Problem: Uniqueness of expansion $\underline{A} = \sum_i A_i \underline{e}_i$

(D.1)

Proof: (Use proof by contradiction).

Given any $\underline{A}$, $\{\underline{e}_1, \underline{e}_2, \cdots, \underline{e}_n\}$ basis, we have $\underline{A} = \sum_i A_i \underline{e}_i$

Assume $\exists$ another set of components $\{A_i'\}$ different from $\{A_i\}$ such that $\underline{A} = \sum_i A_i' \underline{e}_i$.

Then
$$\sum_i A_i \underline{e}_i = \sum_i A_i' \underline{e}_i$$

$$\rightarrow \quad 0 = \sum_i (A_i - A_i')\underline{e}_i$$

Since the $\{A_i'\}$ are assumed to be different than the $\{A_i\}$, there exists at least one index $\underline{i}$ (which we can take to be $\underline{i}=1$) such that $A_{\underline{i}} - A_{\underline{i}}' \neq 0$

$$\rightarrow \quad (A_1 - A_1')\underline{e}_1 = -(A_2 - A_2')\underline{e}_2 + \cdots - (A_n - A_n')\underline{e}_n$$

$$\underline{e}_1 = -\frac{(A_2 - A_2')}{(A_1 - A_1')}\underline{e}_2 + \cdots - \frac{(A_n - A_n')}{(A_1 - A_1')}\underline{e}_n$$

which says that $\underline{e}_1$ is a linear combination of the other basis vectors. But this violates the fact that $\{\underline{e}_1, \underline{e}_2, \cdots, \underline{e}_n\}$ is a basis.

$\rightarrow$ contradiction.

Problem (D.2) Component form of $\underline{0}$ and $-\underline{A}$.

Let $\underline{0} \leftrightarrow [0, 0, \cdots, 0]^T$

Then $\forall\ \underline{A}$ we have

$$\underline{A} + \underline{0} \leftrightarrow [A_1, A_2, \cdots, A_n]^T + [0, 0, \cdots, 0]^T$$
$$= [A_1 + 0, A_2 + 0, \cdots, A_n + 0]^T$$
$$= [A_1, A_2, \cdots, A_n]^T$$
$$\leftrightarrow \underline{A}$$

Given $\underline{A} \leftrightarrow [A_1, A_2, \cdots, A_n]^T$

Let $-\underline{A} \leftrightarrow [-A_1, -A_2, \cdots, -A_n]^T$

Then $\underline{A} + (-\underline{A}) \leftrightarrow [A_1 - A_1, A_2 - A_2, \cdots, A_n - A_n]^T$
$$\leftrightarrow [0, 0, \cdots, 0]$$
$$\leftrightarrow \underline{0}$$

1.3

Let $\vec{C} = b\vec{B}$

$$\vec{C} \cdot \vec{A} = (\vec{A} \cdot \vec{C})^*$$
$$= (\vec{A} \cdot (b\vec{B}))^*$$
$$= (b\,\vec{A} \cdot \vec{B})^*$$
$$= b^* \, (\vec{A} \cdot \vec{B})^*$$
$$= b^* \, \vec{B} \cdot \vec{A}$$

Problem: D.4 Gram-Schmidt orthonormalization

a) $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} = \{\hat{x}, \hat{x}+\hat{y}, \hat{x}+\hat{y}+\hat{z}\}$

$$\hat{f}_1 = \frac{\underline{e}_1}{|\underline{e}_1|} = \frac{\hat{x}}{1} = \boxed{\hat{x}}$$

$$\underline{f}_2 = \underline{e}_2 - (\hat{\underline{f}}_1 \cdot \underline{e}_2)\hat{f}_1$$
$$= \hat{x}+\hat{y} - (\hat{x}\cdot(\hat{x}+\hat{y}))\hat{x}$$
$$= \hat{x}+\hat{y} - 1\cdot\hat{x}$$
$$= \hat{y}$$

