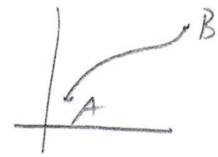
Geodesic in plane: Example C.1



$$I = \int_A^B ds$$

$$= \int_{x_1}^{x_2} dx \sqrt{1+y'^2}$$

$$F(x,y,y') = \sqrt{1+y'^2}$$

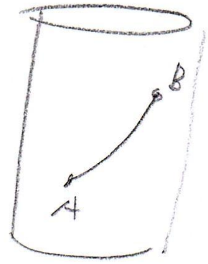
$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \quad \rightarrow \quad \frac{\partial F}{\partial y'} = \text{Const}$$

$$\frac{y'}{\sqrt{1+y'^2}} = \text{Const}$$

$$\rightarrow \boxed{y' = A}$$

Solution: $y = Ax + B$

Geodesic on surface of a cylinder : C.1



$$I = \int_A^B ds$$

$$= \int_{\phi_1}^{\phi_2} d\phi \sqrt{R^2 + z'^2}$$

$$ds^2 = R^2 d\phi^2 + dz^2$$

$$= \int_{\phi_1}^{\phi_2} d\phi \, F(\phi, z, z') \,, \quad F = \sqrt{R^2 + z'^2}$$

$$\frac{d}{d\phi}\left(\frac{\partial F}{\partial z'}\right) - \frac{\partial F}{\partial z} = 0 \rightarrow \frac{\partial F}{\partial z'} = \text{Const}$$

$$\frac{z'}{\sqrt{R^2 + z'^2}} = \text{const}$$

$$\rightarrow \boxed{z' = A}$$

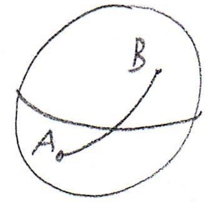
solution: $Z(\phi) = A\phi + B$

Geodesic on 2-sphere (C,2)

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)$$

arc length on sphere of radius R

Take θ as indep. variable:



Take $R = 1$, unit sphere

$$I[\phi] = \int_A^B ds$$

$$= \int_{\theta_A}^{\theta_B} d\theta \sqrt{1 + \sin^2\theta \, \phi'^2}$$

where $\phi' = \dfrac{d\phi}{d\theta}$

$$F(\phi, \phi', \theta) = \sqrt{1 + \sin^2\theta \, \phi'^2}$$

Since F does not depend on ϕ, $\dfrac{\partial F}{\partial \phi'} = \text{const}$

$$\cancel{\frac{d}{d\theta}\left(\frac{\partial F}{\partial \phi'} \right) - \frac{\partial F}{\partial \phi} = 0}$$

$$\rightarrow \cancel{\frac{\partial F}{\partial \phi'} = \text{const}}$$

$$\frac{\partial F}{\partial \phi'} = \frac{1}{\cancel{2}\sqrt{1 + \sin^2\theta \, \phi'^2}} \cancel{2} \sin^2\theta \, \phi' = C$$

$$\frac{\sin^2\theta \, \phi'}{\sqrt{1 + \sin^2\theta \, \phi'^2}} = C$$

$$\sin^4\theta \, \phi'^2 = C^2 \left(1 + \sin^2\theta \, \phi'^2 \right)$$

$$\left[\sin^4\theta - C^2 \sin^2\theta \right] \phi'^2 = C^2$$

$$\rightarrow \quad \phi' = \frac{C}{\sqrt{\sin^4\theta - C^2 \sin^2\theta}}$$

Thus $$\frac{d\phi}{d\theta} = \frac{c}{\sin^2\theta\sqrt{1-\frac{c^2}{\sin^2\theta}}}$$

$$\rightarrow \boxed{\phi(\theta) = \int \frac{c\, d\theta}{\sin^2\theta\sqrt{1-\frac{c^2}{\sin^2\theta}}} + D}$$

To solve integral make trig substitution

$$u = \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$du = \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}\, d\theta = -\frac{d\theta}{\sin^2\theta}$$

$$\begin{aligned} 1+u^2 &= 1+\cot^2\theta \\ &= 1 + \frac{\cos^2\theta}{\sin^2\theta} \\ &= \frac{1}{\sin^2\theta}\left[\cos^2\theta + \sin^2\theta\right] \\ &= \frac{1}{\sin^2\theta} \end{aligned}$$

Thus, $$\begin{aligned} 1 - \frac{c^2}{\sin^2\theta} &= 1 - c^2(1+u^2) \\ &= (1-c^2) - u^2 \\ &= b^2 - u^2 \end{aligned}$$, where $b^2 = 1-c^2$

so $$\boxed{\phi(\theta) = \int \frac{-c\,du}{\sqrt{b^2-u^2}} + D}$$

<u>NOTE</u>: $b^2 = 1-c^2 > 0$ since $$\begin{aligned} 1-c^2 &= 1 - \frac{\sin^4\theta\,\phi'^2}{1+\sin^2\theta\,\phi'^2} \\ &= \frac{1+\sin^2\theta\,\phi'^2 - \sin^4\theta\,\phi'^2}{1+\sin^2\theta\,\phi'^2} \\ &= \frac{1+\sin^2\theta\,\phi'^2\cos^2\theta}{1+\sin^2\theta\,\phi'^2} > 0 \end{aligned}$$

Trig substitution: $u = b\sin w$, $du = b\cos w\,dw$, $b^2 - u^2 = b^2(1-\sin^2 w) = b^2\cos^2 w$

$$\rightarrow \quad \phi(\theta) = -\int \frac{\cancel{b\cos w}\,dw}{\cancel{\sqrt{b^2\cos^2 w}}} + D$$

$$= -w + D$$

$$= -\sin^{-1}\left(\frac{u}{b}\right) + D$$

$$= -\sin^{-1}\left(\frac{\cot\theta}{b}\right) + D$$

Thus, $(D-\phi) = \sin^{-1}\left(\frac{\cot\theta}{b}\right)$

$$b\sin(D-\phi) = \cot\theta$$

$$\cancel{b}(\sin D\cos\phi - \cos D\sin\phi) = \cot\theta$$

$$\boxed{A\cos\phi + B\sin\phi = \cot\theta}$$

Compare to equation of plane passing through orig.

$$\vec{r} = R(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}) \quad \text{on sphere}$$

$$\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z} \quad (\text{unit normal to plane})$$

$$0 = \hat{n}\cdot\vec{r}$$

$$= R[n_x\sin\theta\cos\phi + n_y\sin\theta\sin\phi + n_z\cos\theta]$$

$$= R\sin\theta\, n_z\left[\frac{n_x}{n_z}\cos\phi + \frac{n_y}{n_z}\sin\phi + \cot\theta\right]$$

So $\boxed{a\cos\phi + b\sin\phi + \cot\theta = 0}$

C.3

$$I_1[y] = \int_{x_1}^{x_2} F(y, y', x)\, dx$$

$$I_2[y] = \int_{x_1}^{x_2} F^2(y, y', x)\, dx$$

$$\delta I_1 = \int_{x_1}^{x_2} \delta F \, dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \overbrace{\delta y'}^{\frac{d}{dx}\delta y} \right) dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right] \delta y \, dx + \frac{\partial F}{\partial y'} \underbrace{\cancel{\delta y}}_{0} \Big|_{x_1}^{x_2}$$

$\delta I_1 = 0$ iff $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'} \right) = 0$

$$\delta I_2 = \int_{x_1}^{x_2} 2F \, \delta F \, dx$$

$$= \int_{x_1}^{x_2} 2F \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \overbrace{\delta y'}^{\frac{d}{dx}\delta y} \right)$$

$$= 2 \int_{x_1}^{x_2} \left(F \frac{\partial F}{\partial y} - \frac{d}{dx}\left(F \frac{\partial F}{\partial y'} \right) \right) \delta y + F \frac{\partial F}{\partial y'} \underbrace{\cancel{\delta y}}_{0} \Big|_{x_1}^{x_2}$$

