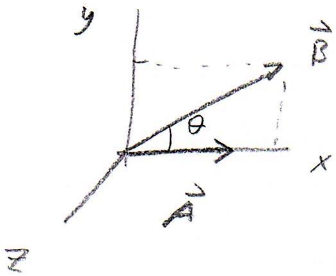
Dot and cross products: A.1

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$
$$\vec{A}\cdot\vec{B} = \sum_i A_i B_i$$

Take $\vec{A}$ along x-axis and $\vec{B}$ in x-y plane:



$$\vec{B} = B_x\hat{x} + B_y\hat{y}$$
$$\vec{A} = A_x\hat{x}$$

$$\vec{A}\cdot\vec{B} = \sum_i A_i B_i$$
$$= A_x B_x$$
$$= AB\cos\theta.$$

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$$ where $\hat{n} \perp$ to $\vec{A},\vec{B}$ and given by RHR.

$$\vec{A}\times\vec{B} = \hat{x}(A_yB_z - A_zB_y) + \hat{y}(A_zB_x - A_xB_z) + \hat{z}(A_xB_y - A_yB_x)$$

$$= \hat{z}A_xB_y$$
$$= \hat{z}AB\sin\theta$$

Triple products: A.2

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = A_i\,\epsilon_{ijk}\,B_j C_k$$

(uses Einstein summation convention)

$$= B_j\,\epsilon_{jki}\,C_k A_i$$

$$= \vec{B}\cdot(\vec{C}\times\vec{A})$$

$$= C_k\,\epsilon_{kij}\,A_i B_j$$

$$= \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \epsilon_{ijk}\,A_j\,\epsilon_{klm}\,B_l C_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,A_j B_l C_m$$

$$= B_i A_j C_j - C_i A_j B_j$$

$$= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B})$$

$$= \cancel{\vec{B}(\vec{A}\cdot\vec{C})} - \cancel{\vec{C}(\vec{A}\cdot\vec{B})}$$

$$+ \cancel{\vec{C}(\vec{B}\cdot\vec{A})} - \vec{A}(\vec{B}\cdot\vec{C})$$

$$+ \vec{A}(\vec{C}\cdot\vec{B}) - \cancel{\vec{B}(\vec{C}\cdot\vec{A})}$$

$$= 0$$ using symmetry of $\vec{A}\cdot\vec{B}$

(A.3) Contravariant / covariant

$$A^{i'} = \sum_i \frac{\partial x^{i'}}{\partial x^i} A^i$$

$$A_{i'} = \sum_i \frac{\partial x^i}{\partial x^{i'}} A_i$$

Now:

coord. differential

$$dx^{i'} = \sum_i \frac{\partial x^{i'}}{\partial x^i} dx^i \quad \longleftrightarrow \quad \text{like } A^i$$

partial derivative

$$\frac{\partial}{\partial x^{i'}} = \sum_i \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i} \quad \longleftrightarrow \quad \text{like } A_i$$

Product rules: A.4

a) $$\vec{\nabla}(fg) = (\vec{\nabla}f)g + f(\vec{\nabla}g)$$

$$\partial_i(fg) = (\partial_i f)g + f(\partial_i g)$$

b) $$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$$

consider

$$[\vec{A}\times(\vec{\nabla}\times\vec{B})]_i = \epsilon_{ijk} A_j \epsilon_{klm} \partial_l B_m$$

$$= \epsilon_{ijk} \epsilon_{lmk} A_j \partial_l B_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j \partial_l B_m$$

$$= A_j \partial_i B_j - A_j \partial_j B_i$$

$$\rightarrow [\vec{B}\times(\vec{\nabla}\times\vec{A})]_i = B_j \partial_i A_j - B_j \partial_j A_i$$

$$[(\vec{A}\cdot\vec{\nabla})\vec{B}]_i = A_j \partial_j B_i$$

$$[(\vec{B}\cdot\vec{\nabla})\vec{A}]_i = B_j \partial_j A_i$$

Thus, $$[\vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}]_i$$

$$= A_j \partial_i B_j + B_j \partial_i A_j$$

$$= \partial_i (A_j B_j)$$

$$= [\vec{\nabla}(\vec{A}\cdot\vec{B})]_i$$

c) $$\vec{\nabla}\times(f\vec{A}) = (\vec{\nabla}f)\times\vec{A} + f\,\vec{\nabla}\times\vec{A}$$