$$\hat{f}_2 = \frac{\underline{f}_2}{|\underline{f}_2|} = \boxed{\hat{y}}$$

$$\underline{f}_3 = \underline{e}_3 - (\hat{\underline{f}}_1 \cdot \underline{e}_3)\hat{\underline{f}}_1 - (\hat{\underline{f}}_2 \cdot \underline{e}_3)\hat{\underline{f}}_2$$
$$= (\hat{x}+\hat{y}+\hat{z}) - (\hat{x}\cdot(\hat{x}+\hat{y}+\hat{z}))\hat{x} - (\hat{y}\cdot(\hat{x}+\hat{y}+\hat{z}))\hat{y}$$
$$= \hat{x}+\hat{y}+\hat{z} - 1\cdot\hat{x} - 1\cdot\hat{y}$$
$$= \hat{z}$$

$$\hat{\underline{f}}_3 = \frac{\underline{f}_3}{|\underline{f}_3|} = \boxed{\hat{z}}$$

b) $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} = \{\hat{x}+\hat{y}+\hat{z}, \hat{x}+\hat{y}, \hat{x}\}$

$$\hat{f}_1 = \frac{\underline{e}_1}{|\underline{e}_1|} = \boxed{\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}}$$

$$\underline{f}_2 = \underline{e}_2 - (\hat{f}_1 \cdot \underline{e}_2)\hat{f}_1$$
$$= \hat{x}+\hat{y} - \left(\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}\right)\cdot(\hat{x}+\hat{y})\left(\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}\right)$$

$$= \hat{x} + \hat{y} - \frac{2}{3}\left(\hat{x} + \hat{y} + \hat{z}\right)$$

$$= \frac{1}{3}\hat{x} + \frac{1}{3}\hat{y} - \frac{2}{3}\hat{z}$$

$$= \frac{1}{3}\left(\hat{x} + \hat{y} - 2\hat{z}\right)$$

$$\hat{f}_2 = \frac{\underline{f}_2}{|\underline{f}_2|} = \frac{\cancel{\frac{1}{3}}\left(\hat{x} + \hat{y} - 2\hat{z}\right)}{\cancel{\frac{1}{3}}\sqrt{1+1+(-2)^2}}$$

$$= \boxed{\frac{\hat{x} + \hat{y} - 2\hat{z}}{\sqrt{6}}}$$

$$\underline{f}_3 = \underline{e}_3 - (\hat{f}_1 \cdot \underline{e}_3)\hat{f}_1 - (\hat{f}_2 \cdot \underline{e}_3)\hat{f}_2$$

$$= \hat{x} - \left(\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}\right)\cdot\hat{x}\left(\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}\right)$$

$$- \left(\frac{\hat{x}+\hat{y}-2\hat{z}}{\sqrt{6}}\right)\cdot\hat{x}\left(\frac{\hat{x}+\hat{y}-2\hat{z}}{\sqrt{6}}\right)$$

$$= \hat{x} - \frac{1}{3}\left(\hat{x} + \hat{y} + \hat{z}\right) - \frac{1}{6}\left(\hat{x} + \hat{y} - 2\hat{z}\right)$$

$$= \frac{1}{6}\left[(6-2-1)\hat{x} + (-2-1)\hat{y} + \underbrace{(-2+2)}_{0}\hat{z}\right]$$

$$= \frac{1}{6}\left[3\hat{x} - 3\hat{y}\right]$$

$$= \frac{1}{2}\left(\hat{x} - \hat{y}\right)$$

$$\hat{f}_3 = \frac{\underline{f}_3}{|\underline{f}_3|}$$

$$= \frac{\cancel{\frac{1}{2}}(\hat{x} - \hat{y})}{\cancel{\frac{1}{2}}\sqrt{1^2+(-1)^2}} = \boxed{\frac{1}{\sqrt{2}}(\hat{x} - \hat{y})}$$

Problem: D.5 Component form of inner product

$$\vec{A} \cdot \vec{B} = \left( \sum_i A_i \hat{e}_i \right) \cdot \left( \sum_j B_j \hat{e}_j \right)$$

$$= \sum_i \sum_j A_i B_j \, \hat{e}_i \cdot \hat{e}_j$$

$$= \sum_i \sum_j A_i B_j \, \delta_{ij}$$

$$= \sum_i A_i B_i$$

Problem: 1.6 Schwarz inequality:

Let $$\underline{C} = \underline{A} - \frac{(\underline{B}\cdot\underline{A})}{(\underline{B}\cdot\underline{B})}\,\underline{B}$$

$$0 \le \underline{C}\cdot\underline{C}$$

$$= \left(\underline{A} - \frac{(\underline{B}\cdot\underline{A})}{(\underline{B}\cdot\underline{B})}\,\underline{B}\right)\cdot\left(\underline{A} - \frac{(\underline{B}\cdot\underline{A})}{(\underline{B}\cdot\underline{B})}\,\underline{B}\right)$$

$$= \underline{A}\cdot\underline{A} + \frac{(\underline{B}\cdot\underline{A})^2}{(\underline{B}\cdot\underline{B})^2}\,\underline{B}\cdot\underline{B} - 2\,\frac{(\underline{A}\cdot\underline{B})^2}{(\underline{B}\cdot\underline{B})}$$

$$= \underline{A}\cdot\underline{A} - \frac{(\underline{A}\cdot\underline{B})^2}{\underline{B}\cdot\underline{B}}$$

Thus, $$\underline{A}\cdot\underline{A} - \frac{(\underline{A}\cdot\underline{B})^2}{(\underline{B}\cdot\underline{B})} \ge 0$$

$$\rightarrow \quad (\underline{A}\cdot\underline{B})^2 \le (\underline{A}\cdot\underline{A})(\underline{B}\cdot\underline{B})$$

$$|\underline{A}\cdot\underline{B}|^2 \le |\underline{A}|^2\,|\underline{B}|^2$$

$$|\underline{A}\cdot\underline{B}| \le |\underline{A}|\,|\underline{B}|$$

Problem: 1.7 Example of a transformation that is <u>not</u> linear

$T : \underline{A} \rightarrow \underline{A} + \underline{C}$ , $\underline{C} = \text{const}$

Then, $T(a\underline{A} + b\underline{B}) = (a\underline{A} + b\underline{B}) + \underline{C}$

vs. $aT(\underline{A}) + bT(\underline{B}) = a(\underline{A} + \underline{C}) + b(\underline{B} + \underline{C})$

$= a\underline{A} + b\underline{B} + (a+b)\underline{C}$

which are not equal to one another.
so $T: \underline{A} \rightarrow \underline{A} + \underline{C}$ is not linear

Problem D.8 show that space of linear transformations is a vector space.

(1.) Closure: Let $S, T$ be two linear transformations. Then $S+T$ is the transformation defined by

$$(S+T)(\underline{A}) \equiv S(\underline{A}) + T(\underline{A})$$

It follows that $(S+T)$ is also linear since

$$\begin{aligned}(S+T)(a\underline{A}+b\underline{B}) &= S(a\underline{A}+b\underline{B}) + T(a\underline{A}+b\underline{B}) \\ &= aS(\underline{A}) + bS(\underline{B}) + aT(\underline{A}) + bT(\underline{B}) \\ &= a\left(S(\underline{A})+T(\underline{A})\right) + b\left(S(\underline{B})+T(\underline{B})\right) \\ &= a(S+T)(\underline{A}) + b(S+T)(\underline{B})\end{aligned}$$

thus, addition of linear transformations is closed

(2.) Commutativity:

$$\begin{aligned}(S+T)(\underline{A}) &= S(\underline{A}) + T(\underline{A}) \\ &= T(\underline{A}) + S(\underline{A}) \\ &= (T+S)(\underline{A})\end{aligned}$$

(3.) Associativity:

$$\begin{aligned}((S+T)+U)(\underline{A}) &= (S+T)(\underline{A}) + U(\underline{A}) \\ &= (S(\underline{A})+T(\underline{A})) + U(\underline{A}) \\ &= S(\underline{A}) + (T(\underline{A})+U(\underline{A})) \\ &= S(\underline{A}) + (T+U)(\underline{A}) \\ &= (S+(T+U))(\underline{A})\end{aligned}$$

(4.) Zero linear transformation:

$$O(\underline{A}) \equiv \underline{0}$$

Then
$$\begin{aligned}(S+O)(\underline{A}) &= S(\underline{A}) + O(\underline{A}) \\ &= S(\underline{A}) + \underline{0} \\ &= S(\underline{A})\end{aligned}$$

