E.11

$$y'' + \pi^2 y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Thus, $0 = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \pi^2 \sum_{n=0}^{\infty} a_n x^n$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \pi^2 \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \pi^2 \sum_{n=0}^{\infty} a_n x^n$$

where
$m = n-2$
$n = m+2$

$$= \sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} + \pi^2 a_n \right) x^n$$

$$\rightarrow \quad a_{n+2} (n+2)(n+1) + \pi^2 a_n = 0$$

$$\boxed{a_{n+2} = \frac{-\pi^2 a_n}{(n+1)(n+2)}}$$

E.2 orthogonality property

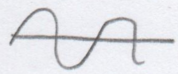
$$\int_{-\pi}^{\pi} dx \, \sin(nx)\sin(mx) = \int_{-\pi}^{\pi} dx \, \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)\frac{1}{2i}\left(e^{imx} - e^{-imx}\right)$$

$$= -\frac{1}{4}\int_{-\pi}^{\pi} dx \left[e^{i(n+m)x} + e^{-i(n+m)x} - e^{-i(n-m)x} - e^{i(n-m)x}\right]$$

Now: $\int_{-\pi}^{\pi} dx\, e^{ipx} = \frac{1}{ip} e^{ipx}\Big|_{-\pi}^{\pi}$ (assuming $p \neq 0$)

$$= \frac{1}{ip}\left(e^{ip\pi} - e^{-ip\pi}\right)$$

$$= \frac{2}{p}\sin(p\pi)$$



$$= 0 \quad (p \neq 0)$$

If $p = 0$, $\int_{-\pi}^{\pi} dx\, e^{ipx} = 2\pi$

Thus, for $n \neq m$

$$\int_{-\pi}^{\pi} dx \, \sin(nx)\sin(mx) = -\frac{1}{4}\left[0 + 0 - 0 - 0\right] = 0$$

For $n = m$:

$$\int_{-\pi}^{\pi} dx \, \sin(nx)\sin(mx) = -\frac{1}{4}\left[0 + 0 - 2\pi - 2\pi\right] = \pi$$

so $\int_{-\pi}^{\pi} dx \, \sin(nx)\sin(mx) = \pi\,\delta_{nm}$

$$\int_{-\pi}^{\pi} dx \cos(nx)\cos(mx) = \int_{-\pi}^{\pi} dx \frac{1}{2}\left(e^{inx}+e^{-inx}\right)\left(\frac{1}{2}\right)\left(e^{imx}+e^{-imx}\right)$$

$$= \frac{1}{4}\int_{-\pi}^{\pi} dx \left[e^{i(n+m)x} + e^{-i(n+m)x} + e^{-i(n-m)x} + e^{+i(n-m)x}\right]$$

$$= \begin{cases} \frac{1}{4}[0+0+0+0] = 0 & (n \neq m) \\ \frac{1}{4}[0+0+2\pi+2\pi] = \pi & (n = m) \end{cases}$$

Thus, $\displaystyle\int_{-\pi}^{\pi} dx \cos(nx)\cos(mx) = \pi\,\delta_{nm}$

$$\int_{-\pi}^{\pi} dx \sin(nx)\cos(mx) = \int_{-\pi}^{\pi} dx \frac{1}{2i}\left(e^{inx}-e^{-inx}\right)\frac{1}{2}\left(e^{imx}+e^{-imx}\right)$$

$$= \frac{1}{4i}\int_{-\pi}^{\pi} dx \left[e^{i(n+m)x} - e^{-i(n+m)x} - e^{-i(n-m)x} + e^{i(n-m)x}\right]$$

$$= \begin{cases} 0 & (n \neq m) \\ \frac{1}{4i}[0-0-2\pi+2\pi] = 0 & (n = m) \end{cases}$$

$$\rightarrow \int_{-\pi}^{\pi} dx \sin(nx)\cos(mx) = 0$$

Prob E.3 Hyperbolic function identities

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

Thus, $\frac{1}{2}(e^{x}+e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

$$= \cosh x$$

$$\frac{1}{2}(e^{x}-e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$= \sinh x$$

Using $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

we have

$$\cos(ix) = \frac{e^{i(ix)}+e^{-i(ix)}}{2} = \frac{e^{-x}+e^{x}}{2} = \cosh x$$

$$\sin(ix) = \frac{e^{i(ix)}-e^{-i(ix)}}{2i} = \frac{e^{-x}-e^{x}}{2i} = -\frac{1}{i}\left(\frac{e^{x}-e^{-x}}{2}\right)$$

$$= -\frac{1}{i}\sinh x$$

Thus, $\sinh x = -i \sin(ix)$

Problem E.4 Legendre polynomial recurrence relation

a) $(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$

$x=0$ is a regular point

$\rightarrow$ $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n$

$y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$

Thus,

$$0 = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n n(n-1)x^n$$

$$- 2\sum_{n=0}^{\infty} a_n n x^n + \ell(\ell+1)\sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) - a_n\left(n(n-1) + 2n - \ell(\ell+1)\right)\right]x^n$$

So $a_{n+2}(n+1)(n+2) - a_n(\underbrace{n(n-1)+2n}_{\substack{=n^2-n+2n\\ =n^2+n\\ =n(n+1)}} - \ell(\ell+1)) = 0$

$$\rightarrow \boxed{a_{n+2} = \frac{a_n\left(n(n+1) - \ell(\ell+1)\right)}{(n+1)(n+2)}} \quad n=0,1,2,\cdots$$

b) Take $a_0 = 0$, $a_1 = 1$, $l = 0$

$a_0 = 0 \rightarrow a_2 = 0, \ a_4 = 0, \ \ldots$

$$l = 0 \rightarrow a_{n+2} = \frac{a_n \, n \cancel{(n+1)}}{\cancel{(n+1)}(n+2)} = \frac{a_n \, n}{n+2}$$

$$a_1 = 1 \rightarrow a_3 = \frac{1}{3}$$

$$a_5 = \frac{a_3 \cdot 3}{5} = \frac{1}{3} \cdot \frac{3}{5} = \frac{1}{5}$$

$$a_7 = \frac{a_5 \cdot 5}{7} = \frac{1}{5} \cdot \frac{5}{7} = \frac{1}{7}$$

etc.

so
$$y(x) = a_1 x + a_3 x^3 + a_5 x^5 + \cdots$$
$$= x + \frac{1}{3} x^3 + \frac{1}{5} x^5 + \cdots$$

c) At $x = \pm 1$, $y(x) = \pm \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots \right)$

Integral test $\displaystyle \pm \int^{\infty} \frac{dn}{(2n+1)} = \pm \frac{1}{2} \int^{\infty} \frac{dx}{x} = \pm \frac{1}{2} \ln x \Big|^{\infty} \rightarrow \pm \infty.$

let $x = 2n+1$
$dx = 2dn$

Thus, this solution diverges at $x = \pm 1$

Problem: 1^{st} few Legendre polynomials

E.5

$$a_{n+2} = \left(\frac{n(n+1) - l(l+1)}{(n+1)(n+2)} \right) a_n$$

$l=0$:

$$a_1 = 0 \rightarrow a_{odd} = 0$$

$$a_{n+2} = \frac{n\cancel{(n+1)}}{\cancel{(n+1)}(n+2)} a_n = \frac{n}{(n+2)} a_n$$

Thus, $a_0 \neq 0 \rightarrow a_2 = \frac{0 \cdot a_0}{2} = 0$

$a_4 = 0$

etc.