$\delta I_2 = 0$ iff $0 = F \dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(F \dfrac{\partial F}{\partial y'} \right)$

$$= F\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right) - \frac{dF}{dx} \frac{\partial F}{\partial y'}$$

so

$$\delta I_1 = 0 \quad \text{iff} \quad \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Longrightarrow \quad \delta I_2 = F\cancelto{0}{\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)} - \frac{dF}{dx}\frac{\partial F}{\partial y'}$$

$$= -\frac{dF}{dx}\frac{\partial F}{\partial y'}$$

$$\delta I_2 = 0 \quad \text{iff} \quad \frac{dF}{dx} = 0 \quad \text{or} \quad \frac{\partial F}{\partial y'} = 0$$

C.4

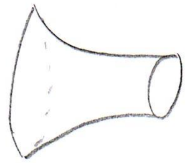
$$I[y] = y(x_0) = \int dx\, \delta(x-x_0)\, y(x)$$

$$\delta I[y] = \int dx\, \delta(x-x_0)\, \delta y(x)$$

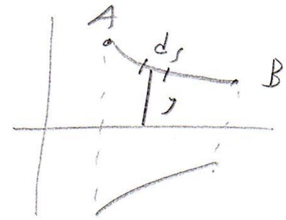
$$= \int dx\, \frac{\delta I[y]}{\delta y(x)}\, \delta y(x)$$

$$\rightarrow \quad \frac{\delta I[y]}{\delta y(x)} = \delta(x-x_0)$$

Soap Film: Example: 6.2



minimize SA of revolution



$$I = \int_A^B 2\pi y \, ds$$

$$= \int_{x_1}^{x_2} 2\pi y\sqrt{1+y'^2} \, dx$$

$$F(x,y,y') = 2\pi y\sqrt{1+y'^2}$$ indep. of x

Method I:

Rewrite

$$I = \int_{y_1}^{y_2} 2\pi y\sqrt{1+x'^2} \, dy$$

$$G(y,x,x') = 2\pi y\sqrt{1+x'^2}$$ indep. of y

$$\frac{\partial G}{\partial x'} = \text{const}$$

$$\frac{2\pi y x'}{\sqrt{1+x'^2}} = C$$

$$\frac{y^2 x'^2}{(1+x'^2)} = A^2$$

$$y^2 x'^2 = A^2(1+x'^2) = A^2 + A^2 x'^2$$

$$(y^2 - A^2)x'^2 = A^2 \rightarrow x' = \frac{A}{\sqrt{y^2 - A^2}}$$

$$dx = \frac{A\,dy}{\sqrt{y^2 - A^2}}$$

$$x = \int \frac{A\,dy}{\sqrt{y^2 - A^2}} + B$$

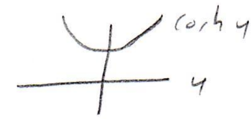
$$= \int \frac{A \cdot \cancel{A \sinh u}\,du}{\cancel{A \sinh u}} + B$$

$$= Au + B$$

$$= A \cosh^{-1}\left(\frac{y}{A}\right) + B$$

$$\rightarrow \left(\frac{x - B}{A}\right) = \cosh^{-1}\left(\frac{y}{A}\right)$$

$$\boxed{y = A \cosh\left(\frac{x - B}{A}\right)}$$



let $y = A \cosh u$

$$y^2 - A^2 = A^2(\cosh^2 u - 1)$$
$$= A^2 \sinh^2 u$$

since $\cosh^2 u - \sinh^2 u = 1$

$$dy = A \sinh u \, du$$

Method II:

$$F(x, y, y') = 2\pi y \sqrt{1 + y'^2}$$

Then $h = y' \dfrac{\partial F}{\partial y'} - F = C$ (const)

$$\rightarrow \quad y' \frac{2\pi y\, y'}{\sqrt{1+y'^2}} - 2\pi y \sqrt{1 + y'^2} = C$$

$$\frac{\cancel{2\pi y\, y'^2} - 2\pi y\,(1 + \cancel{y'^2})}{\sqrt{1+y'^2}} = C$$

$$\frac{-2\pi y}{\sqrt{1+y'^2}} = C$$

$$\frac{y}{\sqrt{1+y'^2}} = A$$

$$y^2 = A^2(1+y'^2)$$

$$\cancel{\frac{y^2}{A^2}} \quad y^2 - A^2 = A^2 y'^2$$

$$y'^2 = \frac{y^2}{A^2} - 1$$

$$\frac{dy}{dx} = y' = \sqrt{\frac{y^2}{A^2} - 1}$$

$$\frac{dy}{\sqrt{\frac{y^2}{A^2} - 1}} = dx$$

$$\rightarrow x = \int \frac{dy}{\sqrt{\frac{y^2}{A^2} - 1}} + B$$

$$= \int \frac{A \cancel{\sinh u}\, du}{\cancel{\sinh u}} + B$$

$$= Au + B$$

$$= A \cosh^{-1}\left(\frac{y}{A}\right) + B$$

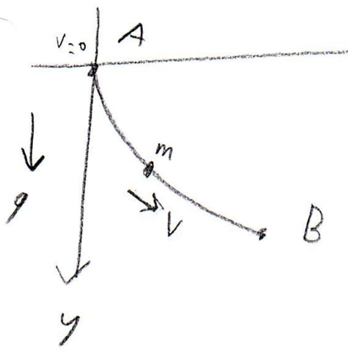
let $y = A \cosh u$

$dy = A \sinh u \, du$

$$\frac{y^2}{A^2} - 1 = \cosh^2 u - 1$$
$$= \sinh^2 u$$

Thus, $\boxed{y = A \cosh\left(\frac{x - B}{A}\right)}$

Brachistochrone: (6.5)



Find curve that gives minimum time for bead to slide down frictionless wire in uniform gravitational field g.

$$I = \int_A^B \frac{ds}{v} \quad \cancel{B}$$

$\frac{1}{2}mv^2 - mgy = E$

Initially $E=0$ (released from rest)

Thus,

$\frac{1}{2}\cancel{m}v^2 - \cancel{m}gy = 0$

$v = \sqrt{2gy}$

Thus, $$I = \int_{x_1}^{x_2} dx \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$$

$$= \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} dx \frac{\sqrt{1+y'^2}}{\sqrt{y}} = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} dy \frac{\sqrt{1+x'^2}}{\sqrt{y}}$$

$$G(y, x, x') = \frac{\sqrt{1+x'^2}}{\sqrt{y}}$$

No x dependence $\rightarrow$ $\frac{\partial G}{\partial x'} = C$

$$\frac{x'}{\sqrt{y}\sqrt{1+x'^2}} = c$$

$$\frac{x'^2}{y(1+x'^2)} = c^2$$

$$x'^2 = yc^2(1+x'^2)$$

$$x'^2(1-yc^2) = yc^2$$

$$\frac{dx}{dy} = x' = \frac{\sqrt{y}\, c}{\sqrt{1-yc^2}}$$

$$dx = \frac{dy\,\sqrt{y}\, c}{\sqrt{1-yc^2}}$$

Introduce parameterize form of the curve.

Let $yc^2 = \sin^2\left(\frac{\theta}{2}\right)$

$= \frac{1}{2}\left(1-\cos\theta\right)$

Then $\sqrt{y}\, c = \frac{1}{\sqrt{2}}\sqrt{1-\cos\theta} = \sin\left(\frac{\theta}{2}\right)$

$\sqrt{1-yc^2} = \sqrt{1-\frac{1}{2}(1-\cos\theta)}$

$= \sqrt{\frac{1}{2}(1+\cos\theta)}$

$= \cos\left(\frac{\theta}{2}\right)$

$\cos 2x = \cos^2 x - \sin^2 x$
$= 1-2\sin^2 x$

$\sin^2 x = \frac{1}{2}\left(1-\cos 2x\right)$

$\sin^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1-\cos x\right)$

$\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{1}{2}\left(1+\cos 2x\right)$

$dy\, c^2 = \cancel{2}\sin\left(\frac{\theta}{2}\right)\frac{1}{\cancel{2}}\cos\left(\frac{\theta}{2}\right)d\theta$

$= \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta$

Thus,

$$dx = \frac{\frac{1}{c^2}\sin\left(\frac{\theta}{2}\right)\cancel{\cos\left(\frac{\theta}{2}\right)}\sin\left(\frac{\theta}{2}\right)d\theta}{\cancel{\cos\left(\frac{\theta}{2}\right)}}$$

