\hat{z}}$

$$\cancel{= R\omega\sin\theta\,\hat{y}}$$

$$\cancel{\vec{\omega}\times(\vec{\omega}\times\vec{R}) = (-\omega\sin\theta\,\hat{x} + \omega\cos\theta\,\hat{z})\times R\omega\sin\theta\,\hat{y}}$$

$$\cancel{= -R\omega^2\sin^2\theta\,\hat{z} - R\omega^2\sin\theta\cos\theta\,\hat{x}}$$

$$\vec{\omega}\times\vec{r} \approx (-\omega\sin\theta\,\hat{x} + \omega\cos\theta\,\hat{z})\times(x\hat{x} + y\hat{y} + z\hat{z})$$

$$= -\omega\sin\theta\,y\,\hat{z} + \omega\sin\theta\,z\,\hat{y}$$

$$+ \omega\cos\theta\,x\,\hat{y} - \omega\cos\theta\,y\,\hat{x}$$

$$= -\omega\cos\theta\,y\,\hat{x} + (\omega\sin\theta\,z + \omega\cos\theta\,x)\hat{y}$$

$$-\omega\sin\theta\,y\,\hat{z}$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = (-\omega \sin\theta\ \hat{x} + \omega\cos\theta\ \hat{z}) \times \Big[-\omega\cos\theta\, y\ \hat{x} + (\omega\sin\theta\, z + \omega\cos\theta\, x)\hat{y} - \omega\sin\theta\, y\ \hat{z} \Big]$$

$$= -\omega\sin\theta\ (\omega\sin\theta\ z + \omega\cos\theta\ x)\ \hat{z} - \omega^2\sin^2\theta\ y\ \hat{y} - \omega^2\cos^2\theta\ y\ \hat{y} - \omega\cos\theta\ (\omega\sin\theta\ z + \omega\cos\theta\ x)\ \hat{x}$$

$$= -\omega\cos\theta\ (\omega\sin\theta\ z + \omega\cos\theta\, x)\ \hat{x} - \omega^2 y\ \hat{y} - \omega\sin\theta\ (\omega\sin\theta\ z + \omega\cos\theta\ x)\ \hat{z}$$

$$\vec{\omega} \times \vec{v} = (-\omega\sin\theta\ \hat{x} + \omega\cos\theta\ \hat{z}) \times (\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z})$$
$$= -\omega\cos\theta\ \dot{y}\ \hat{x} + (\omega\sin\theta\ \dot{z} + \omega\cos\theta\ \dot{x})\hat{y} - \omega\sin\theta\ \dot{y}\ \hat{z}$$

$$\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z} = -g\hat{z} + \omega\cos\theta\ (\omega\sin\theta\ z + \omega\cos\theta\ x)\ \hat{x} + \omega^2 y\ \hat{y} + \omega\sin\theta\ (\omega\sin\theta\ z + \omega\cos\theta\ x)\ \hat{z}$$
$$\cancel{+ R\omega^2\sin^2\theta\ \hat{z}} \quad \cancel{+ R\omega^2\sin\theta\cos\theta\ \hat{x}}$$
$$+ 2\omega\cos\theta\ \dot{y}\ \hat{x} - 2(\omega\sin\theta\ \dot{z} + \omega\cos\theta\ \dot{x})\ \hat{y} + 2\omega\sin\theta\ \dot{y}\ \hat{z}$$

$$= \hat{x}\Big[w\cos\theta\,(w\sin\theta\, z + w\cos\theta\, x) + \cancel{R w^2 \sin\theta \cos\theta} + 2w\cos\theta\, \dot{y} \Big]$$

$$+ \hat{y}\Big[w^2 y - 2(w\sin\theta\, \dot{z} + w\cos\theta\, \dot{x}) \Big]$$

$$+ \hat{z}\Big[-g + \cancel{R w^2 \sin\theta} + 2w\sin\theta\, \dot{y} + w\sin\theta\,(w\sin\theta\, z + w\cos\theta\, x) \Big]$$