(5.) Inverse linear transformation:

Given $S$, define $(-S)$ via

$$(-S)(\underline{A}) = -(S(\underline{A}))$$

Then
$$\begin{aligned}(S+(-S))(\underline{A}) &= S(\underline{A}) + (-S)(\underline{A}) \\ &= S(\underline{A}) - S(\underline{A}) \\ &= \underline{0} \\ &= O(\underline{A})\end{aligned}$$

Scalar multiplication properties

(1.) Closure:

$aT$ defined by $(aT)(\underline{A}) = a \cdot T(\underline{A})$

To show that $aT$ is linear:

$$\begin{aligned}(aT)(b\underline{B} + c\underline{C}) &= a \cdot T(b\underline{B} + c\underline{C}) \\ &= a[bT(\underline{B}) + cT(\underline{C})] \\ &= a(bT(\underline{B})) + a(cT(\underline{C})) \\ &= b(aT(\underline{B})) + c(aT(\underline{C})) \\ &= b(aT)(\underline{B}) + c(aT)(\underline{C}) \quad \checkmark\end{aligned}$$

Thus, scalar multiplication of linear transformations is closed.

(2.) Identity:

$$(1 \cdot T)(\underline{A}) = 1 \cdot T(\underline{A}) = T(\underline{A})$$

(3.) Distributive wrt scalar addition:

$$((a+b)T)(\underline{A}) = (a+b)(T(\underline{A})) = a \cdot T(\underline{A}) + b \cdot T(\underline{A}) = (aT)(\underline{A}) + (bT)(\underline{A}) = (aT + bT)(\underline{A})$$

(4.) Distributive wrT addition of linear transformations.

$$(a(S+T))(\underline{A}) = a \cdot (S+T)(\underline{A}) = a \cdot [S(\underline{A}) + T(\underline{A})] = a \cdot S(\underline{A}) + a \cdot T(\underline{A}) = (aS)(\underline{A}) + (aT)(\underline{A}) = (aS + aT)(\underline{A})$$

(5.) Assoc. wrt scalar multiplication:

$$(a(bT))(\underline{A}) = a \cdot (bT)(\underline{A}) = a \cdot (b \cdot T(\underline{A})) = (ab) \cdot T(\underline{A}) = ((ab)T)(\underline{A})$$

Problem: (1.9)

Component form of addition, scalar multiplication, and multiplication of linear transformation

$(S+T)$:

$$(S+T)(\underline{e}_j) = S(\underline{e}_j) + T(\underline{e}_j)$$
$$= \sum_i S_{ij}\,\underline{e}_i + \sum_i T_{ij}\,\underline{e}_i$$
$$= \sum_i (S_{ij} + T_{ij})\,\underline{e}_i$$

so $S+T \leftrightarrow S_{ij} + T_{ij}$

$(aT)$:

$$(aT)(\underline{e}_j) = a \cdot T(\underline{e}_j)$$
$$= a \sum_i T_{ij}\,\underline{e}_i$$
$$= \sum_i (a T_{ij})\,\underline{e}_i$$

so $aT \leftrightarrow aT_{ij}$

$(ST)$:

$$(ST)(\underline{e}_j) = S(T\,\underline{e}_j)$$
$$= S\Big(\sum_i T_{ij}\,\underline{e}_i\Big)$$
$$= \sum_i T_{ij}\,S(\underline{e}_i)$$
$$= \sum_i T_{ij} \sum_k S_{ki}\,\underline{e}_k$$
$$= \sum_k \Big(\sum_i S_{ki} T_{ij}\Big)\,\underline{e}_k$$

so $ST \leftrightarrow \sum_i S_{ki} T_{ij}$ $(k,j$ th component$)$

Problem: Transpose of product of two matrices

D.10

$$[(ST)^T]_{ij} = (ST)_{ji}$$

$$= \sum_k S_{jk} T_{ki}$$

$$= \sum_k (T^T)_{ik} (S^T)_{kj}$$

$$= (T^T S^T)_{ij}$$

Thus, $(ST)^T = T^T S^T$

$$[(ST)^\dagger]_{ij} = (ST)^*_{ji}$$

$$= \sum_k S^*_{jk} T^*_{ki}$$

$$= \sum_k (T^\dagger)_{ik} (S^\dagger)_{kj}$$

$$= (T^\dagger S^\dagger)_{ij}$$

Thus, $(ST)^\dagger = T^\dagger S^\dagger$

Problem: Inner product of two vectors in terms of row and column matrices

1.11

$$\underline{A} \cdot \underline{B} = \sum_i A_i^* B_i$$

$$= \begin{bmatrix} A_1^* & A_2^* & \cdots & A_n^* \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