$= \frac{1}{c^2}\sin^2\left(\frac{\theta}{2}\right)d\theta$

$= \frac{1}{c^2}\frac{1}{2}\left(1-\cos\theta\right)d\theta$

$$x = \frac{1}{2c^2} \int (1-\cos\theta)\, d\theta$$

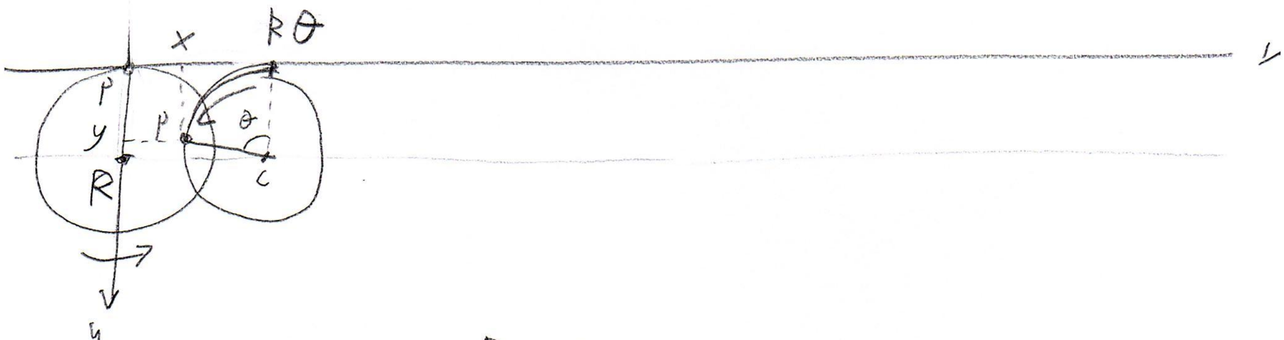
$$= \frac{1}{2c^2}(\theta - \sin\theta) + \alpha$$

$x = 0$ at $\theta = 0$ $\rightarrow$ $0 = \frac{1}{2c^2}(0 - \underbrace{\sin 0}_{0}) + \alpha$ $\rightarrow$ $\alpha = 0$

Thus,

$$\boxed{\begin{aligned} x &= \frac{1}{2c^2}(\theta - \sin\theta) \\ y &= \frac{1}{2c^2}(1-\cos\theta) \end{aligned}}$$

Rolling wheel: (without slipping)



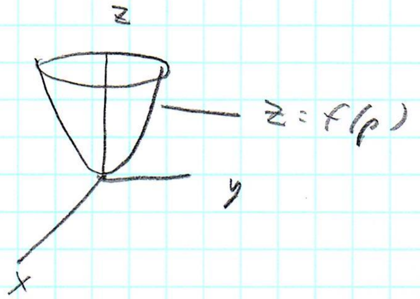
$$x = R\theta - R\sin\theta$$
$$= R(\theta - \sin\theta)$$

$$y = R - R\cos\theta$$
$$= R(1-\cos\theta)$$

which has the same form as above with $R = \frac{1}{2c^2}$

Exercise C.6 Geodesic equation for surface of revolution

a)



$$ds^2\Big|_\Sigma = dx^2 + dy^2 + dz^2 \Big|_{\substack{x=\rho\cos\phi,\\ y=\rho\sin\phi,\\ z=f(\rho)}}$$

$$\rho = \sqrt{x^2+y^2}$$
$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = f(\rho)$$

$$dx = d\rho\cos\phi - \rho\sin\phi\, d\phi$$

$$dy = d\rho\sin\phi + \rho\cos\phi\, d\phi$$

$$\rightarrow dx^2+dy^2 = d\rho^2\cos^2\phi + \rho^2\sin^2\phi\, d\phi^2 \cancel{- 2\rho\sin\phi\cos\phi\, d\rho\, d\phi}$$
$$+ d\rho^2\sin^2\phi + \rho^2\cos^2\phi\, d\phi^2 \cancel{+ 2\rho\sin\phi\cos\phi\, d\rho\, d\phi}$$

$$= d\rho^2 + \rho^2 d\phi^2$$

Also, $\quad dz^2 = [f'(\rho)]^2 d\rho^2$

Thus, $$ds^2\Big|_\Sigma = d\rho^2 + \rho^2 d\phi^2 + [f'(\rho)]^2 d\rho^2$$
$$= \left(1 + [f'(\rho)]^2\right) d\rho^2 + \rho^2 d\phi^2$$

b) $$I[\rho] = \int_A^B ds = \int_A^B \sqrt{(1+f'^2)\, d\rho^2 + \rho^2 d\phi^2}$$

$$= \int_{\phi_A}^{\phi_B} \sqrt{(1+f'^2)\left(\frac{d\rho}{d\phi}\right)^2 + \rho^2}\; d\phi$$

$$F\left(\rho, \frac{d\rho}{d\phi}, \phi\right) = \underbrace{\sqrt{(1+f'^2)\left(\frac{d\rho}{d\phi}\right)^2 + \rho^2}}_{\text{does not depend explicitly on } \phi}$$

$$\rightarrow \quad \frac{d\rho}{d\phi}\left(\frac{\partial F}{\partial\left(\frac{\partial\rho}{\partial\phi}\right)}\right) - F = \text{const} \equiv c_1$$

$$\frac{d\rho}{d\phi}\,\frac{1}{\cancel{2}\sqrt{\ }}\,\cancel{2}(1+f'^2)\frac{d\rho}{d\phi} - F = c_1$$

$$\frac{\left(\frac{d\rho}{d\phi}\right)^2(1+f'^2)}{\sqrt{}} - \sqrt{(1+f'^2)\left(\frac{d\rho}{d\phi}\right)^2+\rho^2} = c_1$$

$$\cancel{\left(\frac{d\rho}{d\phi}\right)^2(1+f'^2)} - \cancel{(1+f'^2)\left(\frac{d\rho}{d\phi}\right)^2} - \rho^2 = c_1\sqrt{}$$

$$-\rho^2 = c_1\sqrt{(1+f'^2)\left(\frac{d\rho}{d\phi}\right)^2+\rho^2}$$

$$\frac{\rho^4}{c_1^2} = (1+f'^2)\left(\frac{d\rho}{d\phi}\right)^2+\rho^2$$

$$\left(\frac{d\rho}{d\phi}\right)^2 = \frac{\frac{\rho^4}{c_1^2}-\rho^2}{1+f'^2}$$

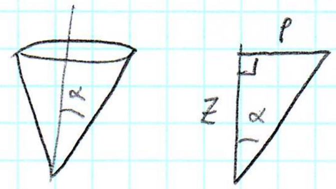
$$= \left(\frac{\rho^2}{c_1^2}\right)\frac{(\rho^2-c_1^2)}{1+f'^2}$$

$$\rightarrow \boxed{\frac{d\rho}{d\phi} = \pm\frac{\rho}{c}\sqrt{\frac{\rho^2-c_1^2}{1+f'^2}}}$$

or $$d\phi = \frac{c_1}{\rho}\sqrt{\frac{1+f'^2}{\rho^2-c_1^2}}\,d\rho$$

$$\boxed{\phi-\phi_0 = c_1\int_{\rho_0}^{\rho}\sqrt{\frac{1+f'^2}{\rho^2-c_1^2}}\,\frac{d\rho}{\rho}}$$

(c)



cone

$$\tan\alpha = \frac{\rho}{z}$$

$$z = \frac{\rho}{\tan\alpha} = \rho\cot\alpha = f(\rho)$$

$$\rightarrow f'(\rho) = \cot\alpha$$

$$\phi - \phi_0 = c_1 \int_{\rho_0}^{\rho} \sqrt{\frac{1 + (\cot\alpha)^2}{\rho^2 - c_1^2}} \frac{d\rho}{\rho}$$

Now: $1 + \cot^2\alpha = \frac{\sin^2\alpha + \cos^2\alpha}{\sin^2\alpha} = \frac{1}{\sin^2\alpha}$

$$\rightarrow \phi - \phi_0 = \left(\frac{c_1}{\sin\alpha}\right) \int_{\rho_0}^{\rho} \frac{d\rho}{\rho\sqrt{\rho^2 - c_1^2}}$$