$\rightarrow P_0(x) = a_0$

Normalize so that $P_0(1) = 1 \rightarrow a_0 = 1$

$$\boxed{P_0(x) = 1}$$

$l=1$:

$$a_0 = 0 \rightarrow a_{even} = 0$$

$$a_{n+2} = \left(\frac{n(n+1) - 2}{(n+1)(n+2)} \right) a_n$$

$$a_3 = \left(\cancel{\frac{1 \cdot 2 - 2}{2 \cdot 3}} \right) a_1 = 0, \rightarrow a_5 = 0, a_7 = 0, \text{etc.}$$

Thus, $P_1(x) = a_1 x$

Normalize: $P_1(1) = 1 \rightarrow a_1 \cdot 1 = 1 \rightarrow a_1 = 1$

$$\boxed{P_1(x) = x}$$

$\underline{l=2}$: $\quad a_1 = 0 \rightarrow a_{odd} = 0$

$$a_{n+2} = \left(\frac{n(n+1) - 2\cdot 3}{(n+1)(n+2)} \right) a_n$$

$$\rightarrow a_2 = \left(\frac{\cancel{0\cdot 1}^{0} - \cancel{2}\cdot 3}{1\cdot \cancel{2}} \right) a_0 = -3a_0$$

$$a_4 = \cancel{\left(\frac{2\cdot 3 - 2\cdot 3}{3\cdot 4} \right)}^{0} a_2 = 0, \quad a_6 = 0, \ldots$$

so $P_2(x) = a_0 - 3a_0 x^2$

$= a_0(1-3x^2)$

$\underline{\text{Normalize:}}$ $\quad P_2(1) = 1 = a_0(1-3\cdot 1^2) = -2a_0$

$$a_0 = -\frac{1}{2}$$

$$\text{so} \quad \boxed{P_2(x) = -\frac{1}{2}(1-3x^2) = \frac{1}{2}(3x^2-1)}$$

$\underline{l=3}$: $\quad a_0 = 0 \rightarrow a_{even} = 0$

$$a_{n+2} = \left(\frac{n(n+1) - 3\cdot 4}{(n+1)(n+2)} \right) a_n$$

$$\rightarrow a_3 = \left(\frac{1\cdot 2 - 3\cdot 4}{2\cdot 3} \right) a_1 = -\frac{5}{3} a_1$$

$$a_5 = \cancel{\left(\frac{3\cdot 4 - 3\cdot 4}{4\cdot 5} \right)}^{0} a_3 = 0, \quad a_7 = 0, \text{ etc.}$$

Thus, $P_3(x) = a_1 x - \frac{5}{3} a_1 x^3$

$= a_1\left(x - \frac{5}{3}x^3\right)$

Normalize: $P_3(1) = 1 \rightarrow a_1\left(1 - \frac{5}{3} \cdot 1^3\right) = 1$

$$a_1\left(\frac{-2}{3}\right) = 1$$

$$a_1 = \frac{-3}{2}$$

Thus,

$$\boxed{P_3(x) = \frac{-3}{2}\left(x - \frac{5}{3}x^3\right) = \frac{1}{2}\left(5x^3 - 3x\right)}$$

Problem: Legendre polynomials for l = negative integer

E.6

$$a_{n+2} = \left(\frac{n(n+1) - l(l+1)}{(n+1)(n+2)} \right) a_n$$

Suppose $l = -1, -2, \ldots$

a) $$a_{n+2} = \left(\frac{n(n+1) + |l|(-|l|+1)}{(n+1)(n+2)} \right) a_n$$

Series terminates when

$$n(n+1) + |l|(-|l|+1) = 0$$

$$n(n+1) = (|l|-1)|l|$$

or $$\boxed{n = |l| - 1}$$

b) Thus, $l = -1 \leftrightarrow n = 0 \rightarrow P_0(x) = P_{-1}(x)$

$l = -2 \leftrightarrow n = 1 \rightarrow P_1(x) = P_{-2}(x)$

$l = -3 \leftrightarrow n = 2 \rightarrow P_2(x) = P_{-3}(x)$

so no loss in generality assuming $l = 0, 1, 2, \ldots$

Problem: E.7 Orthogonality and normalization of Legendre polynomials

Want to show: $\displaystyle\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'}$

Orthogonality:

Legendre's equation

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$$

$$\frac{d}{dx}\left[(1-x^2)y'\right] + \ell(\ell+1)y = 0$$

Thus,

$$\frac{d}{dx}\left[(1-x^2)P_\ell'\right] + \ell(\ell+1)P_\ell = 0$$

$$\frac{d}{dx}\left[(1-x^2)P_{\ell'}'\right] + \ell'(\ell'+1)P_{\ell'} = 0$$

Multiply 1st by $P_{\ell'}$, the second by P_ℓ, then subtract and integrate:

$$0 = \int_{-1}^{1} dx \left\{ P_{\ell'} \frac{d}{dx}\left[(1-x^2)P_\ell'\right] - P_\ell \frac{d}{dx}\left[(1-x^2)P_{\ell'}'\right] + \left(\ell(\ell+1) - \ell'(\ell'+1)\right) P_\ell P_{\ell'} \right\}$$

$$= \left(\ell(\ell+1) - \ell'(\ell'+1)\right) \int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x)$$

$$+ \int_{-1}^{1} dx \left\{ P_{\ell'} \frac{d}{dx}\left[(1-x^2)P_\ell'\right] - P_\ell \frac{d}{dx}\left[(1-x^2)P_{\ell'}'\right]\right\}$$

The last two terms can be integrated by parts

$$\int_{-1}^{1} dx\ P_{\ell'} \frac{d}{dx}\left[(1-x^2)P_\ell'\right] = \int_{-1}^{1} dx\ \frac{d}{dx}\left\{P_{\ell'}(1-x^2)P_\ell'\right\}$$

$$- \int_{-1}^{1} dx\ P_{\ell'}'(1-x^2)P_\ell'$$

$$= P_{\ell'}\underbrace{(1-x^2)}_{0}P_\ell'\Big|_{-1}^{1} - \int_{-1}^{1} dx\,(1-x^2)\,P_{\ell'}'P_\ell'$$

similarly

$$\int_{-1}^{1} dx\ P_\ell \frac{d}{dx}\left[(1-x^2)P_{\ell'}'\right] = -\int_{-1}^{1} dx\ (1-x^2)\,P_\ell' P_{\ell'}'$$

so they cancel when subtracted from one another.

Thus,
$$0 = \left(\ell(\ell+1) - \ell'(\ell'+1)\right)\int_{-1}^{1} dx\ P_\ell(x)P_{\ell'}(x)$$

$$= \int_{-1}^{1} dx\,P_\ell(x)\,P_{\ell'}(x)$$

assuming $\ell \neq \ell'$

If $\ell = \ell'$, want to show $\int_{-1}^{1} dx\ P_\ell^2(x) = \frac{2}{2\ell+1}$

For $\ell = \ell' = 0$: $\quad \text{LHS} = \int_{-1}^{1} dx\ P_0^2(x) = 2$

$$\text{RHS} = \frac{2}{2\cdot 0+1} = 2$$

Thus, consider $\ell \geq 1$

Use Rodrigues' formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}\left[(x^2-1)^\ell\right]$$

so

$$\int_{-1}^{1} dx\, P_\ell^2(x) = \frac{1}{(2^\ell \ell!)^2}\int_{-1}^{1} dx\, \frac{d^\ell}{dx^\ell}\left[(x^2-1)^\ell\right]\frac{d^\ell}{dx^\ell}\left[(x^2-1)^\ell\right]$$

$$= \frac{1}{(2^\ell \ell!)^2}\left\{\int_{-1}^{1} dx\, \frac{d}{dx}\left\{\frac{d^\ell}{dx^\ell}\left[(x^2-1)^\ell\right]\frac{d^{\ell-1}}{dx^{\ell-1}}\left[(x^2-1)^\ell\right]\right\}\right.$$

$$\left. - \int_{-1}^{1} dx\, \frac{d^{\ell+1}}{dx^{\ell+1}}\left[(x^2-1)^\ell\right]\frac{d^{\ell-1}}{dx^{\ell-1}}\left[(x^2-1)^\ell\right]\right\}$$

$$= \frac{1}{(2^\ell \ell!)^2}\left\{\left.\cancelto{0}{\frac{d^\ell}{dx^\ell}\left[(x^2-1)^\ell\right]\frac{d^{\ell-1}}{dx^{\ell-1}}\left[(x^2-1)^\ell\right]}\right|_{-1}^{1}\right.$$

since there will be one factor of (x^2-1) remaining

$$\left. - \int_{-1}^{1} dx\, \frac{d^{\ell+1}}{dx^{\ell+1}}\left[(x^2-1)^\ell\right]\frac{d^{\ell-1}}{dx^{\ell-1}}\left[(x^2-1)^\ell\right]\right\}$$

$$= \frac{-1}{(2^\ell \ell!)^2}\int_{-1}^{1} dx\, \frac{d^{\ell+1}}{dx^{\ell+1}}\left[(x^2-1)^\ell\right]\frac{d^{\ell-1}}{dx^{\ell-1}}\left[(x^2-1)^\ell\right]$$

$\vdots$ [repeat integration by parts]

$$= \frac{(-1)^\ell}{(2^\ell \ell!)^2}\int_{-1}^{1} dx\, \frac{d^{2\ell}}{dx^{2\ell}}\left[(x^2-1)^\ell\right]\,(x^2-1)^\ell$$

<u>Now:</u>

$$\frac{d^{2l}}{dx^{2l}}\left[(x^2-1)^l\right] = \frac{d^{2l}}{dx^{2l}}\left[x^{2l} + O(x^{2l-1})\right]$$

$$= (2l)!$$

so $$\int_{-1}^{1} dx\, P_l^2(x) = \frac{(-1)^l (2l)!}{\left[2^l\, l!\right]^2} \int_{-1}^{1} dx\, (x^2-1)^l$$

