Problem: B.1 show $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$

Pf: $\alpha = \sum_{i_1 < i_2 < \cdots < i_p} \alpha_{i_1 i_2 \cdots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$

$\beta = \sum_{j_1 < j_2 < \cdots < j_q} \beta_{j_1 j_2 \cdots j_q} dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_q}$

$$\alpha \wedge \beta = \sum_i \sum_j \alpha_{i \cdots} \beta_{j \cdots} (dx^{i_1} \wedge \cdots \wedge dx^{i_p}) \wedge \underbrace{(dx^{j_1} \wedge \cdots \wedge dx^{j_q})}_{\text{need to move each of these through } p \text{ coord differentials}}$$

$$\underbrace{(-1)^p \cdots (-1)^p}_{q \text{ times}}$$

$= (-1)^{p + \cdots + p}$

$= (-1)^{pq}$

$= (-1)^{pq} \sum_i \sum_j \alpha_{i \cdots} \beta_{j \cdots} (dx^{j_1} \wedge \cdots \wedge dx^{j_q})(dx^{i_1} \wedge \cdots \wedge dx^{i_p})$

$= (-1)^{pq} \sum_i \sum_j \beta_{j \cdots} \alpha_{i \cdots} (dx^{j_1} \wedge \cdots \wedge dx^{j_q})(dx^{i_1} \wedge \cdots \wedge dx^{i_p})$

$= (-1)^{pq} \beta \wedge \alpha$

Problem: B.2 Show $d(\alpha\wedge\beta) = (d\alpha)\wedge\beta + (-1)^p \alpha\wedge d\beta$

$$[d(\alpha\wedge\beta)]_{\underbrace{ij\cdots kl\cdots m}_{p+q+1}} = (p+q+1)\, \partial_{[i}\, (\alpha\wedge\beta)_{j\cdots kl\cdots m]}$$

$$= (p+q+1)\, \partial_{[i} \left(\frac{(p+q)!}{p!\,q!}\, \alpha_{j\cdots k}\, \beta_{l\cdots m]} \right)$$

$$= \frac{(p+q+1)(p+q)!}{p!\,q!} \left\{ \left(\partial_{[i}\, \alpha_{j\cdots k}\right) \beta_{l\cdots m]} + \alpha_{[j\cdots k}\, \partial_{i}\, \beta_{l\cdots m]} \right\}$$

move to front through p indices

$$= \frac{(p+q+1)(p+q)!}{p!\,q!} \left\{ \frac{1}{p+1} (d\alpha)_{[ij\cdots k}\, \beta_{l\cdots m]} + (-1)^p\, \alpha_{[j\cdots} \underbrace{\left(\partial_k\, \beta_{l\cdots m]}\right)}_{\frac{1}{q+1}(d\beta)_{kl\cdots m}} \right\}$$

$$= \frac{(p+q+1)!}{p!\,q!} \left\{ \frac{1}{\cancel{(p+1)}} \frac{\cancel{(p+1)!}^{\,p!}\, q!}{(p+q+1)!} (d\alpha\wedge\beta)_{ij\cdots kl\cdots m} + (-1)^p \frac{1}{\cancel{(q+1)}} \frac{p!\, \cancel{(q+1)!}^{\,q!}}{(p+q+1)!} (\alpha\wedge d\beta)_{ij\cdots kl\cdots m} \right\}$$

$$= (d\alpha\wedge\beta)_{ij\cdots kl\cdots m} + (-1)^p (\alpha\wedge d\beta)_{ij\cdots kl\cdots m}$$