---

so $$\ddot{x} = w\cos\theta\,(w\sin\theta\, z + w\cos\theta\, x) + \cancel{R w^2 \sin\theta\cos\theta} + 2w\cos\theta\,\dot{y}$$

$$\ddot{y} = w^2 y - 2(w\sin\theta\,\dot{z} + w\cos\theta\,\dot{x})$$

$$\ddot{z} = \left(-g + \cancel{R w^2 \sin\theta}\right) + 2w\sin\theta\,\dot{y} + w\sin\theta\,(w\sin\theta\, z + w\cos\theta\, x)$$

0th order:

$\ddot{x}_0 = 0$, $\ddot{y}_0 = 0$, $\ddot{z}_0 = -g_0$

$$\boxed{z_0(t) = v_0 t - \tfrac{1}{2} g t^2}$$ using $z(0) = 0$, $\dot{z}(0) = v_0$

$$\boxed{\begin{aligned} y_0(t) &= 0 \\ x_0(t) &= 0 \end{aligned}}$$ using $x(0) = 0$, $\dot{x}(0) = 0$, $y(0) = 0$, $\dot{y}(0) = 0$

1st order:

$\ddot{x} = 0$, $\ddot{y} = -2w\sin\theta\,\dot{z}$, $\ddot{z} = -g$

Thus, $$\boxed{\ddot{y} = -2w\sin\theta\,(v_0 - g_0 t)}$$

$$\ddot{y} = -2w\sin\theta\, v_0 + 2w\sin\theta\, g_0 t$$

$$\rightarrow \dot{y} = -2w\sin\theta\, v_0 t + w\sin\theta\, g_0 t^2$$

$$\rightarrow \boxed{y = -w\sin\theta\, v_0 t^2 + \tfrac{1}{3} w\sin\theta\, g_0 t^3} \quad (1^{st}\text{ order})$$

Have to go to 2nd order to find x: ④

$$\ddot{x} = \overset{1^{st}}{\omega} \cos\theta \left(\overset{1^{st}}{\omega \sin\theta}\, \overset{0^{th}}{z} + \underset{1^{st}}{\omega \cos\theta}\, \overset{2^{nd}}{x} \right)$$

$$+ \cancel{R\omega^2 \sin\theta \cos\theta} + 2\omega \cos\theta\, \dot{y}$$

$$\approx \omega^2 \sin\theta \cos\theta\, z + \cancel{R\omega^2 \sin\theta \cos\theta} + 2\omega\cos\theta\, \dot{y}$$

$$= \omega^2 \sin\theta \cos\theta \left[\cancel{R} + v_0 t - \tfrac{1}{2} g_0 t^2 \right]$$

$$+ 2\omega\cos\theta \left[-2\omega \sin\theta\, v_0 t + \omega \sin\theta\, g t^2 \right]$$

$$= \cancel{R\omega^2 \sin\theta\cos\theta} + \underline{\omega^2 \sin\theta \cos\theta\, v_0 t} - \underline{\tfrac{1}{2}\omega^2 \sin\theta\cos\theta\, g t^2}$$

$$\underline{-4\omega^2 \sin\theta\cos\theta\, v_0 t} + 2\omega^2 \sin\theta \cos\theta\, g_0 t^2$$

$$= \cancel{R\omega^2 \sin\theta\cos\theta} - 3\omega^2 \sin\theta\cos\theta\, v_0 t + \tfrac{3}{2}\omega^2 \sin\theta \cos\theta\, g t^2$$

$$= \omega^2 \sin\theta\cos\theta \left[\cancel{R} - 3 v_0 t + \tfrac{3}{2} g t^2 \right]$$

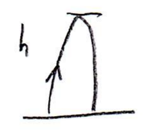
$$= \omega^2 \sin\theta \cos\theta \left[\cancel{R} - 3 z(t) \right]$$

$$\rightarrow \quad \dot{x} = \omega^2 \sin\theta \cos\theta \left[\cancel{Rt} - \tfrac{3}{2} v_0 t^2 + \tfrac{1}{2} g t^3 \right] \qquad 2^{nd} \text{ order}$$

$$\rightarrow \quad \boxed{x = \omega^2 \sin\theta \cos\theta \left[\cancel{\tfrac{1}{2}Rt^2} - \tfrac{1}{2} v_0 t^3 + \tfrac{1}{8} g t^4 \right]}$$