Need to evaluate:

$$K_l \equiv \int_{-1}^{1} dx\, (x^2-1)^l$$

$$= \underbrace{(x^2-1)^l x \Big|_{-1}^{1}}_{0} - \int_{-1}^{1} 2lx^2 (x^2-1)^{l-1}\, dx$$

$u = (x^2-1)^l$
$dv = dx \rightarrow v = x$
$du = l(x^2-1)^{l-1}\, 2x\, dx$
$= 2lx(x^2-1)^{l-1}\, dx$

$$= -2l \int_{-1}^{1} x^2 (x^2-1)^{l-1}\, dx$$

$$= -2l \int_{-1}^{1} \left[(x^2-1)+1\right](x^2-1)^{l-1}\, dx$$

$$= -2l \left\{ \int_{-1}^{1} (x^2-1)^l\, dx + \int_{-1}^{1} (x^2-1)^{l-1}\, dx \right\}$$

$$= -2l \left\{ K_l + K_{l-1} \right\}$$

so $$(1+2l) K_l = -2l\, K_{l-1}$$

$$K_l = \frac{-2l}{1+2l} K_{l-1}$$

$$I_0 = \cancel{4} \int_{-1}^{1} dx \,(x^2-1)^0 = 2$$

$$I_1 = -\frac{2}{1+2} I_0 = -\frac{2\cdot 2}{3}$$

$$I_2 = \frac{-2\cdot 2}{1+4} I_1 = \frac{-2\cdot 2}{5}\left(\frac{-2\cdot 2}{3}\right) = \frac{2^2\, 4}{5\cdot 3}$$

$$I_3 = \frac{-6}{1+6} I_2 = -\frac{6}{7}\left(\frac{2^2\, 4}{5\cdot 3}\right) = -\frac{2\cdot\ 2\cdot 4\cdot 6}{3\cdot 5\cdot 7}$$

$$I_4 = -\frac{8}{1+8} I_3 = -\frac{8}{9}\left(\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\right)(-2) = +\frac{2\ (2\cdot 4\cdot 6\cdot 8)}{(3\cdot 5\cdot 7\cdot 9)}$$

so
$$I_\ell = (-1)^\ell\, 2\, \frac{(2\ell)(2\ell-2)\cdots 2}{(2\ell+1)(2\ell-1)\cdots 1}$$

$$= \frac{(-1)^\ell\, 2\, [(2\ell)(2\ell-2)\cdots 2]^2}{(2\ell+1)!}$$

$$= \frac{(-1)^\ell\, 2\, [2^\ell\, \ell(\ell-1)\cdots 1]^2}{(2\ell+1)!}$$

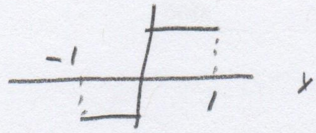
$$= \frac{(-1)^\ell\, 2^{2\ell+1}\, (\ell!)^2}{(2\ell+1)!}$$

$$\rightarrow \int_{-1}^{1} dx\, P_\ell^2(x) = \frac{\cancel{(-1)^\ell}(2\ell)!}{[2^\ell\, \cancel{\ell!}]^2}\ \frac{\cancel{(-1)^\ell}\, 2^{2\ell+1}\, \cancel{(\ell!)^2}}{(2\ell+1)!}$$

$$= \boxed{\frac{2}{2\ell+1}}$$

Problem E.8 Legendre polynomial expansion

$$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$



Expand in terms of Legendre polynomials:

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \quad , \quad A_l = \frac{2l+1}{2} \int_{-1}^{1} dx\, f(x) P_l(x)$$

Since $f(x)$ is odd, only have contributions from odd A_l. [i.e., $A_0, A_2, \dots = 0$]

l = odd:

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} dx \underbrace{f(x)}_{\text{odd}} \underbrace{P_l(x)}_{\text{odd}}$$

$$= 2l+1 \int_0^1 dx\, f(x) P_l(x)$$

$$= 2l+1 \int_0^1 dx\, P_l(x)$$

Thus,

$$A_1 = 3 \int_0^1 dx\, P_1(x)$$

$$= 3 \int_0^1 dx\, x$$

$$= 3 \frac{x^2}{2} \Big|_0^1$$

$$= \boxed{\frac{3}{2}}$$

$$A_3 = 7 \int_0^1 dx \, P_3(x)$$

$$= 7 \int_0^1 dx \, \frac{1}{2}(5x^3 - 3x)$$

$$= \frac{7}{2} \int_0^1 dx \, (5x^3 - 3x)$$

$$= \frac{7}{2} \left(5\frac{x^4}{4} - 3\frac{x^2}{2} \right) \Big|_0^1$$

$$= \frac{7}{2} \left(\frac{5}{4} - \frac{3}{2} \right)$$

$$= \frac{7}{2} \left(\frac{5}{4} - \frac{6}{4} \right)$$

$$= \boxed{-\frac{7}{8}}$$

$$A_5 = 11 \int_0^1 dx \, P_5(x)$$

$$= 11 \int_0^1 dx \, \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$= \frac{11}{8} \left(63\frac{x^6}{6} - 70\frac{x^4}{4} + 15\frac{x^2}{2} \right) \Big|_0^1$$

$$= \frac{11}{8} \left(\frac{63}{6} - \frac{70}{4} + \frac{15}{2} \right)$$

$$= \frac{11}{8} \left(\frac{21}{2} - \frac{35}{2} + \frac{15}{2} \right)$$

$$= \boxed{\frac{11}{16}}$$

Thus, $F(x) = \frac{3}{2} P_1(x) - \frac{7}{8} P_3(x) + \frac{11}{16} P_5(x) + \cdots$

Problem: Recurrence relations for Legendre polynomials

E.9

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Take $\frac{d}{dt}$: $\quad LHS = -\frac{1}{2}\left(1-2xt+t^2\right)^{-3/2}\left(-2x+2t\right)$

$$= \left(1-2xt+t^2\right)^{-3/2}(x-t)$$

$$RHS = \sum_{n=0}^{\infty} P_n(x)\, n\, t^{n-1}$$

$$= \sum_{n=1}^{\infty} P_n(x)\, n\, t^{n-1} \qquad m = n-1, \quad n = m+1$$

$$= \sum_{m=0}^{\infty} P_{m+1}(x)(m+1)\, t^m$$

$$= \sum_{n=0}^{\infty} P_{n+1}(x)(n+1)\, t^n$$

Thus, $$\boxed{\left(1-2xt+t^2\right)^{-3/2}(x-t) = \sum_{n=0}^{\infty} P_{n+1}(x)(n+1)\, t^n} \quad (1)$$

Take $\frac{d}{dx}$: $\quad LHS = -\frac{1}{2}\left(1-2xt+t^2\right)^{-3/2}\left(-2t\right)$

$$= t\left(1-2xt+t^2\right)^{-3/2}$$

$$RHS = \sum_{n=0}^{\infty} P_n'(x)\, t^n$$

Thus, $$\boxed{\left(1-2xt+t^2\right)^{-3/2} t = \sum_{n=0}^{\infty} P_n'(x)\, t^n} \quad (2)$$

Taking (2):

$$\frac{1}{\sqrt{1-2xt+t^2}}\, t = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)\, t^n$$

$$\sum_{n=0}^{\infty} P_n(x)\, t^{n+1} = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)\, t^n$$

$$0 = \sum_{n=0}^{\infty} P_n'(x)\left(t^n - 2x t^{n+1} + t^{n+2}\right) - P_n(x)\, t^{n+1}$$

$$= \sum_{n=0}^{\infty}\left(P_n'(x)\, t^n - \left(2x P_n'(x) + P_n(x)\right)t^{n+1} + P_n'(x)\, t^{n+2}\right)$$

$$= \underbrace{P_0'(x)}_{0} + \sum_{n=0}^{\infty}\left[\left(P_{n+1}'(x) - 2x P_n'(x) - P_n(x)\right)t^{n+1} + P_n'(x)\, t^{n+2}\right]$$

$$= \left(P_1'(x) - 2x \underbrace{P_0'(x)}_{0} - P_0(x)\right)t$$

$$+ \sum_{n=0}^{\infty}\left(P_{n+2}' - 2x P_{n+1}' - P_{n+1} + P_n'\right)t^{n+2}$$

$$= \underbrace{\left(P_1'(x) - P_0(x)\right)}_{=0} t + \sum_{n=0}^{\infty}\left(P_{n+2}' - 2x P_{n+1}' + P_n' - P_{n+1}\right)t^{n+2}$$

$$\longrightarrow \quad P_{n+2}' - 2x P_{n+1}' + P_n' - P_n = 0$$

$$P_{n+1} = P_n' - 2x P_{n+1}' + P_{n+2}'$$

$$\rightarrow \boxed{P_n = P_{n-1}' - 2x P_n' + P_{n+1}'} \qquad (E.30b)$$

(1) and (2):