$$= A^\dagger B$$

Problem: (D.12) Determinant of a matrix in terms of Levi-Civita symbol

$$\det T = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_n} \epsilon_{i_1 i_2 \cdots i_n} T_{1 i_1} T_{2 i_2} \cdots T_{n i_n}$$

a) 3x3 matrix

$$\det T = \sum_i \sum_j \sum_k \epsilon_{ijk} T_{1i} T_{2j} T_{3k}$$

$$= \epsilon_{123} T_{11} T_{22} T_{32} + \epsilon_{132} T_{11} T_{23} T_{32}$$
$$+ \epsilon_{231} T_{12} T_{23} T_{31} + \epsilon_{213} T_{12} T_{21} T_{33}$$
$$+ \epsilon_{312} T_{13} T_{21} T_{32} + \epsilon_{321} T_{13} T_{22} T_{31}$$

$$= T_{11}(T_{22} T_{32} - T_{23} T_{32})$$
$$+ T_{12}(T_{23} T_{31} - T_{21} T_{33})$$
$$+ T_{13}(T_{21} T_{32} - T_{22} T_{31})$$

b) $\det T = \det \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix}$

$$= T_{11}(T_{22} T_{33} - T_{32} T_{23})$$
$$- T_{12}(T_{21} T_{33} - T_{31} T_{23})$$
$$+ T_{13}(T_{21} T_{32} - T_{31} T_{22}$$

agrees with part (a)

Using Laplace development off of 1st row

Problem: D.13 Inverse matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Check:

$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det T} C^T$$

$$\det T = a(ei-fh) - b(di-fg) + c(dh-eg)$$

$$C_{11} = (ei-fh)$$
$$C_{12} = -(di-fg)$$
$$C_{13} = (dh-eg)$$
$$C_{21} = -(bi-hc)$$
$$C_{22} = (ai-cg)$$
$$C_{23} = -(ah-bg)$$
$$C_{31} = (bf-ec)$$
$$C_{32} = -(af-cd)$$
$$C_{33} = (ae-bd)$$

$$C = \begin{bmatrix} (ei-fh) & -(di-fg) & (dh-eg) \\ -(bi-hc) & (ai-cg) & -(ah-bg) \\ (bf-ec) & -(af-cd) & (ae-bd) \end{bmatrix}$$

$$C^T = \begin{bmatrix} (ei-fh) & -(bi-hc) & (bf-ec) \\ -(di-fg) & (ai-cg) & -(af-cd) \\ (dh-eg) & -(ah-bg) & (ae-bd) \end{bmatrix}$$

Thus,

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{a(ei-fh)-b(di-fg)+c(dh-eg)} \times$$

$$\times \begin{bmatrix} ei-fh & -(bi-hc) & bf-ec \\ -(di-fg) & ai-cg & -(af-cd) \\ dh-eg & -(ah-bg) & ae-bd \end{bmatrix}$$

Problem: Rotation as an orthog. matrix

D.14

$$R_z(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[R_z(\phi)]^{-1} = R_z(-\phi)$$

$$= \begin{bmatrix} \cos(-\phi) & \sin(-\phi) & 0 \\ -\sin(-\phi) & \cos(-\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [R_z(\phi)]^T$$

Thus, $M^{-1} = M^T$ so the matrix $M = R_z(\phi)$ is orthogonal.