~~Let $u = \frac{1}{\rho}$~~

~~Note~~: $\sin^2 x + \cos^2 x = 1$

$$\tan^2 x + 1 = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

Thus, let $\rho = c_1 \sec x$

$$\rightarrow \rho^2 - c_1^2 = c_1^2 \sec^2 x - c_1^2 = c_1^2 \tan^2 x$$

$$\sqrt{\rho^2 - c_1^2} = c_1 \tan x$$

$$d\rho = c_1 \frac{\sin x \, dx}{\cos^2 x} = c_1 \tan x \sec x \, dx$$

$$\phi - \phi_0 = \frac{\cancel{c_1}}{\sin\alpha} \int_{\sec^{-1}[\rho_0/c_1]}^{\sec^{-1}[\rho/c_1]} \frac{\cancel{c_1 \tan x \sec x} \, dx}{\cancel{c_1} \cancel{\sec x} \cancel{c_1 \tan x}}$$

$$= \frac{1}{\sin\alpha} \left(\sec^{-1}[\rho/c_1] - \sec^{-1}[\rho_0/c_1] \right)$$

So $\sin\alpha(\phi - \phi_0) + \sec^{-1}[\rho_0/c_1] = \sec^{-1}[\rho/c_1]$

$$\phi \sin\alpha + c_2 = \sec^{-1}[\rho/c_1] =$$

$$\sec(\phi \sin\alpha + c_2) = \frac{\rho}{c_1}$$

$$\frac{1}{\cos(\phi \sin\alpha + c_2)} = \frac{\rho}{c_1}$$

$$\rightarrow \boxed{\rho = \frac{c_1}{\cos(\phi \sin\alpha + c_2)}}$$

d) To show that the above equation is equivalent to a line $y = mx + b$

$$\rho \cos(\phi \sin\alpha + c_2) = c_1$$

$$\rho\left[\cos(\phi \sin\alpha) \cos c_2 - \sin(\phi \sin\alpha) \sin c_2\right] = c_1$$

$$\underbrace{\rho \cos(\phi \sin\alpha)}_{\bar{x}} \cos c_2 - \underbrace{\rho \sin(\phi \sin\alpha)}_{\bar{y}} \sin c_2 = c_1$$

$$\boxed{A\bar{x} + B\bar{y} = C}$$

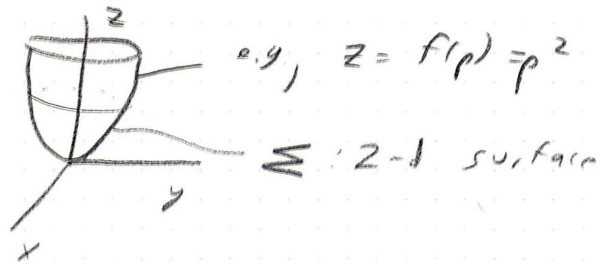
where $A = \cos c_2$
$B = -\sin c_2$
$C = c_1$

and $\bar{x} = \rho \cos(\phi \sin\alpha)$
$\bar{y} = \rho \sin(\phi \sin\alpha)$

$y = x^2$
$\frac{dy}{dx} = 2x$

Exercise: C.6 Geodesics on a surface of revolution

Consider a 2-d surface of revolution obtained by rotating the curve $z = f(\rho)$ around the z-axis:



a) Write down the line element $ds^2|_\Sigma$ on the surface Σ by substituting $z = f(\rho)$ into the 3-d line element $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$.

$$ds^2|_\Sigma = d\rho^2 + \rho^2 d\phi^2 + (f'(\rho)\,d\rho)^2$$

$$= [1 + (f'(\rho))^2]\,d\rho^2 + \rho^2 d\phi^2$$

$$= g_{\rho\rho}\,d\rho^2 + g_{\phi\phi}\,d\phi^2$$

where $g_{\rho\phi} = g_{\phi\rho} = 0$

b) Using the results of problem B.6 obtain the geodesic equations for ρ, ϕ for an affine parameterization $t = as + b$ where s = arc length

$$\int_{t_1}^{t_2} K(\rho, \phi, \dot\rho, \dot\phi)\,dt\,, \quad K = \frac{1}{2}\left[(1 + f'^2)\dot\rho^2 + \rho^2\dot\phi^2\right]$$

$\underline{\rho}$: $0 = \dfrac{d}{dt}\left(\dfrac{\partial K}{\partial \dot\rho}\right) - \dfrac{\partial K}{\partial \rho}$

$$= \frac{d}{dt}\left[(1 + f'^2)\dot\rho\right] - \left[f'f''\dot\rho^2 + \rho\dot\phi^2\right]$$

$\underline{\phi}$: $0 = \dfrac{d}{dt}\left(\dfrac{\partial K}{\partial \dot\phi}\right) - \cancel{\dfrac{\partial K}{\partial \phi}} = \dfrac{d}{dt}\left[\rho^2\dot\phi\right]$

c) solve the ϕ equation for $\dot\phi$

$$\rho^2 \dot\phi = A$$

d) Instead of solving the ρ equation directly, note that since K is indep of t

$$\dot\rho \frac{\partial K}{\partial \dot\rho} + \dot\phi \frac{\partial K}{\partial \dot\phi} - K = \text{const} \equiv B$$

~~then~~ and obtain in this way a first-order equation for $\rho, \dot\rho$

$$\dot\rho (1+f'^2)\dot\rho + \dot\phi \rho^2 \dot\phi - \tfrac{1}{2}\left[(1+f'^2)\dot\rho^2 + \rho^2\dot\phi^2\right] = B$$

$$\frac{1}{2}\left[(1+f'^2)\dot\rho^2 + \rho^2\dot\phi^2\right] = B$$

so $\boxed{B = K}$ = const

~~Solve: $\frac{1}{2}$~~

e) Substituting your solution for $\dot\phi$ from part c) solve the ρ equation by quadratures

$$(1+f'^2)\dot\rho^2 + \frac{\rho^4\dot\phi^2}{\rho^2} = 2B$$

$$(1+f'^2)\dot\rho^2 + \frac{A^2}{\rho^2} = 2B$$

$$\dot\rho^2 = \frac{\left(2B - \frac{A^2}{\rho^2}\right)}{(1+f'^2)}$$

$$\frac{d\rho}{dt} = \pm\sqrt{\frac{2B - A^2/\rho^2}{1+f'^2}}$$

so $\boxed{t - t_0 = \int_{\rho_0}^{\rho} \pm\, d\rho \sqrt{\frac{1+f'^2}{2B - A^2/\rho^2}}}$

NOTE:

$$\int \pi \, dt, \quad \pi = \frac{1}{2}\left[(1+f'^2)\dot{\rho}^2 + \rho^2\dot{\phi}^2\right]$$

is <u>not</u> parametrization invariant.

$$\int G \, dt, \quad G = \sqrt{(1+f'^2)\dot{\rho}^2 + \rho^2\dot{\phi}^2}$$

is parametrization invariant. ($G\,dt$ is independent of t)

~~WA~~ G indep. of t $\rightarrow$

$$\dot{\rho}\frac{\partial G}{\partial \dot{\rho}} + \dot{\phi}\frac{\partial G}{\partial \dot{\phi}} - G = \text{const}$$

$$\dot{\rho}\frac{1}{\cancel{2}\sqrt{\ }}\cancel{2}\dot{\rho}(1+f'^2) + \dot{\phi}\frac{1}{\cancel{2}\sqrt{\ }}\cancel{2}\dot{\phi}\rho^2 - \sqrt{\ } = \text{const}$$

$$\frac{\dot{\rho}^2(1+f'^2)}{\sqrt{\ }} + \frac{\rho^2\dot{\phi}^2}{\sqrt{\ }} - \sqrt{\ } = \text{const}$$

multiply by $\sqrt{\ }$:

$$\rightarrow \cancel{\dot{\rho}^2(1+f'^2) + \rho^2\dot{\phi}^2} - \cancel{\left(\dot{\rho}^2(1+f'^2) - \rho^2\dot{\phi}^2\right)} = \text{const}\sqrt{\ }$$

$$\boxed{0 = \text{const}}$$

Thus, dividing by $\sqrt{\ }$, taking const $= 0$

$$\frac{\dot{\rho}^2(1+f'^2)}{(\sqrt{\ })^2} + \frac{\rho^2\dot{\phi}^2}{(\sqrt{\ })^2} = 1$$

$$(1+f'^2)\left(\frac{d\rho}{ds}\right)^2 + \rho^2\left(\frac{d\phi}{ds}\right)^2 = 1$$

$$\Longleftrightarrow \boxed{(1+f'^2)\,d\rho^2 + \rho^2(d\phi)^2 = ds^2}$$

which is the expression for the line elt.