$$(x-t)\sum_{n=0}^{\infty} P_n'(x)\, t^{n-1} = \sum_{n=0}^{\infty} P_{n+1}(x)\,(n+1)\,t^n$$

$$0 = \sum_{n=0}^{\infty}\left[P_n'(x)\left(x t^{n-1} - t^n\right) - P_{n+1}(x)(n+1)t^n\right]$$

$$= \sum_{n=0}^{\infty}\left[\left(-P_n' - P_{n+1}(n+1)\right)t^n + P_n' x t^{n-1}\right]$$

$$\cancel{= P_0'}$$

$$= \sum_{n=0}^{\infty}\left[\left(-P_n' - P_{n+1}(n+1)\right)t^n + P_{n+1}' x t^n\right] + \cancel{P_0'}\, x\, t^{-1} \to 0$$

$$= \sum_{n=0}^{\infty}\left[-P_n' - (n+1)P_{n+1} + x P_{n+1}'\right]t^n$$

$$\to \quad -P_n' - (n+1)P_{n+1} + x P_{n+1}' = 0$$

$$(n+1)P_{n+1} = -P_n' + x P_{n+1}'$$

$$\to \boxed{n P_n = -P_{n-1}' + x P_n'} \qquad (E.30c)$$

Add (E.30b) (E.30c)

$$\boxed{(n+1)P_n = -x P_n' + P_{n+1}'} \qquad (E.30d)$$

Add 2(E.30c) and (E.30b):

$$\boxed{(1+2n)P_n = -P'_{n-1} + P'_{n+1}} \qquad (E.30e)$$

Take (E.30d) replace $n \to n-1$

Take (E.30c) and multiply by $-x$,

$$n P_{n-1} = -x P'_{n-1} + P'_n$$

$$-x n P_n = +x P'_{n-1} - x^2 P'_n$$

$$\boxed{n(P_{n-1} - xP_n) = (1-x^2)P'_n} \qquad (3.30f)$$

Taking (1):

$$(1-2xt+t^2)^{-3/2}(x-t) = \sum_{n=0}^{\infty} P_{n+1}(n+1)t^n$$

$$\sum_{n=0}^{\infty} P_n t^n (x-t) = (1-2xt+t^2)\sum_{n=0}^{\infty} P_{n+1}(n+1)t^n$$

$$0 = \sum_{n=0}^{\infty} xP_n t^n - P_n t^{n+1} - (n+1)P_{n+1}t^n + 2x(n+1)P_{n+1}t^{n+1} - (n+1)P_{n+1}t^{n+2}$$

$$= \sum_{n=0}^{\infty}\Big[\big(xP_n - (n+1)P_{n+1}\big)t^n + \big(2x(n+1)P_{n+1} - P_n\big)t^{n+1} - (n+1)P_{n+1}t^{n+2}\Big]$$

$$= \underbrace{\cancel{(xP_0 - P_1)}}_{0} + \sum_{n=0}^{\infty}\Big[\big(xP_{n+1} - (n+2)P_{n+2} + 2x(n+1)P_{n+1} - P_n\big)t^{n+1} - (n+1)P_{n+1}t^{n+2}\Big]$$

$P_1 = x$, $P_0 = 1$

$$= \underbrace{\left(x P_1 - 2P_2 + 2x P_1 - P_0\right)}_{\substack{x^2 - (3x^2-1) + 2x^2 - 1 \\ = x^2 - 3x^2 + 1 + 2x^2 - 1 \\ = 0 \ \checkmark}} t + \sum_{n=0}^{\infty} \Big[x P_{n+2} - (n+3) P_{n+3} + 2x(n+2) P_{n+2} - P_{n+1} - (n+1) P_{n+1} \Big] t^{n+2}$$

$1 = P_0$

$x = P_1$

$\frac{1}{2}(3x^2-1) = P_2$

$$= \sum_{n=0}^{\infty} \underbrace{\Big[-(n+3) P_{n+3} + \big(x + 2x(n+2)\big) P_{n+2} - (n+2) P_{n+1} \Big]}_{=0} t^{n+2}$$

~~set~~ $n+3 \Rightarrow n+1$

so $n \to n-2$

$$-(n+1) P_{n+1} + (x + 2xn) P_n - n P_{n-1} = 0$$

$\cancel{(n+1) P_{n+1}}$

$$\boxed{(n+1) P_{n+1} = x(1+2n) P_n - n P_{n-1}} \qquad (E.30a)$$

Problem: Expressions for $P_\ell^m(x)$ solve the assoc. Legendre equation

E.10

Assoc. Legendre equation

$$(1-x^2)y'' - 2xy' + \left[\ell(\ell+1) - \frac{m^2}{(1-x^2)} \right] y = 0$$

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m P_\ell}{dx^m} \qquad (m>0)$$

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

— so if $P_\ell^m(x)$ is a sol'n, so is $P_\ell^{-m}(x)$

Consider: $y(x) = (1-x^2)^{m/2} u(x)$

substitute into assoc Legendre equation:

$$y' = \frac{m}{2}(1-x^2)^{\frac{m}{2}-1}(-2x)\,u(x) + (1-x^2)^{m/2} u'(x)$$

$$= -mx(1-x^2)^{\frac{m}{2}-1} u(x) + (1-x^2)^{m/2} u'(x)$$

$$y'' = -m(1-x^2)^{\frac{m}{2}-1} u(x) - mx\left(\frac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-2}(-2x)\,u(x)$$
$$- mx(1-x^2)^{\frac{m}{2}-1} u'(x) + \frac{m}{2}(1-x^2)^{\frac{m}{2}-1}(-2x)\,u'(x)$$
$$+ (1-x^2)^{m/2} u''(x)$$

$$= -m(1-x^2)^{\frac{m}{2}-1} u(x) + x^2 m(m-2)(1-x^2)^{\frac{m}{2}-2} u(x)$$
$$- 2mx(1-x^2)^{\frac{m}{2}-1} u'(x) + (1-x^2)^{m/2} u''(x)$$

$$= -m(1-x^2)^{\frac{m}{2}-2} u(x)\left[(1-x^2) - x^2(m-2)\right]$$
$$- 2mx(1-x^2)^{\frac{m}{2}-1} u'(x) + (1-x^2)^{m/2} u''(x)$$

Thus,

$$0 = (1-x^2)y'' - 2xy' + \left[l(l+1) - \frac{m^2}{(1-x^2)} \right] y$$

$$= -m(1-x^2)^{\frac{m}{2}-1} u(x) \left[(1-x^2) - x^2(m-2) \right]$$

$$- 2mx(1-x^2)^{\frac{m}{2}} u'(x) + (1-x^2)^{\frac{m}{2}+1} u''(x)$$

$$+ 2mx^2(1-x^2)^{\frac{m}{2}-1} u(x) - 2x(1-x^2)^{m/2} u'(x)$$

$$+ \left[l(l+1) - \frac{m^2}{(1-x^2)} \right] (1-x^2)^{m/2} u(x)$$

<u>Factor out $(1-x^2)^{\frac{m}{2}}$:</u>

$$0 = -m(1-x^2)^{-1} u(x) \left[(1-x^2) - x^2(m-2) \right]$$

$$+ 2mx^2(1-x^2)^{-1} u(x) + \left[l(l+1) - m^2(1-x^2)^{-1} \right] u(x)$$

$$- 2mx\, u'(x) - 2x\, u'(x) + (1-x^2) u''(x)$$

$$= (1-x^2) u''(x) - 2x(m+1) u'(x)$$

$$+ u(x) \underbrace{\left[-m + m x^2(m-2)(1-x^2)^{-1} + 2mx^2(1-x^2)^{-1} + l(l+1) - m^2(1-x^2)^{-1} \right]}$$

$$[\ ] = l(l+1) - m + \left(m^2x^2 \cancel{- 2mx^2} \cancel{+ 2mx^2} - m^2 \right)(1-x^2)^{-1}$$

$$= l(l+1) - m - m^2 \cancel{(1-x^2)(1-x^2)^{-1}}$$

$$= l(l+1) - m(m+1)$$

Thus,

$$\boxed{0 = (1-x^2)u''(x) - 2(m+1)\,x\,u'(x) + [\ell(\ell+1) - m(m+1)]\,u(x)} \quad (\star)$$

<u>Now:</u>

$\boxed{m=0}$: $\rightarrow$ $\underbrace{0 = (1-x^2)u'' - 2x\,u'(x) + \ell(\ell+1)\,u(x)}_{\text{Legendre's equation}}$

$\rightarrow u(x) = P_\ell(x)$ (polynomial solution)

Differentiate top equation $(\star)$

$$0 = -2x\,u'' + (1-x^2)(u')'' - 2(m+1)u' - 2(m+1)x(u')' + [\ell(\ell+1) - m(m+1)]\,u'$$

$$= (1-x^2)(u')'' + [-2x - 2(m+1)x]\,u'' + [\ell(\ell+1) - (m+2)(m+1)]\,u'$$

$$\underbrace{= (1-x^2)(u')'' - 2\big((m+1)+1\big)(u')' + [\ell(\ell+1) - (m+1)(m+2)]\,u'}_{\text{which is the same as } (\star) \text{ with } u \to u' \text{ and } m \to (m+1)}$$

Thus, if $P_\ell(x)$ is a solution of $(\star)$ with $m=0$

$P_\ell'(x)$ " $m=1$

and $P_\ell''(x)$ " $m=2$

etc.