$$= [d\alpha\wedge\beta + (-1)^p \alpha\wedge d\beta]_{ij\cdots kl\cdots m}$$

B.3

Problem: $d(d\alpha) = 0$

a) In 3-dimensions, let $\alpha = 0$ be a ~~Zero~~ 0-form

Then $d\alpha \longleftrightarrow \vec{\nabla}\alpha$

Also $d(\text{1-form}) = \vec{\nabla}\times(\text{1-form})$

So $d(d\alpha) = d(\vec{\nabla}\alpha)$

$= \vec{\nabla}\times\vec{\nabla}\alpha$

b) In 3-dimensions, let $\alpha = 1$ be a 1-form

Then $d(\text{1-form}) = \epsilon_{ijk}(\vec{\nabla}\times\vec{\alpha})_k$

$\underbrace{d(\text{2-form})}_{\text{3-form}} = 3\,\partial_{[i}\beta_{jk]}$

Contract with ϵ^{ijk} to get a ~~S~~ scalar

$\epsilon^{ijk}(d\beta)_{ijk} = 3\,\epsilon^{ijk}\partial_i(\beta_{jk})$

$= 3\,\partial_i(\epsilon^{ijk}\beta_{jk})$

$= 3\,\vec{\nabla}\cdot\vec{V}$

where $\vec{V} = \epsilon^{ijk}\beta_{jk}$

But $\beta_{jk} = \epsilon_{jkl}(\vec{\nabla}\times\vec{\alpha})_l$

$\rightarrow \quad \vec{V} = \epsilon^{ijk}\epsilon_{jkl}(\vec{\nabla}\times\vec{\alpha})_l$

$= \epsilon^{ijk}\epsilon_{lij}(\vec{\nabla}\times\vec{\alpha})_l$

$= (\delta^i_l\delta^j_j - \delta^i_j\delta^j_l)(\vec{\nabla}\times\vec{\alpha})_l$

$= 3(\vec{\nabla}\times\vec{\alpha})_i - (\vec{\nabla}\times\vec{\alpha})_i = 2(\vec{\nabla}\times\vec{\alpha})_i$

Thus,

$$\underset{\substack{| \\ 1\text{form}}}{d(d\alpha)} = 3!\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{\alpha})$$

so $d(d\alpha) = 0 \iff \boxed{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\alpha}) = 0}$

B.4

Problem: Closed but globally not exact

$$\alpha = \frac{1}{x^2+y^2}(-y\,dx + x\,dy)$$

a) Check that $d\alpha = 0$

$$d\alpha = -\frac{1}{(x^2+y^2)^2}(2x\,dx + 2y\,dy)\wedge(-y\,dx + x\,dy)$$

$$+ \frac{1}{(x^2+y^2)}\left[-dy\wedge dx + dx\wedge dy\right]$$

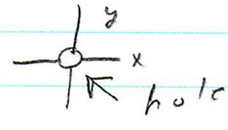
$$= -\frac{1}{(x^2+y^2)^2}\left[2x^2\,dx\wedge dy + 2y^2\,dx\wedge dy\right]$$

$$+ \frac{1}{(x^2+y^2)}\left[dx\wedge dy + dx\wedge dy\right]$$

$$= -\frac{1}{(x^2+y^2)^2}\,2(x^2+y^2)\,dx\wedge dy + \frac{2\,dx\wedge dy}{(x^2+y^2)}$$

$$= \boxed{0}$$

b) Globally not exact since space is topologically non-trivial



c) $x^2+y^2 = r^2$

$x = r\cos\phi\ ,\ y = r\sin\phi$

$$\rightarrow \alpha = \frac{1}{r^2}\Big[-r\sin\phi\,(dr\cos\phi - r\sin\phi\,d\phi)$$

$$+ r\cos\phi\,(dr\sin\phi + r\cos\phi\,d\phi)\Big]$$

$$= \frac{1}{r^2}\left[\cancel{-r\sin\phi\cos\phi\,dr} + r^2\sin^2\phi\,d\phi \right.$$
$$\left. \cancel{+\,r\cos\phi\sin\phi\,dr} + r^2\cos^2\phi\,d\phi\right]$$

$$= d\phi$$

so $$d\phi = \frac{1}{x^2+y^2}\left(-y\,dx + x\,dy\right)$$

$\uparrow$

locally defined (not single-valued)
for closed loops
enclosing the origin.

B.5

<u>Problem:</u>

$$\alpha = A\,dx + B\,dy \quad \text{in } 2\text{-d}$$

Frobenius: $\underbrace{d\alpha \wedge \alpha}_{\text{3-form}} = 0$

3-form so automatically zero!