Arbitrary parametrization (C.7)

(a) $\frac{d}{d\lambda}\left[\frac{1}{L} g_{ab}\dot{x}^b\right] = \frac{1}{2L} g_{bc,a}\dot{x}^b\dot{x}^c$ $\qquad \frac{1}{L}\dot{x}^b = \frac{d\lambda}{ds}\frac{dx^b}{d\lambda}$

$-\frac{1}{L^2}\left(\frac{dL}{d\lambda}\right) g_{ab}\dot{x}^b + \frac{1}{L}\frac{d}{d\lambda}\left(g_{ab}\dot{x}^b\right) = \frac{1}{2L} g_{bc,a}\dot{x}^b\dot{x}^c$ $\qquad = \frac{dx^b}{ds}$

$-\frac{1}{L}\left(\frac{dL}{d\lambda}\right) g_{ab}\dot{x}^b + \frac{d}{d\lambda}\left(g_{ab}\dot{x}^b\right) = \frac{1}{2} g_{bc,a}\dot{x}^b\dot{x}^c$

So $\boxed{\frac{d}{d\lambda}\left(g_{ab}\dot{x}^b\right) = \frac{1}{2} g_{bc,a}\dot{x}^b\dot{x}^c + \frac{1}{L}\left(\frac{dL}{d\lambda}\right) g_{ab}\dot{x}^b}$

extra term

$L = \sqrt{g_{ab}(x)\dot{x}^a\dot{x}^b}$ $\qquad L = \frac{ds}{d\lambda}$

$\frac{dL}{d\lambda} = \frac{1}{2\sqrt{\ }}\left(g_{ab,c}\dot{x}^c\dot{x}^a\dot{x}^b + g_{ab}\ddot{x}^a\dot{x}^b + g_{ab}\dot{x}^a\ddot{x}^b\right)$ $\qquad \lambda = as + b$

$= \frac{1}{2\sqrt{\ }}\left(g_{ab,c}\dot{x}^a\dot{x}^b\dot{x}^c + 2 g_{ab}\dot{x}^a\ddot{x}^b\right)$ $\qquad G$

Consider $I[x,y] = \int_{t_1}^{t_2} \sqrt{\dot{x}^2+\dot{y}^2}\, dt$

Show that euler equations are

$\ddot{x} = \frac{1}{G}\frac{d}{dt}\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)\dot{x}$ $\qquad g_{ab} = \delta_{ab}$

$= \frac{1}{\sqrt{\ }}\frac{1}{2}\frac{1}{\sqrt{\ }}\left(2\dot{x}\ddot{x} + 2\dot{y}\ddot{y}\right)\dot{x}$

$\boxed{\ddot{x} = \frac{1}{(\dot{x}^2+\dot{y}^2)}\left(\dot{x}\ddot{x} + \dot{y}\ddot{y}\right)\dot{x}}$ $\qquad x''$

$\boxed{\ddot{y} = \frac{1}{(\dot{x}^2+\dot{y}^2)}\left(\dot{x}\ddot{x} + \dot{y}\ddot{y}\right)\dot{y}}$

$$\frac{\dot x}{\sqrt{\dot x^2+\dot y^2}} = c_1 \quad , \quad \frac{\dot y}{\sqrt{\dot x^2+\dot y^2}} = c_2$$

$$0 = \frac{\ddot x}{\sqrt{\ }} - \frac{1}{2} \cancel{\quad} \dot x \frac{1}{(\quad)^{3/2}} (2\dot x\ddot x + 2\dot y\ddot y)$$

$$= \frac{\ddot x}{\sqrt{\ }} - \frac{\dot x(\dot x\ddot x + \dot y\ddot y)}{(\quad)^{3/2}}$$

$$= \ddot x - \frac{\dot x(\dot x\ddot x + \dot y\ddot y)}{(\dot x^2+\dot y^2)}$$

Thus, $$\ddot x = \frac{1}{(\dot x^2+\dot y^2)} (\dot x\ddot x + \dot y\ddot y)\dot x$$

similarly $$\ddot y = \frac{1}{(\dot x^2+\dot y^2)} (\dot y\ddot y + \dot x\ddot x)\dot y$$

(b) $\boxed{L = \sqrt{g_{ab}(x)\, \dot{x}^a \dot{x}^b}} = \frac{ds}{d\lambda}$ $\boxed{L = g_{ab}\dot{x}^a \dot{x}^b}$

$$\frac{\partial L}{\partial \dot{x}^a} = \frac{1}{\cancel{2}} \frac{1}{\sqrt{\ }} \cancel{2} g_{ab} \dot{x}^b$$

$$= \frac{1}{L} g_{ab} \dot{x}^b$$

$\int ds \cdot 1$

$$\cancel{\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a}}$$

$$\frac{\partial L}{\partial x^a} = \frac{1}{2} \frac{1}{\sqrt{\ }} \frac{\partial g_{bc}}{\partial x^a} \dot{x}^b \dot{x}^c$$

$$= \frac{1}{2L} g_{bc,a} \dot{x}^b \dot{x}^c$$

Thus, $\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a}$

$$\frac{d}{d\lambda}\left[\frac{1}{L} g_{ab}\dot{x}^b\right] = \frac{1}{2L} g_{bc,a} \dot{x}^b \dot{x}^c$$

$$\frac{d\lambda}{ds} \times \left\{\frac{d}{d\lambda}\left[\frac{d\lambda}{ds} g_{ab}\dot{x}^b\right] = \frac{1}{2}\frac{d\lambda}{ds} g_{bc,a} \dot{x}^b \dot{x}^c\right\}$$

$$\Leftarrow \boxed{\frac{d}{ds}\left[g_{ab}\frac{dx^b}{ds}\right] = \frac{1}{2} g_{bc,a} \frac{dx^b}{ds}\frac{dx^c}{ds}}$$

$L = f = \frac{ds}{d\lambda}$ $s \to a\bar{s} + b$

$\boxed{\text{Welcome @ 121}}$

(2)

(c) $E = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b$

$$\int \left(\frac{1}{2} g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} \right) d\lambda$$

$$\frac{\partial E}{\partial \dot{x}^a} = g_{ab} \dot{x}^b \quad ; \quad \frac{\partial E}{\partial x^a} = \frac{1}{2} g_{bc,a} \dot{x}^b \dot{x}^c$$

$$\boxed{\frac{d}{d\lambda}\left[g_{ab} \dot{x}^b \right] = \frac{1}{2} g_{bc,a} \dot{x}^b \dot{x}^c}$$

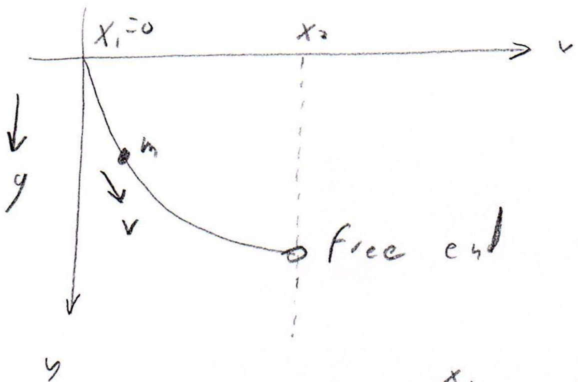
which agrees with

$$\frac{d}{ds}\left[g_{ab} \frac{dx^b}{ds} \right] = \frac{1}{2} g_{bc,a} \frac{dx^b}{ds} \frac{dx^c}{ds}$$

for $\lambda = as + b$

obtained by varying $L = \sqrt{g_{ab}\dot{x}^a \dot{x}^b}$

Brachistochrone with one free end point (C.8)



$$I = \int_A^B \frac{ds}{v} \quad \text{where}$$

$$v = \sqrt{2gy} \quad \text{as before}$$

Thus,
$$I = \int_{x_1}^{x_2} dx \, \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$$

$$= \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} dx \, \frac{\sqrt{1+y'^2}}{\sqrt{y}}$$

$$F(y, y', x) = \frac{\sqrt{1+y'^2}}{\sqrt{y}} \quad \text{indep. of } x$$

$$h \equiv y' \frac{\partial F}{\partial y'} - F \quad = \text{const}$$

$$C = y' \frac{1}{\cancel{2}\sqrt{1+y'^2}} \frac{\cancel{2}y'}{\sqrt{y}} - \frac{\sqrt{1+y'^2}}{\sqrt{y}}$$

$$= \frac{y'^2}{\sqrt{1+y'^2}\sqrt{y}} - \frac{\sqrt{1+y'^2}}{\sqrt{y}}$$

$$= \frac{1}{\sqrt{y}\sqrt{1+y'^2}} \left[y'^2 - (1+y'^2) \right]$$

$$= \frac{1}{\sqrt{y(1+y'^2)}}$$

$$\rightarrow \quad c^2 = \frac{1}{y(1+y'^2)} \quad \text{or} \quad 1+y'^2 = \frac{1}{c^2 y}$$

$$y'^2 = \frac{1}{c^2 y} - 1 = \frac{1 - c^2 y}{c^2 y}$$

$$\rightarrow \quad y' = \frac{\sqrt{1 - c^2 y}}{c\sqrt{y}} = \frac{dy}{dx}$$

$$\boxed{dx = \frac{c\sqrt{y}}{\sqrt{1 - c^2 y}}\, dy}$$

same differential equation that we found before.