Thus,

$$\boxed{0 = (1-x^2)u''(x) - 2(m+1)\,x\,u'(x) + [\ell(\ell+1) - m(m+1)]\,u(x)} \quad (\star)$$

<u>Now:</u>

$\boxed{m=0}$: $\rightarrow$ $$\underbrace{0 = (1-x^2)u'' - 2x\,u'(x) + \ell(\ell+1)\,u(x)}_{\text{Legendre's equation}}$$

$\rightarrow u(x) = P_\ell(x)$ (polynomial solution)

Differentiate top equation $(\star)$

$$0 = -2x\,u'' + (1-x^2)(u')'' - 2(m+1)u' - 2(m+1)x(u')' + [\ell(\ell+1) - m(m+1)]\,u'$$

$$= (1-x^2)(u')'' + [-2x - 2(m+1)x]\,u'' + [\ell(\ell+1) - (m+2)(m+1)]\,u'$$

$$\underbrace{= (1-x^2)(u')'' - 2\big((m+1)+1\big)(u')' + [\ell(\ell+1) - (m+1)(m+2)]\,u'}_{\substack{\text{which is the same as } (\star) \text{ with } u \to u' \\ \text{and } m \to (m+1)}}$$

Thus, if $P_\ell(x)$ is a solution of $(\star)$ with $m=0$

$P_\ell'(x)$ " $m=1$

and $P_\ell''(x)$ " $m=2$

etc.

Thus,

$$\frac{d^m P_\ell}{dx^m} = u(x)$$ is a solution of ☆ for arbitrary $m = 0, 1, 2, \cdots$

$$\rightarrow \quad P_\ell^m(x) = \text{Const} \cdot (1-x^2)^{m/2} \frac{d^m P_\ell}{dx^m}$$

is general solution of assoc. Legendre's equation.
for $m = 0, 1, 2, \cdots$

Problem: $Y_{l,-m}(\theta,\phi)$ and $Y_{lm}(\pi-\theta,\phi+\pi)$

E.11

$$Y_{lm}(\theta,\phi) = N_l^m P_l^m(\cos\theta)e^{im\phi} \quad , \quad N_l^m = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}$$

Thus, $$Y_{l,-m}(\theta,\phi) = N_l^{-m} P_l^{-m}(\cos\theta)e^{-im\phi}$$

$$= N_l^{-m}(-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta)e^{-im\phi}$$

$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}(-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta)e^{-im\phi}$$

$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}(-1)^m P_l^m(\cos\theta)e^{-im\phi}$$

$$= (-1)^m N_l^m P_l^m(\cos\theta)e^{-im\phi}$$

$$= (-1)^m Y_{lm}^*(\theta,\phi)$$

Also, $$Y_{lm}(\pi-\theta,\phi+\pi) = N_l^m P_l^m(\cos(\pi-\theta))e^{im(\phi+\pi)}$$

Now: $$\cos(\pi-\theta) = \cos\pi\cos\theta + \underbrace{\sin\pi}_{0}\sin\theta$$
$$= -\cos\theta$$

$$e^{im\pi} = (e^{i\pi})^m = (-1)^m$$

Thus, $$Y_{lm}(\pi-\theta,\phi+\pi) = N_l^m P_l^m(-\cos\theta)(-1)^m e^{im\phi}$$

Now:
$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

$$\rightarrow \quad P_\ell^m(-x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}(-1)^{\ell+m}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

$$= (-1)^{\ell+m} P_\ell^m(x)$$

So
$$Y_{\ell m}(\pi-\theta, \phi+\pi) = N_\ell^m (-1)^{\ell+m} P_\ell^m(x)(-1)^m e^{im\phi}$$
$$= (-1)^\ell N_\ell^m P_\ell^m(x) e^{im\phi}$$
$$= (-1)^\ell Y_{\ell m}(\theta,\phi)$$

Summarizing:

$$Y_{\ell,-m}(\theta,\phi) = (-1)^m Y_{\ell m}^*(\theta,\phi)$$
$$Y_{\ell m}(\pi-\theta, \phi+\pi) = (-1)^\ell Y_{\ell m}(\theta,\phi)$$

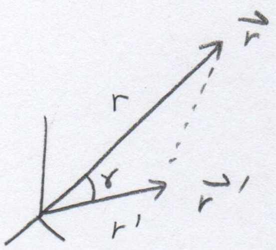
Problem: Potential for point source

E.12

Using generating function for Legendre polynomials:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n \quad , \quad |t| < 1$$

consider
$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}$$



$$= \frac{1}{r\sqrt{1+\left(\frac{r'}{r}\right)^2-2\left(\frac{r'}{r}\right)\cos\gamma}}$$

assume $r > r'$

$$= \frac{1}{r_>\sqrt{1+\left(\frac{r_<}{r_>}\right)^2-2\left(\frac{r_<}{r_>}\right)\cos\gamma}}$$

$$= \frac{1}{r_>}\sum_{n=0}^{\infty} P_n(\cos\gamma)\left(\frac{r_<}{r_>}\right)^n$$

$$= \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

Now: addition theorem →

$$\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi) = \left(\frac{2l+1}{4\pi}\right) P_l(\cos\gamma)$$

or
$$P_l(\cos\gamma) = \left(\frac{4\pi}{2l+1}\right)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

Thus,

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty} \frac{r_<^{\,l}}{r_>^{\,l+1}} \left(\frac{4\pi}{2l+1}\right) \sum_{m=-l}^{l} Y_{lm}^{*}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}^{*}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

Problem: Solution of modified Bessel's equation

E.13

Bessel: $$y''(x) + \frac{1}{x} y'(x) + \left(1 - \frac{\nu^2}{x^2}\right) y(x) = 0$$

Modified Bessel: $$y''(x) + \frac{1}{x} y'(x) - \left(1 + \frac{\nu^2}{x^2}\right) y(x) = 0$$

Suppose $y(x)$ is a solution of Bessel's equation.

Define $\bar{y}(x) = y(ix)$

Then $$\bar{y}'(x) = \frac{d\bar{y}}{dx} = y'(ix)\, i$$

$$\bar{y}''(x) = y''(ix)\, i^2 = -y''(ix)$$

$$\rightarrow \quad \bar{y}''(x) + \left(\frac{1}{x}\right) \bar{y}'(x) - \left(1 + \frac{\nu^2}{x}\right) \bar{y}(x)$$

$$= -y''(ix) + \frac{1}{x} i y'(ix) - \left(1 + \frac{\nu^2}{x^2}\right) y(ix)$$

$$= -y''(ix) - \frac{1}{ix} y'(ix) - \left(1 - \frac{\nu^2}{(ix)^2}\right) y(ix)$$

$$= -\left[y''(ix) + \frac{1}{ix} y'(ix) + \left(1 - \frac{\nu^2}{(ix)^2}\right) y(ix) \right.$$

$= 0$ (since y is a solution of Bessel's eqn)