<u>Check:</u>

$$d\alpha = \left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy\right)\wedge dx + \left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy\right)\wedge dy$$

$$= \frac{\partial A}{\partial y}dy\wedge dx + \frac{\partial B}{\partial x}dx\wedge dy$$

$$= \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)dx\wedge dy$$

$$\rightarrow d\alpha\wedge\alpha = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)dx\wedge dy \wedge (A\,dx + B\,dy)$$

$$= \boxed{0}.$$

B.6

Problem:

~~Sources - www.google.com~~

$\alpha = yz\,dx + xz\,dy + dz$ in 3-d

a) $d\alpha = z\,dy\wedge dx + y\,dz\wedge dx$
$+ z\,dx\wedge dy + x\,dz\wedge dy$

$= \cancel{-z\,dx\wedge dy} + y\,dz\wedge dx$
$\cancel{+ z\,dx\wedge dy} + x\,dz\wedge dy$

$= dz\wedge(y\,dx + x\,dy)$

$d\alpha\wedge\alpha = dz\wedge(y\,dx + x\,dy)\wedge(yz\,dx + xz\,dy$
$+ \cancel{dz})$ 0

$= dz\wedge x\,dy\wedge yz\,dx$

$+ dz\wedge y\,dx\wedge xz\,dy$

$= -xyz\,(dx\wedge dy\wedge dz) + xyz\,(dx\wedge dy\wedge dz)$

$= \boxed{0}$

Thus $\rightarrow$ integrable

b) $\mu\alpha = e^{xy}[yz\,dx + xz\,dy + dz]$

$d\varphi = d[z e^{xy}]$
$= dz\,e^{xy} + z e^{xy} y\,dx + z e^{xy} x\,dy$

$= e^{xy}[dz + yz\,dx + xz\,dy]$

$= \mu\alpha$

3.7

Problem: $dx \wedge dy = r\, dr \wedge d\phi$

$x = r\cos\phi \quad , y = r\sin\phi$

$dx = \cos\phi\, dr - r\sin\phi\, d\phi$
$dy = \sin\phi\, dr + r\cos\phi\, d\phi$

$$dx \wedge dy = (\cos\phi\, dr - r\sin\phi\, d\phi) \wedge (\sin\phi\, dr + r\cos\phi\, d\psi)$$

$$= r\cos^2\phi\, dr \wedge d\phi - r\sin^2\phi\, d\phi \wedge dr$$

$$= r(\cos^2\phi + \sin^2\phi)\, dr \wedge d\phi$$
$$= r\, dr \wedge d\phi$$

Problem: 8.8

Stokes':

$$\int_R \underset{\substack{\uparrow \\ \left(\substack{p-1 \\ \text{form}}\right)}}{d\alpha} = \oint_{\partial R} \alpha$$

$p=1$:

$\alpha = 0\text{-form} \rightarrow U$ function

$d\alpha = dU \rightarrow \vec{\nabla} U$

$RHS = U(2) - U(1)$

$$LHS = \int_1^2 \sum_i \underset{\substack{\parallel \\ U}}{(d\alpha)_i} \underbrace{\frac{dx^i}{ds}}_{(d\vec{s})^i} ds$$

$$= \int_1^2 (\vec{\nabla} U)\cdot d\vec{s}$$

so $\boxed{\int_1^2 \vec{\nabla} U \cdot d\vec{s} = U(2) - U(1)}$ Fund. thm of gradients

$p=2$:

$\alpha = 1\text{-form} \rightarrow$ vector $A_i = \alpha_i$

$d\alpha = 2\text{-form} = (d\alpha)_{ij} = \sum_k \epsilon_{ijk} (\vec{\nabla}\times\vec{A})_k$

$$RHS = \oint_{C=\partial S} \sum_i \alpha_i \left(\frac{dx^i}{ds}\right) ds = \oint_{C=\partial S} \vec{A}\cdot d\vec{s}$$

$$LHS = \int_S \sum_{i<j} (d\alpha)_{ij} \frac{\partial(x^i, x^j)}{\partial(u,v)}\, du\, dv$$

$$= \int_S \sum_{i<j} \sum_k \epsilon_{ijk} (\vec{\nabla}\times\vec{A})_k \frac{\partial(x^i, x^j)}{\partial(u,v)}\, du\, dv$$

$$= \int_S \sum_k (\vec{\nabla}\times\vec{A})_k \underbrace{\sum_{i<j} \epsilon_{ijk} \frac{\partial(x^i, x^j)}{\partial(u,v)} \, du\, dv}_{n_k \, da}$$

$$= \int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\, da$$

so $\boxed{\int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\, da = \oint_{\partial S} \vec{A}\cdot d\vec{s}}$ stokes!