$\rightarrow$ brachistochrone solution

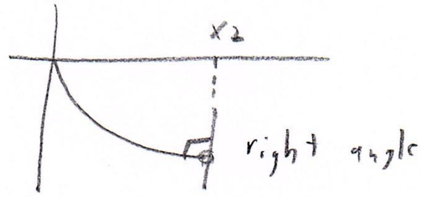
Natural boundary condition:

$$\left.\frac{\partial F}{\partial y'}\right|_{x_2} = 0$$

$$\rightarrow \quad 0 = \frac{1}{\cancel{2}\sqrt{1+y'^2}} \left.\frac{\cancel{2}y'}{\sqrt{y}}\right|_{x_2}$$

$$= \left.\frac{y'}{\sqrt{y}\sqrt{1+y'^2}}\right|_{x_2}$$

so $\boxed{y'|_{x_2} = 0}$ $\rightarrow$ slope $= 0$ at x_2



Verification of Simplification #3 (C.9)

To show

$$\frac{d}{dx}\left(\frac{\partial \underline{F}}{\partial y_i'}\right) - \frac{\partial \underline{F}}{\partial y_i} = 0 \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) - \frac{\partial F}{\partial y_i} = 0$$

for $i = 1, 2, \cdots, n-1$ as a consequence of $\frac{\partial F}{\partial y_n} = 0$

Proof: $\underline{F}(y_1, \cdots, y_{n-1}; y_1', \cdots, y_{n-1}'; x)$

$$\equiv F(y_1, \cdots y_n; y_1', \cdots, y_{n-1}')\Big|_{y_n = y_n(y_1, \cdots, y_{n-1}; y_1', \cdots, y_{n-1}'; x)}$$

where F indep of y_n'

By ~~Eu~~ ~~E~~uler equation:

$$\frac{d}{dx}\cancel{\left(\frac{\partial F}{\partial y_n'}\right)}^{0} - \frac{\partial F}{\partial y_n} = 0$$

$$\rightarrow \frac{\partial F}{\partial y_n} = 0$$

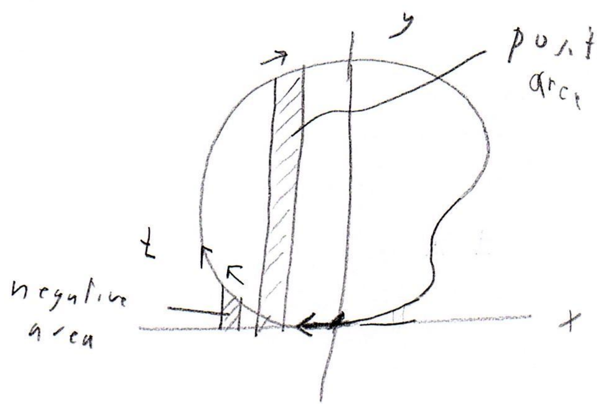
Now: $$\frac{\partial \underline{F}}{\partial y_i'} = \frac{\partial F}{\partial y_i'} + \underbrace{\cancel{\frac{\partial F}{\partial y_n}}}_{0} \frac{\partial y_n}{\partial y_i'} = \frac{\partial F}{\partial y_i'}$$

$$\frac{\partial \underline{F}}{\partial y_i} = \frac{\partial F}{\partial y_i} + \underbrace{\cancel{\frac{\partial F}{\partial y_n}}}_{0} \frac{\partial y_n}{\partial y_i} = \frac{\partial F}{\partial y_i}$$

Thus, $$0 = \frac{d}{dx}\left(\frac{\partial \underline{F}}{\partial y_i'}\right) - \frac{\partial \underline{F}}{\partial y_i} \qquad i = 1, 2, \cdots, n-1$$

$$= \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) - \frac{\partial F}{\partial y_i}$$

Isoperimetric Area problem: $\boxed{\text{Example C.4}}$



$t \in [t_1, t_2] = [0, 1]$

$x = x(t),\ y = y(t)$

BCs: $x(0) = 0,\ y(0) = 0,\ \dot{x}(0) = -c,\ \dot{y}(0) = 0$

$x(1) = 0,\ y(1) = 0,\ \dot{x}(1) = -c,\ \dot{y}(1) = 0$

$$\text{Area} = \oint y\,dx = \int_{t_1}^{t_2} y\dot{x}\,dt$$

$$\text{Perimeter} = \oint ds = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt = l \quad \leftarrow \text{constraint}$$

$$F(x, y, \dot{x}, \dot{y}) = y\dot{x}$$

$$G(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2}$$

Extremise $\bar{I}[x, y, \lambda] = \int_{t_1}^{t_2} dt\,(F + \lambda G) - \lambda l$

$$F + \lambda G = y\dot{x} + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$$

(positive homog function of degree 1 in $\dot{x}, \dot{y}$)

Euler equations for $\delta \bar{I}$:

$\underline{\delta\lambda}$: $\int_{t_1}^{t_2} dt\,\sqrt{\dot{x}^2 + \dot{y}^2} = l$ (constraint)

$\underline{\delta x}$: $\frac{d}{dt}\left(\frac{\partial(F + \lambda G)}{\partial \dot{x}}\right) - \cancel{\frac{\partial(F + \lambda G)}{\partial x}}^{\,0} = 0$

$$\frac{d}{dt}\left[y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] = 0$$

so $\boxed{y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \text{const}}$

NOTE: $ds = \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$

so $\frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{dt}{ds}$

In terms of arc length ds

$$y + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = y + \lambda \dot{x}\frac{dt}{ds} = \boxed{y + \lambda \frac{dx}{ds} = \text{const}}$$

$\underline{\delta y}$: $$\frac{d}{dt}\left(\frac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) - \dot{x} = 0$$

In terms of arc length $$\left[\frac{d}{dt}\left(\lambda \frac{dy}{ds}\right) - \dot{x} = 0\right]\frac{dt}{ds}$$

$$\frac{d}{ds}\left(\lambda \frac{dy}{ds}\right) - \frac{dx}{ds} = 0$$

$$\frac{d}{ds}\left[\lambda \frac{dy}{ds} - x\right] = 0$$

so $$\boxed{\lambda \frac{dy}{ds} - x = \text{const}}$$

$\underline{\text{Combine}}$: $$y + \lambda \frac{dx}{ds} = c_1$$
$$x - \lambda \frac{dy}{ds} = c_2$$