Problem: Prove $Tr(ST) = Tr(TS)$

D.15

By definition $Tr(T) = \sum_i T_{ii}$

Thus,

$$
\begin{aligned}
Tr(ST) &= \sum_i (ST)_{ii} \\
&= \sum_i \sum_j S_{ij} T_{ji} \\
&= \sum_j \sum_i T_{ji} S_{ij} \\
&= \sum_j (TS)_{jj} \\
&= Tr(TS)
\end{aligned}
$$

**Problem:** Eigenvector / values of 2-d rotation matrices

D.16

$$R(\phi) = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

$$R(\phi)\,\underline{v} = \lambda\,\underline{v}$$

$$(R(\phi) - \lambda \mathbb{1})\,\underline{v} = \underline{0}$$

$$\begin{aligned} 0 &= \det\left(R(\phi) - \lambda \mathbb{1}\right) \\ &= (\cos\phi - \lambda)^2 + \sin^2\phi \\ &= \cos^2\phi + \lambda^2 - 2\lambda\cos\phi + \sin^2\phi \\ &= \lambda^2 - 2\lambda\cos\phi + 1 \end{aligned}$$

$$\lambda = \frac{2\cos\phi \pm \sqrt{4\cos^2\phi - 4}}{2}$$

$$\begin{aligned} &= \cos\phi \pm \sqrt{\cos^2\phi - 1} \\ &= \cos\phi \pm \sqrt{-(1-\cos^2\phi)} \\ &= \cos\phi \pm \sqrt{-\sin^2\phi} \\ &= \cos\phi \pm i\sin\phi \quad = e^{\pm i\phi} \end{aligned}$$

$\lambda$ is real iff $\sin\phi = 0 \iff \phi = n\pi$

e.g., rotation by $0, \pi, 2\pi, \ldots$

$\underline{\phi = 0}$: $\lambda = 1, 1$

$\underline{\phi = \pi}$: $\lambda = -1, -1$

etc.

Eigen vectors:

$\lambda = e^{i\phi}$:

$$\begin{pmatrix} \cos\phi - e^{i\phi} & \sin\phi \\ -\sin\phi & \cos\phi - e^{i\phi} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -i\sin\phi & \sin\phi \\ -\sin\phi & -i\sin\phi \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\sin\phi \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If $\phi \neq n\pi$ then $\sin\phi \neq 0$ and $-iv_1 + v_2 = 0$

$v_2 = iv_1$

$$\underline{e}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

If $\phi = n\pi$ then $v_1, v_2$ can be anything as

$R(0)\underline{v} = \mathbb{1}\underline{v} = \underline{v}$ and $R(\pi)\underline{v} = -\underline{v}$ $\forall\, \underline{v}$

$\lambda = e^{-i\phi}$:

$$\begin{pmatrix} \cos\phi - e^{-i\phi} & \sin\phi \\ -\sin\phi & \cos\phi - e^{-i\phi} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} +i\sin\phi & \sin\phi \\ -\sin\phi & i\sin\phi \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\sin\phi \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If $\phi \neq n\pi$ then $\sin\phi \neq 0$ and

$$i v_1 + v_2 = 0 \rightarrow v_2 = -i v_1$$

$$\underline{e}_2 = \frac{1}{\sqrt{2}} \boxed{\begin{bmatrix} 1 \\ -i \end{bmatrix}}$$

If $\phi = n\pi$ then $\sin\phi = 0$ and $v_1, v_2$ can be anything as we saw for $\underline{e}_1$

Problem: Eigenvector and eigenvalues of a 2-d rotation ①

D.17

$$R(\phi) = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

$$(R - \lambda 1) = \begin{bmatrix} \cos\phi - \lambda & \sin\phi \\ -\sin\phi & \cos\phi - \lambda \end{bmatrix}$$

$$0 = \det(R - \lambda 1)$$

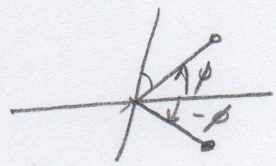
$$= (\cos\phi - \lambda)^2 + \sin^2\phi$$

$$= \cos^2\phi - 2\lambda\cos\phi + \lambda^2 + \sin^2\phi$$

$$= 1 - 2\lambda\cos\phi + \lambda^2$$

$$\rightarrow \lambda = \frac{2\cos\phi \pm \sqrt{4\cos^2\phi - 4\cdot 1\cdot 1}}{2}$$