<u>p=3:</u> $\alpha = 2\text{-form} \longmapsto$ vector $A^i = \sum_{j<k} \epsilon^{ijk} \alpha_{jk}$

$d\alpha = 3\text{-form} = (d\alpha)_{ijk} =$

$$\sum_{i,j,k} \epsilon^{ijk} (d\alpha)_{ijk} = \sum_{i,j,k} 3\,\epsilon^{ijk}\, \partial_i \alpha_{jk}$$

$$= \sum_{i,j,k} 3\, \partial_i (\epsilon^{ijk} \alpha_{jk})$$

$$= 3\sum_i \partial_i \Big(\underbrace{\sum_{j,k} \epsilon^{ijk} \alpha_{jk}}_{2A^i}\Big) = 3!\,(\vec{\nabla}\cdot\vec{A})$$

so $\sum_{i,j,k} \epsilon^{ijk} (d\alpha)_{ijk} = 3!\,(\vec{\nabla}\cdot\vec{A}) \iff (d\alpha)_{ijk} = (\vec{\nabla}\cdot\vec{A})\,\epsilon_{ijk}$

Using $\sum_{i,j,k} \epsilon^{ijk} \epsilon_{ijk} = 3!$

<u>Now:</u> $A^i = \frac{1}{2}\sum_{j,k} \epsilon^{ijk} \alpha_{jk} = \frac{1}{2}\epsilon^{ijk}\alpha_{jk}$

$$2A^i \epsilon^{ilm} = \epsilon^{ijk}\epsilon^{ilm}\alpha_{jk}$$
$$= (\delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl})\alpha_{jk} = \alpha_{lm} - \alpha_{ml} = 2\alpha_{lm}$$

Thus, $\alpha_{lm} = A^i \epsilon^{ilm}$ so $\alpha_{ij} = \sum_k \epsilon_{ijk} A^k$

or $\boxed{\cancel{\alpha_{ij} = \sum_k \epsilon_{ijk} A^k}}$

$$RHS = \oint_{\partial R} \alpha$$

$$= \oint_{\partial S} \sum_{i<j} \alpha_{ij} \frac{\partial(x^i, x^j)}{\partial(u,v)}\, du\, dv$$

$$= \oint_{\partial S} \sum_{i<j} \sum_k \epsilon_{ijk} A^k \frac{\partial(x^i, x^j)}{\partial(u,v)}\, du\, dv$$

$$= \oint_{\partial S} \sum_k A^k \underbrace{\sum_{i<j} \epsilon_{ijk} \frac{\partial(x^i, x^j)}{\partial u\, \partial v}\, du\, dv}_{n_k\, da}$$

$$=$$

$$= \oint_{\partial S} \vec{A} \cdot \hat{n}\, da$$

$$LHS = \int_R d\alpha$$

$$= \int_R \sum_{i<j<k} (d\alpha)_{ijk} \frac{\partial(x^i, x^j, x^k)}{\partial(u,v,w)}\, du\, dv\, dw$$

$$= \int_R (\vec{\nabla} \cdot \vec{A}) \underbrace{\sum_{i<j<k} \epsilon_{ijk} \frac{\partial(x^i, x^j, x^k)}{\partial(u,v,w)}\, du\, dv\, dw}_{dV}$$

$$= \int_R (\vec{\nabla} \cdot \vec{A})\, dV$$

Thus, $\boxed{\int_R (\vec{\nabla} \cdot \vec{A})\, dV = \oint_{\partial R} \vec{A} \cdot \hat{n}\, da}$