$\rightarrow$ $$xy + \lambda x \frac{dx}{ds} = c_1 x$$
$$xy - \lambda y \frac{dy}{ds} = c_2 y$$

subtract: $$\lambda\left(\underbrace{x\frac{dx}{ds}}_{\frac{1}{2}\frac{d}{ds}(x^2)} + \underbrace{y\frac{dy}{ds}}_{\frac{1}{2}\frac{d}{ds}(y^2)}\right) = c_1 x - c_2 y$$

$$\frac{1}{2}\lambda \frac{d}{ds}(x^2+y^2) = c_1 x - c_2 y$$

$$y + \lambda \frac{dx}{ds} = A$$

$$\lambda \frac{dy}{ds} - x = B$$

B.C. $\left.\frac{dx}{ds}\right|_{s=0,\ell} = -1$, $\left.\frac{dy}{ds}\right|_{s=0,\ell} = 0$, $x\big|_{s=0,\ell} = 0$, $y\big|_{s=0,\ell} = 0$

$$\rightarrow \quad \lambda \cdot 0 = B \rightarrow \boxed{B = 0}$$

$$0 + \lambda(-1) = A \rightarrow \boxed{A = -\lambda}$$

Thus

$$\boxed{\begin{aligned} \lambda \frac{dy}{ds} - x &= 0 \\ y + \lambda \frac{dx}{ds} &= -\lambda \end{aligned}} \rightarrow y = -\lambda\left(1 + \frac{dx}{ds}\right)$$

substitute into (1): $-\lambda^2\left(\frac{d^2x}{ds^2}\right) - x = 0 \Rightarrow$

$$\frac{d^2x}{ds^2} = -\frac{x}{\lambda^2}$$

so $\boxed{x(s) = D \sin\left(\frac{s}{|\lambda|}\right) + E \cos\left(\frac{s}{|\lambda|}\right)}$

$x(0) = 0 \rightarrow \boxed{E = 0}$

$\frac{dx}{ds}(0) = -1 \rightarrow -1 = \frac{D}{|\lambda|} \cos\left(\frac{0}{|\lambda|}\right) \rightarrow \boxed{D = -|\lambda|}$

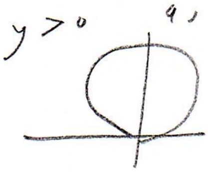
so $\boxed{x(s) = -|\lambda| \sin\left(\frac{s}{|\lambda|}\right)}$

$$y = -\lambda\left(1 + \frac{dx}{ds}\right)$$

$$= -\lambda\left(1 - \cos\left(\frac{s}{|\lambda|}\right)\right)$$

so $\boxed{y(s) = -\lambda\left(1 - \cos\left(\frac{s}{|\lambda|}\right)\right)}$

To get $y > 0$ as in figure, need $\lambda < 0 \rightarrow \boxed{\lambda = -R}$



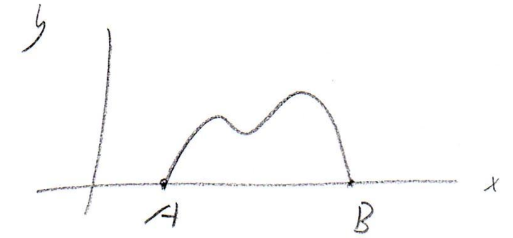
radius of circle)

Then
$$\boxed{\begin{aligned} x &= -R\sin\left(\frac{s}{R}\right) \\ y &= R\left(1 - \cos\left(\frac{s}{R}\right)\right) \end{aligned}}$$

when $\frac{s = l}{R} = 2\pi \rightarrow \boxed{R = \frac{l}{2\pi}}$, $\boxed{\lambda = -\frac{l}{2\pi}}$

Dido's isoperimetric problem:

Maximize area between curve and x-axis, subject to the constraint that the curve has fixed length l.



Assume: $l > x_2 - x_1$

but $l < \frac{\pi}{2}(x_2 - x_1)$

$C = 2\pi r = \pi d$
$= \pi(x_2 - x_1)$

$$I = \int_A^B y\,dx = \int_{x_1}^{x_2} y\,dx$$

$$J = \int_A^B ds = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx = l$$

Isoperimetric variational problem

$$\delta(I + \lambda J) = 0$$

$$\rightarrow \quad F(x,y,y') + \lambda G(x,y,y') = y + \lambda\sqrt{1+y'^2}$$

No explicit x-dependence, so

$$h \equiv y'\frac{\partial}{\partial y'}(F+\lambda G) - (F+\lambda G) = \text{const}$$

$$y'\frac{\lambda y'}{\sqrt{1+y'^2}} - \left(y + \lambda\sqrt{1+y'^2}\right) = C$$

$$\frac{\lambda y'^2 - \left(y + \lambda\sqrt{1+y'^2}\right)\sqrt{1+y'^2}}{\sqrt{1+y'^2}} = C$$

$$\frac{\cancel{\lambda y'^2} - y\sqrt{1+y'^2} - \lambda(1+y'^2)}{\sqrt{1+y'^2}} = C$$

$$-y - \frac{\lambda}{\sqrt{1+y'^2}} = C$$

$$-\frac{\lambda}{\sqrt{1+y'^2}} = y + c$$

$$\frac{\lambda^2}{(1+y'^2)} = (y+c)^2$$

$$\frac{\lambda^2}{(y+c)^2} = 1+y'^2$$

$$\rightarrow \quad y' = \sqrt{\frac{\lambda^2}{(y+c)^2} - 1}$$

$$\frac{dy}{dx} = \frac{1}{(y+C)}\sqrt{\lambda^2 - (y+c)^2}$$

$$\int dx = \int dy \frac{(y+c)}{\sqrt{\lambda^2 - (y+c)^2}}$$

$$x = \int dy \frac{(y+c)}{\sqrt{\lambda^2 - (y+c)^2}} + D$$

$$= \int \frac{\cancel{\lambda\cos\theta}}{\cancel{\lambda\cos\theta}} \lambda \sin\theta \, d\theta + D$$

$$= -\lambda\cos\theta + D$$

let

$y + c = \lambda\sin\theta$

$dy = \lambda\cos\theta \, d\theta$

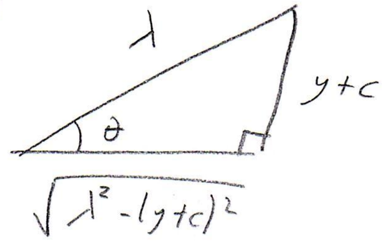
$\lambda^2 - (y+c)^2 = \lambda^2(1-\sin^2\theta)$

$= \lambda^2\cos^2\theta$

$$x - D = -\lambda \cos\theta$$

$$-\frac{(x-D)}{\lambda} = \cos\theta$$

$$-\frac{(x-D)}{\cancel{\lambda}} = \frac{\sqrt{\lambda^2 - (y+c)^2}}{\cancel{\lambda}}$$



$$\sin\theta = \frac{y+c}{\lambda}$$

$$\cos\theta = \frac{\sqrt{\lambda^2 - (y+c)^2}}{\lambda}$$

$$(x-D)^2 = \lambda^2 - (y+c)^2$$

Thus, $(x-D)^2 + (y+c)^2 = \lambda^2$

Circle with center $(D, -c)$ and radius λ

$$(x_1 - D)^2 + (0+c)^2 = \lambda^2 \rightarrow (x_1 - D)^2 = \lambda^2 - c^2$$

$$\cancel{x_1 - D \quad \lambda^2}$$

$$(x_2 - D)^2 + (0+c)^2 = \lambda^2 \rightarrow (x_2 - D)^2 = \lambda^2 - c^2$$

Thus $(x_1 - D) = \pm(x_2 - D)$

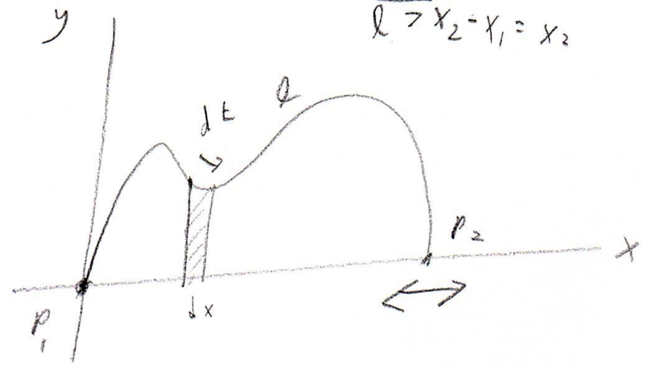
$+$: $x_1 - D = x_2 - D \rightarrow x_1 = x_2$