~~Time to hit~~

Time of flight: $T = \dfrac{2v_0}{g} = \dfrac{2}{g}\underbrace{\sqrt{2gh}}_{v_0} = \boxed{2\sqrt{\dfrac{2h}{g}}}$ 0^{th} order.



$$\Delta y = y(T)$$

$$= -\omega\sin\theta\, v_0 \left(\frac{2v_0}{g}\right)^2 + \frac{1}{3}\omega\sin\theta\, g\left(\frac{2v_0}{g}\right)^3$$

$$= -4\omega\sin\theta\,\frac{v_0^3}{g^2} + \frac{8}{3}\omega\sin\theta\, g\,\frac{v_0^3}{g^3}$$

$$= -\frac{4}{3}\omega\sin\theta\,\frac{v_0^3}{g^2}$$

$$= -\frac{4}{3}\omega\sin\theta\,\frac{1}{g^2}(2gh)^{3/2} \qquad \text{using } v_0 = \sqrt{2gh}$$

$$= \boxed{-\frac{8}{3}\omega\sin\theta\sqrt{\frac{2h^3}{g}}}$$

$$\Delta x = x(T)$$

$$= \omega^2\sin\theta\cos\theta\left[\underbrace{\cancel{\frac{1}{2}R\left(\frac{2v_0}{g_0}\right)^2}}_{\cancel{2R\frac{v_0^2}{g_0^2}}} - \underbrace{\frac{1}{2}v_0\left(\frac{2v_0}{g}\right)^3}_{-4\frac{v_0^4}{g^3}} + \underbrace{\frac{1}{8}g\left(\frac{2v_0}{g}\right)^4}_{2\frac{v_0^4}{g^3}}\right]$$

$$= \omega^2\sin\theta\cos\theta\left[\cancel{2R\frac{v_0^2}{g_0^2}} - 2\frac{v_0^4}{g^3}\right]$$

$$= \omega^2\sin\theta\cos\theta\left[\cancel{2R\left(\frac{2gh}{g_0^2}\right)} - 2\left(\frac{4g^2h^2}{g^3}\right)\right]$$

$$= \omega^2\frac{\sin(2\theta)}{2}\left[\cancel{4\left(\frac{Rh}{g_0}\right)} - 8\left(\frac{h^2}{g}\right)\right]$$

$h = 100$ km
$R = 6400$ km

$$= \boxed{-4\omega^2\sin(2\theta)\frac{h^2}{g}} = -8\omega^2\sin\theta\cos\theta\frac{h^2}{g}$$

(c) Numerical example

26° north latitude $\rightarrow$ $\theta = 90° - 26° = 64°$

$h = 100\ \text{km} = 10^5\ \text{m}$

$$D = \sqrt{\Delta x^2 + \Delta y^2}$$

where $\Delta x = -8\,\omega^2 \sin\theta \cos\theta \,\dfrac{h^2}{g}$

with $\omega = \dfrac{2\pi}{1\,\text{day}}$

$g \approx 9.8\ \text{m/s}^2$ $\left(\text{since } \dfrac{|\delta g|}{g_0} \leq 0.003 = 0.3\% \text{ at equator} \right)$

$$\Delta y = -\frac{8}{3}\sqrt{\frac{2h^3}{g}}\,\omega \sin\theta$$

$$\rightarrow \boxed{\begin{aligned} \Delta x &= -17\ \text{m} \\ \Delta y &= -2.49 \times 10^3\ \text{m} \end{aligned}}$$

$$\rightarrow \boxed{D = 2.49\ \text{km}}$$

$$\phi = \operatorname{atan}\left(\frac{17}{2.49 \times 10^3}\right)$$

$$= \boxed{+179.6°}$$