Thus, $\bar{y}(x) = y(ix)$ is a solution of the modified Bessel's equation

E.14

Prob: $\Gamma(z+1) = z\Gamma(z)$

a) Proof:

$$\Gamma(z) = \int_0^\infty dx \, x^{z-1} e^{-x}$$

$$\Gamma(z+1) = \int_0^\infty dx \, x^{z} e^{-x}$$

Let: $u = x^z$, $dv = e^{-x}dx$
$du = zx^{z-1}dx$ $v = -e^{-x}$

$$\rightarrow \Gamma(z+1) = \int_0^\infty dx \, x^z e^{-x}$$

$$= \int_0^\infty u \, dv$$

$$= uv \Big|_0^\infty - \int_0^\infty v \, du$$

$$= -x^z e^{-x} \Big|_0^\infty + \int_0^\infty e^{-x} z x^{z-1} dx$$

needs Real(z) > 0

$$= \boxed{0} + z\int_0^\infty dx \, e^{-x} x^{z-1}$$

$$= z\Gamma(z)$$

Thus, $\boxed{\Gamma(z+1) = z\Gamma(z)}$

Prob: Γ function

b) $\Gamma(z) = \int_0^\infty dx \; x^{z-1} e^{-x}$

$\Gamma(1) = \int_0^\infty dx \; x^0 e^{-x}$

$= \int_0^\infty dx \, e^{-x}$

$= -e^{-x} \Big|_0^\infty$

$= -(0-1)$

$= \boxed{1}$

$\Gamma(\tfrac{1}{2}) = \int_0^\infty dx \; x^{\frac{1}{2}-1} e^{-x}$

$= \int_0^\infty dx \; x^{-\frac{1}{2}} e^{-x}$

<u>let:</u> $u^2 = x$ $\qquad 2u\,du = dx$

$x^{-\frac{1}{2}} = u^{-1}$

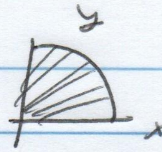
$x = 0, \infty \iff u = 0, \infty$

$= \int_0^\infty 2\cancel{u}\,du \; \cancel{u^{-1}} e^{-u^2}$

$= 2\int_0^\infty du \; e^{-u^2}$

<u>Now:</u> $I \equiv \int_0^\infty dx\, e^{-x^2}$

$I^2 = \int_0^\infty dx \int_0^\infty dy \; e^{-(x^2+y^2)}$



$$I^2 = \frac{1}{4}\int_0^\infty r\,dr \int_0^{2\pi} d\phi \;\; e^{-r^2}$$

$$= \frac{2\pi}{4}\int_0^\infty r\,dr\; e^{-r^2}$$

$$= \frac{\pi}{2}\int_0^\infty \frac{1}{2}\,du\; e^{-u}$$

$u = r^2$

$du = 2r\,dr$

$$= \frac{\pi}{4}\int_0^\infty du\; e^{-u}$$

$$= \frac{\pi}{4}$$

Thus, $I = \frac{\sqrt{\pi}}{2}$

$\rightarrow \Gamma\left(\frac{1}{2}\right) = 2I = \sqrt{\pi}$

so $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

$\Gamma(1) = 1 = 0!$

$\Gamma(2) = 1! = 1$

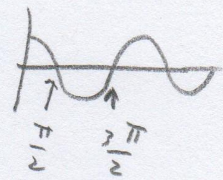
$\Gamma(3) = 2!$

Problem: Zeros of $J_\nu(x)$

E.15

$x \gg 1, \nu$: $\quad J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$

zeroes when argument of cosine $= (2n-1)\frac{\pi}{2}$:



$$x - \frac{\nu\pi}{2} - \frac{\pi}{4} = (2n-1)\frac{\pi}{2}$$

$$x - \frac{\nu\pi}{2} - \frac{\pi}{4} = n\pi - \frac{\pi}{2}$$

$[n = 1, 2, \ldots]$

$$x = \frac{\nu\pi}{2} + n\pi - \frac{\pi}{2} + \frac{\pi}{4}$$

$$= n\pi + \nu\frac{\pi}{2} - \frac{\pi}{4}$$

$$= \boxed{n\pi + \left(\nu - \frac{1}{2}\right)\frac{\pi}{2}}$$

Problem: E.16 Orthog of Bessel Functions:

Want to show $\int_0^a d\rho\, \rho\, J_\nu(x_{\nu n}\rho/a)\, J_\nu(x_{\nu n'}\rho/a) = 0$
for $n \neq n'$

Let $f(\rho) = J_\nu(x_{\nu n}\rho/a)$, $k = x_{\nu n}/a$
$g(\rho) = J_\nu(x_{\nu n'}\rho/a)$, $k' = x_{\nu n'}/a$

Bessel's equation:

$$R''(\rho) + \frac{1}{\rho} R'(\rho) + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R(\rho) = 0$$

$$\rho^2 R''(\rho) + \rho R'(\rho) + (k^2\rho^2 - \nu^2) R(\rho) = 0$$

$$\rho \frac{d}{d\rho}\left[\rho \frac{dR}{d\rho}\right] + (k^2\rho^2 - \nu^2) R(\rho) = 0$$

$$\rightarrow \quad \rho \frac{d}{d\rho}\left[\rho \frac{df}{d\rho}\right] + (k^2\rho^2 - \nu^2) f(\rho) = 0$$

$$\rho \frac{d}{d\rho}\left[\rho \frac{dg}{d\rho}\right] + (k'^2\rho^2 - \nu^2) g(\rho) = 0$$

Multiply 1st equation by $\frac{1}{\rho} g(\rho)$, the 2nd by $\frac{1}{\rho} f(\rho)$, then subtract and integrate.

$$0 = \int_0^a d\rho \left\{ g \frac{d}{d\rho}\left[\rho \frac{df}{d\rho}\right] + \left(k^2\rho - \cancel{\frac{\nu^2}{\rho}}\right) fg - f \frac{d}{d\rho}\left[\rho \frac{dg}{d\rho}\right] - \left(k'^2\rho - \cancel{\frac{\nu^2}{\rho}}\right) fg \right\}$$

$$= \int_0^a d\rho \left\{ g \frac{d}{d\rho}\left[\rho \frac{df}{d\rho}\right] - f \frac{d}{d\rho}\left[\rho \frac{dg}{d\rho}\right] \right\} + (k^2 - k'^2) \int_0^a d\rho\, \rho\, f(\rho) g(\rho)$$

Integrate the 1st two terms by parts

$$\int_0^a d\rho \; g \frac{d}{d\rho}\left[\rho \frac{df}{d\rho}\right] = \int_0^a d\rho \left[\frac{d}{d\rho}\left\{ g(\rho)\, \rho \frac{df}{d\rho} \right\} - \rho \frac{dg}{d\rho} \frac{df}{d\rho}\right]$$

$$= \cancelto{0}{\rho\, g(\rho) \frac{df}{d\rho}} \Big|_0^a - \int_0^a d\rho \; \rho \frac{dg}{d\rho} \frac{df}{d\rho}$$

at 0: since $\rho = 0$

at a: $g(a) = 0$

$$= - \int_0^a d\rho \; \rho \frac{dg}{d\rho} \frac{df}{d\rho}$$

equal

Also

$$\int_0^a d\rho \; f \frac{d}{d\rho}\left[\rho \frac{dg}{d\rho}\right] = - \int_0^a d\rho \; \rho \frac{df}{d\rho} \frac{dg}{d\rho}$$

Thus, these two terms cancel when subtracted.

$$\rightarrow \quad 0 = (k^2 - k'^2) \int_0^a d\rho \; \rho\, f(\rho)\, g(\rho)$$

$$= \int_0^a d\rho \; \rho\, f(\rho)\, g(\rho) \qquad \text{for } k \neq k'$$

(which holds for $n \neq n'$)

Thus,

$$0 = \int_0^a d\rho \; \rho \; J_\nu(x_{\nu n} \rho / a)\; J_\nu(x_{\nu n'} \rho / a)$$

Problem E.17 Normalization of Bessel Functions

$$\int_0^a d\rho\, \rho\, J_\nu^2(x_{\nu n}\rho/a) = \int_0^{x_{\nu n}} \left(\frac{a}{x_{\nu n}}\right) dx\, \frac{a x}{x_{\nu n}}\, J_\nu^2(x)$$

Let: $x = x_{\nu n}\rho/a$

$dx = \frac{x_{\nu n}}{a}\, d\rho$

$\rho = 0, a \rightarrow x = 0,\ x_{\nu n}$

$$= \left(\frac{a}{x_{\nu n}}\right)^2 \int_0^{x_{\nu n}} dx\, x\, J_\nu^2(x)$$

$$= \left(\frac{a}{x_{\nu n}}\right)^2 \left[\cancelto{0}{\frac{1}{2} x^2 J_\nu^2(x) \Big|_0^{x_{\nu n}}} - \int_0^{x_{\nu n}} dx\, x^2 J_\nu(x) J_\nu'(x) \right]$$

$$= -\left(\frac{a}{x_{\nu n}}\right)^2 \int_0^{x_{\nu n}} dx\, x^2 J_\nu(x) J_\nu'(x)$$