$$= \cos\phi \pm \sqrt{\cos^2\phi - 1}$$




$$= \cos\phi \pm \sqrt{-(1-\cos^2\phi)}$$

$$= \cos\phi \pm i\sin\phi$$

$$= e^{\pm i\phi}$$

NOTE: $\lambda$ is real iff $\phi = 0 \rightarrow \lambda = 1, 1$

$\phi = 180° \rightarrow \lambda = -1, -1$

$\lambda = e^{i\phi}$:

$$\begin{bmatrix} \cos\phi - e^{i\phi} & \sin\phi \\ -\sin\phi & \cos\phi - e^{i\phi} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now: $\cos\phi - e^{i\phi} = \cancel{\cos\phi} - (\cancel{\cos\phi} + i\sin\phi)$

$$= -i\sin\phi$$

$$\rightarrow \sin\phi \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\phi = 0, \pi, 2\pi, 3\pi, \ldots$ then $v_1, v_2$ can be anything.

But for other values of $\phi$, $\sin\phi \neq 0$

$$\rightarrow \left.\begin{aligned} -iv_1 + v_2 &= 0 \\ -v_1 - iv_2 &= 0 \end{aligned}\right\} \text{ same}$$

Thus, $v_2 = iv_1$ so $\boxed{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} = \hat{e}_1}$

$\lambda = e^{-i\phi}$:

$$\begin{bmatrix} \cos\phi - e^{-i\phi} & \sin\phi \\ -\sin\phi & \cos\phi - e^{-i\phi} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\sin\phi \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rightarrow$ $iv_1 + v_2 = 0$ $\rightarrow$ $v_2 = -iv_1$

(assuming $\sin\phi \neq 0$) $-v_1 + iv_2 = 0$

so $\boxed{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \hat{e}_2}$

Problem: D.18 Diagonalizing $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ from Example B.2

From the example we found $\lambda_{+,-} = 1, -1$
with normalized eigen vectors

$$v_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_- = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Let $S^{-1} = \begin{bmatrix} v_+ & v_- \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ be the matrix of eigen vectors.

Then $S = \left(\frac{1}{-1}\right) \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

$\det(S^{-1}) = \frac{1}{2}(-1-1)$
$= -1$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= S^{-1}$$

$$= S^{T}$$

Since $S^{-1} = S^{T}$ the matrix $S$ is an orthogonal matrix

Also $STS^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

Problem: Diagonalize the Hermitian matrix $T = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

D.19

$$T \underline{v} = \lambda \underline{v}$$

$$(T - \lambda \mathbb{1}) \underline{v} = \underline{0}$$

$$0 = \det (T - \lambda \mathbb{1})$$

$$= \det \begin{bmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)^2 - 1$$

$$= \cancel{1} + \lambda^2 - 2\lambda - \cancel{1}$$

$$= \lambda (\lambda - 2)$$

$$\rightarrow \boxed{\lambda = 0 \,, \quad \lambda = 2} \qquad \text{(real)}$$

$\underline{\lambda = 0}$:

$$\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + i v_2 = 0$$

$$-i v_1 + v_2 = 0 \quad \rightarrow \quad v_2 = i v_1$$

$$\underline{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$\underline{\lambda = 2}$:

$$\begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-v_1 + i v_2 = 0 \quad \rightarrow \quad v_1 = i v_2$$

$$-i v_1 - v_2 = 0 \quad \rightarrow \quad v_2 = -i v_1$$

$$\underline{e}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

check orthonormality:

$$\underline{e}_1 \cdot \underline{e}_2 = \frac{1}{2} \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}(i - i)$$

$$= \boxed{0}$$

$$\underline{e}_1 \cdot \underline{e}_1 = \frac{1}{2} \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= \frac{1}{2}(1 + 1)$$

$$= 1$$

$$\underline{e}_2 \cdot \underline{e}_2 = \frac{1}{2} \begin{bmatrix} -i & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}(1 + 1)$$

$$= 1$$

**Define:** $S^{-1} = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 \end{bmatrix}$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

$$\det(S^{-1}) = 1 \rightarrow S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \quad (S^{-1} = S^{\dagger})$$

**check:**

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

Then

$$S\,T\,S^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} 0 & 2i \\ 0 & 2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

= diagonal matrix of eigenvalues