$-$: $x_1 - D = -(x_2 - D) \rightarrow x_1 + x_2 = 2D$

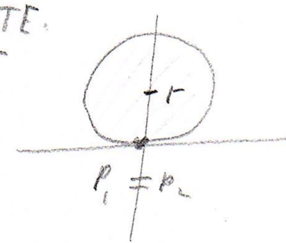
$$D = \frac{1}{2}(x_1 + x_2)$$

Isoperimetric problem with a free end: C.10

NOTE: $\ell > x_2 - x_1 = x_2$



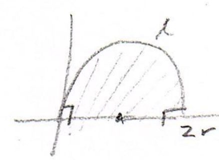
NOTE:



$$r = \frac{\ell}{2\pi}$$

$$A = \pi r^2 = \pi \frac{\ell^2}{4\pi^2} = \frac{\ell^2}{4\pi}$$

vs.



$$\ell = \pi r$$

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{\ell}{\pi}\right)^2 = \frac{\ell^2}{2\pi}$$

B.C.s $\quad P_1 : (0,0)$

$\quad P_2 : (x_2, 0)$, x_2 : free

(see below for natural BC)

Parametric representation

$x(t),\ y(t) : t \in [t_1, t_2]$

$$\text{Area} = \int_{t_1}^{t_2} y\dot{x}\,dt$$

$$\text{Length} = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt \quad \text{using} \quad \frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\bar{F} = F + \lambda G = y\dot{x} + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$$

Natural BC: $\left.\dfrac{\partial \bar{F}}{\partial \dot{x}}\right|_{t_2} = 0 \quad [x_2 \text{ free};\ y_2 = 0]$

$\delta\lambda$: constraint $\quad \ell = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$

δx: $\quad \dfrac{d}{dt}\left(\dfrac{\partial \bar{F}}{\partial \dot{x}}\right) - \dfrac{\partial \bar{F}}{\partial x} = 0 \iff 0 = \dfrac{d}{dt}\left(y + \dfrac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) - 0$

$$\boxed{y + \lambda\frac{dx}{ds} = A}$$

δy: $\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) - \frac{\partial F}{\partial y} = 0 \Leftrightarrow 0 = \frac{d}{dt}\left(\frac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) - \dot{x}$

$$= \frac{d}{dt}\left[\frac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} - x\right]$$

Thus, $\boxed{\lambda \frac{dy}{ds} - x = B}$

BCs:
$s=0$: $x(0)=0$, $y(0)=0$
$s=l$: $x(l)=x_2$, $y(l)=0$

$$0 = \left.\frac{\partial F}{\partial \dot{x}}\right|_{s=l} = \left.\left(y + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)\right|_{s=l} = \left.\left(y + \lambda\frac{dx}{ds}\right)\right|_{s=l}$$

$\rightarrow \boxed{A=0}$

Thus, $y + \lambda \frac{dx}{ds} = 0$

$\lambda \frac{dy}{ds} - x = B \quad \rightarrow \quad x = \lambda \frac{dy}{ds} - B$

Thus, $0 = y + \lambda \frac{dx}{ds}$

$$= y + \lambda \frac{d}{ds}\left[\lambda \frac{dy}{ds} - B\right]$$

$$= y + \lambda^2 \frac{d^2y}{ds^2}$$

so $\frac{d^2y}{ds^2} = -\frac{1}{\lambda^2} y \rightarrow \boxed{y = D\sin\left[\frac{s}{|\lambda|}\right] + E\cos\left[\frac{s}{|\lambda|}\right]}$

B.C: $y|_{s=0,l} = 0 \rightarrow 0 = D\,\underbrace{\sin(0)}_{0} + E\cos(0) \rightarrow \boxed{E=0}$

$0 = D\sin\left[\frac{l}{|\lambda|}\right] \rightarrow \boxed{\frac{l}{|\lambda|} = n\pi}$

$n = 0, 1, 2, \dots$

Thus, $|\lambda| = \frac{\ell}{n\pi}$, $n = 0, 1, 2, \ldots$ and $\boxed{y = D \sin\left[\frac{n\pi s}{\ell}\right]}$

★ (need $n = 1$ for $y > 0$ for $s \in [0, \ell]$)

Also: $X(s) = \lambda \frac{dy}{ds} - B$

$$= \pm \frac{\ell}{n\pi} \frac{n\pi}{\ell} D \cos\left[\frac{n\pi s}{\ell}\right] - B = \pm D \cos\left[\frac{n\pi s}{\ell}\right] - B$$

BC: $x(0) = 0 \quad \to \quad 0 = \pm D \cos[0] - B$

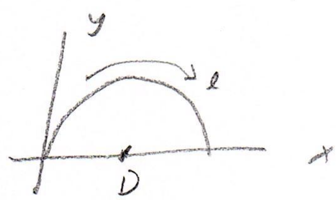
$$\boxed{B = \pm D}$$

So $X = \pm D \left(\cos\left[\frac{n\pi s}{\ell}\right] - 1 \right)$

$\to \boxed{X = D\left(1 - \cos\left[\frac{n\pi s}{\ell}\right]\right)}$ choose negative sign to have $X > 0$

NOTE: $(x - D)^2 + y^2 = D^2 \cos^2[\;] + D^2 \sin^2[\;]$
$= D^2$

So D = radius
$(D, 0)$ = center of circle



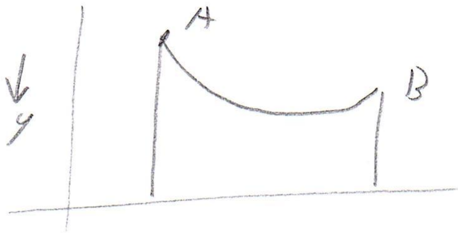
$n = 1$

$\to D = \frac{1}{2} x(\ell) = \frac{D}{2}\left(1 - \cos[n\pi]\right) = \frac{1}{\cancel{2}}\cancel{2} D$ ✓

$\ell = \frac{1}{2}(2\pi r) = \pi r = \pi D \to \boxed{D = \frac{\ell}{\pi}}$

Thus,
$$\boxed{\begin{aligned} X &= \frac{\ell}{\pi}\left(1 - \cos\left(\frac{\pi s}{\ell}\right)\right) \\ y &= \frac{\ell}{\pi} \sin\left(\frac{\pi s}{\ell}\right) \end{aligned}}$$

Hanging Cable: C.11 Minimize gravitational PE subject to constraint that cable has length l.



$$I \equiv PE = \int_A^B dm \cdot g y \qquad \Big| \; dm = \mu\, ds$$

μ mass/length

$$= \mu g \int_A^B y\, ds$$

$$= \mu g \int_{x_1}^{x_2} dx\, y\sqrt{1+y'^2}$$

$$J \equiv l = \int_A^B ds = \int_{x_1}^{x_2} dx \sqrt{1+y'^2}$$

Isoperimetric problem:

Extremize $I + \lambda J$

$\rightarrow$ EL equations for $F + \lambda G = \mu g y\sqrt{1+y'^2} + \lambda\sqrt{1+y'^2}$

$$= (\mu g y + \lambda)\sqrt{1+y'^2}$$

since $F+\lambda G$ does not depend explicitly on x

$$h \equiv y' \frac{\partial(F+\lambda G)}{\partial y'} - (F+\lambda G) = C$$

$$y' \frac{y'(\mu g y+\lambda)}{\sqrt{1+y'^2}} - (\mu g y + \lambda)\sqrt{1+y'^2} = C$$

$$\frac{\cancel{y'^2(\mu g y + \lambda)} - (\mu g y + \lambda)(1+\cancel{y'^2})}{\sqrt{1+y'^2}} = C$$

Thus,

$$\frac{\mu g y + \lambda}{\sqrt{1+y'^2}} = -C$$

$$\frac{y + \frac{\lambda}{\mu g}}{\sqrt{1+y'^2}} \equiv A \quad \left(A = -\frac{C}{\mu g} \right)$$

$$\rightarrow \quad \left(y + \frac{\lambda}{\mu g}\right)^2 = A^2\left(1+y'^2\right)$$

$$\left(y + \frac{\lambda}{\mu g}\right)^2 = A^2 + A^2 y'^2$$

$$y'^2 = \left(\frac{y + \frac{\lambda}{\mu g}}{A}\right)^2 - 1$$

$$\frac{dy}{dx} = y' = \sqrt{\frac{\left(y + \frac{\lambda}{\mu g}\right)^2}{A^2} - 1}$$

NOTE: This is same equation as that for soap film with $y \rightarrow y + \frac{\lambda}{\mu g}$, so solution is the same

Solution: $$\boxed{y + \frac{\lambda}{\mu g} = A \cosh\left(\frac{x-B}{A}\right)}$$