Now: $\int u\, dv = uv - \int v\, du$

where $u = J_\nu^2(x)$

$dv = x\, dx$

$V = \frac{1}{2} x^2$

$du = 2 J_\nu(x) J_\nu'(x)\, dx$

Bessel's equation

$$J_\nu'' + \frac{1}{x} J_\nu' + \left(1 - \frac{\nu^2}{x^2}\right) J_\nu = 0$$

$$\rightarrow \quad x^2 J_\nu'' + x J_\nu' + (x^2 - \nu^2) J_\nu = 0$$

$$x^2 J_\nu = \nu^2 J_\nu - x J_\nu' - x^2 J_\nu''$$

Thus,

$$\int_0^{x_{\nu n}} dx\, x^2 J_\nu(x) J_\nu'(x) = \int_0^{x_{\nu n}} dx \left[\nu^2 J_\nu - x J_\nu' - x^2 J_\nu'' \right] J_\nu'$$

$$= \nu^2 \int_0^{x_{\nu n}} dx\, \underbrace{J_\nu J_\nu'}_{\frac{1}{2}\frac{d}{dx}(J_\nu^2)} - \int_0^{x_{\nu n}} dx \Big(\underbrace{x (J_\nu')^2 + x^2 J_\nu'' J_\nu'}_{\frac{1}{2}\frac{d}{dx}\left[x^2 (J_\nu')^2\right]} \Big)$$

$$= \frac{1}{2}\nu^2 \, J_\nu^2(x) \Big|_0^{x_{\nu n}} - \frac{1}{2} x^2 (J_\nu')^2 \Big|_0^{x_{\nu n}}$$

$$= \underbrace{\underbrace{\frac{1}{2}\nu^2}_{\substack{=0 \\ \text{for } \nu=0}} \Big(\underbrace{J_\nu^2(\cancel{x_{\nu n}})}_{0} - \underbrace{J_\nu^2(0)}_{\substack{= 0 \text{ for } \nu \neq 0 \\ = 1 \text{ for } \nu = 0}} \Big)}_{=0} - \frac{1}{2} x_{\nu n}^2 \left[J_\nu'(x_{\nu n})\right]^2$$

$$= -\frac{1}{2} x_{\nu n}^2 \left[J_\nu'(x_{\nu n})\right]^2$$

~~Thus~~ $\cancel{\int_0^a d\rho\,\rho\, J_\nu^2(x_{\nu n}\rho/a) = \frac{1}{2} \left[J_\nu'(x_{\nu n})\right]^2}$

<u>But:</u> $J_\nu'(x_{\nu n}) = \frac{\nu}{x_{\nu n}} \cancel{J_\nu(x_{\nu n})} - J_{\nu+1}(x_{\nu n})$

Thus, $\text{RHS} = -\frac{1}{2} x_{\nu n}^2 \, J_{\nu+1}^2(x_{\nu n})$

$$\rightarrow \boxed{\int_0^a d\rho\,\rho\, J_\nu^2(x_{\nu n}\rho/a) = \frac{1}{2} a^2 J_{\nu+1}^2(x_{\nu n})}$$

Problem: Expression for $j_0(x)$

E.18

$$j_0(x) = \sqrt{\frac{\pi}{2x}}\, J_{\frac{1}{2}}(x)$$

$$= \sqrt{\frac{\pi}{2x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1+\frac{1}{2})} \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}$$

$$= \sqrt{\frac{\pi}{2x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1+\frac{1}{2})} \left(\frac{x}{2}\right)^{2n+1} \left(\frac{x}{2}\right)^{-\frac{1}{2}}$$

$$= \frac{\sqrt{\pi}}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1+\frac{1}{2})} \left(\frac{x}{2}\right)^{2n+1}$$

Now:

$$\Gamma(z) = \int_0^{\infty} dx\; x^{z-1} e^{-x}, \qquad \mathrm{Re}(z) > 0$$

$$\Gamma(n+1+\tfrac{1}{2}) = ?$$

Note:

$$\Gamma(\tfrac{1}{2}) = \int_0^{\infty} dx\; x^{-\frac{1}{2}} e^{-x}$$

let $u = x^{\frac{1}{2}}$

$x = u^2$

$dx = 2u\,du$

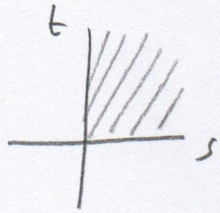
$$= \int_0^{\infty} 2u\,du\, \frac{1}{u} e^{-u^2}$$

$$= 2\int_0^{\infty} du\, e^{-u^2}$$

Let $I = \int_0^{\infty} du\, e^{-u^2}$

$$I^2 = \int_0^{\infty} ds\, e^{-s^2} \int_0^{\infty} dt\, e^{-t^2}$$

$$= \int_0^{\infty}\int_0^{\infty} ds\,dt\; e^{-(s^2+t^2)}$$



$$= \frac{1}{4}\int_0^{2\pi} d\phi \int_0^{\infty} r\,dr\; e^{-r^2}$$

$$= \frac{1}{4}\cdot 2\pi \int_0^{\infty} r\,dr\; e^{-r^2} \qquad u = r^2,\; du = 2r\,dr$$

$$= \frac{\pi}{2}\,\frac{1}{2}\int_0^{\infty} du\, e^{-u}$$

$$= \frac{\pi}{4}\left(-e^{-u}\right)\Big|_0^{\infty}$$

$$= \frac{\pi}{4}$$

so $I^2 = \frac{\pi}{4} \rightarrow I = \frac{\sqrt{\pi}}{2} \rightarrow \boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$

(n=0) $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right)$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad \text{using } \Gamma(z+1) = z\,\Gamma(z)$$

(n=1) $\Gamma\left(\frac{5}{2}\right) = \Gamma\left(1+\frac{3}{2}\right)$

$$= \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

(n=2) $\Gamma\left(\frac{7}{2}\right) = \Gamma\left(1+\frac{5}{2}\right)$

$$= \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{5\cdot 3\cdot 1}{(2)^3}\Gamma\left(\frac{1}{2}\right)$$

So $\Gamma\left(n+1+\frac{1}{2}\right) = \frac{(2n+1)(2n-1)\cdots 1}{2^{n+1}}\Gamma\left(\frac{1}{2}\right) = \frac{(2n+1)(2n-1)\cdots 1}{2^{n+1}}\sqrt{\pi}$

Thus,

$$j_0(x) = \frac{\sqrt{\pi}}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1+\frac{1}{2})} \left(\frac{x}{2}\right)^{2n+1}$$

$$= \frac{\cancel{\sqrt{\pi}}}{x} \sum_{n=0}^{\infty} \frac{(-1)^n \quad 2^{n+1}}{n!\,\cancel{\sqrt{\pi}}\,(2n+1)(2n-1)(2n-3)\cdots 1} \left(\frac{x}{2}\right)^{2n+1}$$

$$= \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,2^n\,(2n+1)(2n-1)(2n-3)\cdots 1}\, x^{2n+1}$$

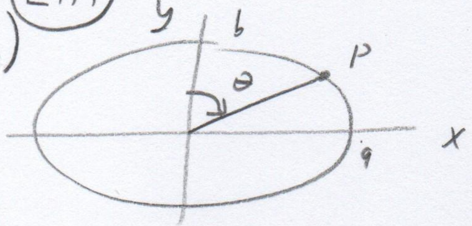
$$= \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{2n+1}}{\left[2n(2n-2)(2n-4)\cdots 2\right]\left[(2n+1)(2n-1)(2n-3)\cdots 1\right]}$$

$$= \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{x} \sin x$$

Problem: Arc length of ellipse

(E.19)



$$x = a \sin\theta$$
$$y = b \cos\theta$$

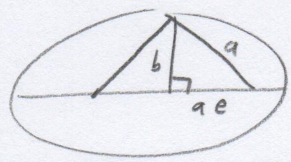
$$dx = a \cos\theta \, d\theta$$
$$dy = -b \sin\theta \, d\theta$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Thus,
$$ds = \sqrt{dx^2 + dy^2}$$
$$= \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} \; d\theta$$
$$= \sqrt{a^2(1-\sin^2\theta) + b^2\sin^2\theta} \; d\theta$$
$$= a\sqrt{1-\sin^2\theta + \left(\frac{b}{a}\right)^2\sin^2\theta} \; d\theta$$
$$= a\sqrt{1 - \left(1-\left(\frac{b}{a}\right)^2\right)\sin^2\theta} \; d\theta$$
$$= a\sqrt{1-k^2\sin^2\theta} \; d\theta$$

where
$$k = \sqrt{1-\left(\frac{b}{a}\right)^2}$$
$$= \sqrt{1-(1-e^2)}$$
$$= e.$$

~~b~~



$$a^2 = a^2e^2 + b^2$$
$$b^2 = a^2(1-e^2)$$

Thus,
$$s(\phi_1,\phi_2) = \int_{\theta=\phi_1}^{\phi_2} ds$$
$$= \int_{\theta=\phi_1}^{\phi_2} a\sqrt{1-e^2\sin^2\theta} \; d\theta$$
$$= a\left[\int_0^{\phi_2}\sqrt{1-e^2\sin^2\theta}\; d\theta - \int_0^{\phi_1}\sqrt{1-e^2\sin^2\theta}\; d\theta\right]$$
$$= a\left[E(\phi_2,e) - E(\phi_1,e)\right]$$

b) Total arc length:

$$s = \int_0^{2\pi} ds = 4\int_0^{\pi/2} ds$$

$$= 4a\left[E(\tfrac{\pi}{2}, e) - E(0, e)\right]$$

$$= 4a\, E(e)$$

complete elliptic integral

c) 1st order correction to total arclength for nearly circular ellipse

$$s = 4a\, E(e)$$

$$= 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2\theta}\; d\theta$$

$$\approx 4a \int_0^{\pi/2} \left(1 - \tfrac{1}{2} e^2 \sin^2\theta\right) d\theta$$

$$= 4a\left[\frac{\pi}{2} - \frac{1}{2} e^2 \int_0^{\pi/2} \sin^2\theta\, d\theta\right]$$

$$= 2\pi a - 2ae^2 \int_0^{\pi/2} \frac{1}{2}(1 - \cos 2\theta)\, d\theta$$

$$= 2\pi a - ae^2 \int_0^{\pi/2} (1 - \cos 2\theta)\, d\theta$$

$$= 2\pi a - ae^2\left(\frac{\pi}{2} - \frac{1}{2}\underbrace{\sin(2\theta)\Big|_0^{\pi/2}}_{0}\right)$$

$$= 2\pi a - \frac{a\pi e^2}{2}$$

$$= 2\pi a\left(1 - \frac{1}{4}e^2\right)$$

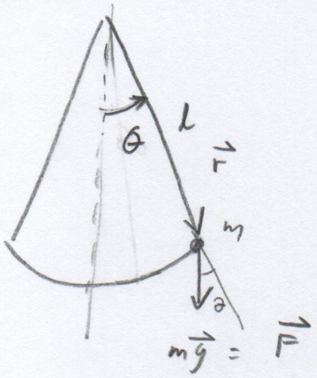
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 1 - 2\sin^2\theta$$

$$\rightarrow \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

Problem: Period of simple pendulum in terms of elliptic integrals (1)

E.20



Equation of motion:

$$I = ml^2$$

$$\alpha = \ddot{\theta}$$

$$\tau = \vec{r} \times \vec{F}$$

$$= -lmg \sin\theta \quad (\text{CW rotation})$$

Thus, $\tau = I\alpha$

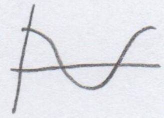
$$-lmg \sin\theta = ml^2 \ddot{\theta}$$

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$

a) Conservation of energy:

$$T = \frac{1}{2} I \omega^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = mgl(1-\cos\theta)$$



$$E = T + U$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1-\cos\theta)$$

$E = mgl(1-\cos\theta_0)$ ← released from rest at $\theta = \theta_0$

Thus, $mgl(\cancel{1}-\cos\theta_0) = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(\cancel{1}-\cos\theta)$

$$\frac{1}{2} \cancel{m} l^2 \dot{\theta}^2 = \cancel{m} gl(\cos\theta - \cos\theta_0)$$

$$\dot{\theta}^2 = 2\left(\frac{g}{l}\right)(\cos\theta - \cos\theta_0)$$

$(\theta < \theta_0)$

so RHS > 0

$$\frac{d\theta}{dt} = \dot{\theta} = -\sqrt{2}\sqrt{\frac{g}{l}}\sqrt{\cos\theta - \cos\theta_0}$$ ←

chose minus sign since θ is initially decreasing

$$\rightarrow \quad dt = \frac{-d\theta}{\sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_0}}$$

Period:

$$T(\theta_0) = 4\int_{\theta=\theta_0}^{\theta=0} dt$$

$$= 4\int_{\theta=\theta_0}^{\theta=0} \frac{-d\theta}{\sqrt{\frac{2g}{l}}\sqrt{\cos\theta-\cos\theta_0}}$$

$$= 4\sqrt{\frac{l}{2g}}\int_{\theta=0}^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}}$$

Now: need to write integral as an elliptic integral

NOTE: $\cos\theta = \cos\left(2\left(\frac{\theta}{2}\right)\right)$

$$= \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

and $\cos\theta_0 = 1 - 2\sin^2\left(\frac{\theta_0}{2}\right)$

so $\sqrt{\cos\theta - \cos\theta_0} = \sqrt{2\left(\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right)}$

$$= \sqrt{2}\,\sin\left(\frac{\theta_0}{2}\right)\sqrt{1-\left(\frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_0}{2}\right)}\right)^2}$$

Thus,

$$T(\theta_0) = 4\sqrt{\frac{l}{2g}}\,\frac{1}{\sqrt{2}}\,\frac{1}{\sin\left(\frac{\theta_0}{2}\right)}\int_{\theta=0}^{\theta_0}\frac{d\theta}{\sqrt{1-\left(\frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_0}{2}\right)}\right)^2}}$$

Now let

$$x = \frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_0}{2}\right)}$$

$$\rightarrow dx = \frac{1}{\sin\left(\frac{\theta_0}{2}\right)} \frac{1}{2}\cos\left(\frac{\theta}{2}\right) \, d\theta$$

$$= \frac{1}{2\sin\left(\frac{\theta_0}{2}\right)} \sqrt{1 - \sin^2\left(\frac{\theta}{2}\right)} \, d\theta$$

$$= \frac{1}{2\sin\left(\frac{\theta_0}{2}\right)} \sqrt{1 - \sin^2\left(\frac{\theta_0}{2}\right) x^2} \, d\theta$$

Also:

$$\theta = 0 \leftrightarrow x = 0$$
$$\theta = \theta_0 \leftrightarrow x = 1$$

thus

$$T(\theta_0) = 4\sqrt{\frac{l}{2g}} \frac{1}{\sqrt{2}} \frac{1}{\cancel{\sin\left(\frac{\theta_0}{2}\right)}} \int_{x=0}^{1} \frac{2\cancel{\sin\left(\frac{\theta_0}{2}\right)} \, dx}{\sqrt{1 - \sin^2\left(\frac{\theta_0}{2}\right) x^2} \sqrt{1 - x^2}}$$

$$= 4\sqrt{\frac{l}{2g}} \sqrt{2} \int_{x=0}^{1} \frac{dx}{\sqrt{1-x^2} \sqrt{1 - \sin^2\left(\frac{\theta_0}{2}\right) x^2}}$$

$$= 4\sqrt{\frac{l}{2g}} \sqrt{2} \, F\left(\frac{\pi}{2}, \sin\left(\frac{\theta_0}{2}\right)\right)$$

$$= 4\sqrt{\frac{l}{2g}} \sqrt{2} \, K\left(\sin\left(\frac{\theta_0}{2}\right)\right)$$

$$= \boxed{4\sqrt{\frac{l}{g}} \, K\left(\sin\left(\frac{\theta_0}{2}\right)\right)}$$

b) Assume $\theta_0 \ll 1$ so that $\sin\left(\frac{\theta_0}{2}\right) \approx \frac{\theta_0}{2}$

Then

$$K\left(\frac{\theta_0}{2}\right) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \left(\frac{\theta_0}{2}\right)^2 \sin^2\theta}}$$

$$\approx \int_0^{\pi/2} d\theta \left(1 + \frac{1}{2}\left(\frac{\theta_0}{2}\right)^2 \sin^2\theta\right)$$

$$= \frac{\pi}{2} + \frac{1}{2}\left(\frac{\theta_0}{2}\right)^2 \int_0^{\pi/2} d\theta \sin^2\theta$$

$$= \frac{\pi}{2} + \frac{1}{2}\left(\frac{\theta_0}{2}\right)^2 \int_0^{\pi/2} d\theta \left(\frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{\pi}{2} + \frac{1}{2}\left(\frac{\theta_0}{2}\right)^2 \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2}\cancel{\sin 2\theta\Big|_0^{\pi/2}}\right)$$

$$= \frac{\pi}{2}\left(1 + \frac{1}{16}\theta_0^2\right)$$

Thus,

$$T(\theta_0) \approx 4\sqrt{\frac{l}{g}}\ \frac{\pi}{2}\left(1 + \frac{1}{16}\theta_0^2\right)$$

$$= \boxed{2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\theta_0^2\right)}$$

$\rightarrow 2\pi\sqrt{\frac{l}{g}}$ when θ_0^2 is negligible.

Problem: Show $\frac{d}{dy} \operatorname{sn} y = \operatorname{cn} y \cdot \operatorname{dn} y$

(E.21)

Proof:
$$\frac{d}{dy} \operatorname{sn} y = \frac{d}{dy} \sin\phi$$
$$= \cos\phi \, \frac{d\phi}{dy}$$
$$= \operatorname{cn} y \cdot \operatorname{dn} y$$

using $\frac{d\phi}{dy} = \operatorname{dn} y$.