$$[\vec{\nabla}\times(f\vec{A})]_i = \epsilon_{ijk}\partial_j(fA_k)$$

$$= \epsilon_{ijk}(\partial_j f)A_k + f\epsilon_{ijk}\partial_j A_k$$

$$= [(\vec{\nabla}f)\times\vec{A}]_i + f(\vec{\nabla}\times\vec{A})_i$$

$$= [(\vec{\nabla}f)\times\vec{A} + f\,\vec{\nabla}\times\vec{A}]_i$$

d) $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$

$$
\begin{aligned}
LHS &= \epsilon_{ijk}\, \partial_j (\epsilon_{klm} A_l B_m) \\
&= \epsilon_{ijk}\, \epsilon_{lmk}\, \partial_j (A_l B_m) \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\big((\partial_j A_l) B_m + A_l (\partial_j B_m)\big) \\
&= \delta_{il}\delta_{jm} (\partial_j A_l) B_m + \delta_{il}\delta_{jm} A_l\, \partial_j B_m \\
&\quad - \delta_{im}\delta_{jl} (\partial_j A_l) B_m - \delta_{im}\delta_{jl} A_l\, \partial_j B_m \\
&= B_j \partial_j A_i + A_i \partial_j B_j - B_i \partial_j A_j - A_j \partial_j B_i \\
&= \left[(\vec{B} \cdot \vec{\nabla})\vec{A} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A}) - (\vec{A} \cdot \vec{\nabla})\vec{B}\right]_i \\
&= RHS
\end{aligned}
$$

) $\vec{\nabla} \cdot (f\vec{A}) = (\vec{\nabla} f) \cdot \vec{A} + f\, \vec{\nabla} \cdot \vec{A}$

$$
\begin{aligned}
\partial_i (f A_i) &= (\partial_i f) A_i + f\, \partial_i A_i \\
&= (\vec{\nabla} f) \cdot \vec{A} + f\, \vec{\nabla} \cdot \vec{A}
\end{aligned}
$$

f) $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

$$
\begin{aligned}
\partial_i (\epsilon_{ijk} A_j B_k) &= \epsilon_{ijk} \big((\partial_i A_j) B_k + A_j\, \partial_i B_k\big) \\
&= (\vec{\nabla} \times \vec{A})_k B_k - A_j (\vec{\nabla} \times \vec{B})_j \\
&= (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\vec{\nabla} \times \vec{B})
\end{aligned}
$$

Curl of a curl in Cartesian coords: (A.5)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \epsilon_{ijk} \, \partial_j (\epsilon_{klm} \, \partial_l A_m)$$

$$= \epsilon_{ijk} \, \epsilon_{klm} \, \partial_j \partial_l A_m$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \, \partial_j \partial_l A_m$$

$$= \partial_j \partial_i A_j - \partial_j \partial_j A_i$$

$$= \partial_i (\partial_j A_j) - \nabla^2 A_i$$

$$= \partial_i (\vec{\nabla} \cdot \vec{A}) - \nabla^2 A_i$$

$$= \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

in Cartesian coordinates since the basis vectors $\hat{x}, \hat{y}, \hat{z}$ are constants.

Directional derivatives of $\hat{e}_i$ in sph-polar coordinates (Example A.2)

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$\hat{r} = \overrightarrow{\left(\frac{\partial}{\partial r}\right)} = \frac{\partial x}{\partial r}\hat{x} + \frac{\partial y}{\partial r}\hat{y} + \frac{\partial z}{\partial r}\hat{z}$$

$$= \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \frac{1}{r}\overrightarrow{\left(\frac{\partial}{\partial \theta}\right)} = \frac{1}{r}\left[\frac{\partial x}{\partial \theta}\hat{x} + \frac{\partial y}{\partial \theta}\hat{y} + \frac{\partial z}{\partial \theta}\hat{z}\right]$$

$$= \frac{1}{r}\left[r\cos\theta\cos\phi\,\hat{x} + r\cos\theta\sin\phi\,\hat{y} - r\sin\theta\,\hat{z}\right]$$

$$= \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = \frac{1}{r\sin\theta}\overrightarrow{\left(\frac{\partial}{\partial \phi}\right)} = \frac{1}{r\sin\theta}\left[\frac{\partial x}{\partial \phi}\hat{x} + \frac{\partial y}{\partial \phi}\hat{y} + \frac{\partial z}{\partial \phi}\hat{z}\right]$$

$$= \frac{1}{r\sin\theta}\left[-r\sin\theta\sin\phi\,\hat{x} + r\sin\theta\cos\phi\,\hat{y}\right]$$

$$= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

Thus,

$$\begin{aligned} \hat{r} &= \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \\ \hat{\theta} &= \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \\ \hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{aligned}$$

Inverse relations:

$$\begin{aligned} \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} &= \cancel{\sin\theta\cos\theta\cos\phi\,\hat{x}} + \cancel{\sin\theta\cos\theta\sin\phi\,\hat{y}} + \cos^2\theta\,\hat{z} \\ &\quad - \cancel{\sin\theta\cos\theta\cos\phi\,\hat{x}} - \cancel{\sin\theta\cos\theta\sin\phi\,\hat{y}} + \sin^2\theta\,\hat{z} \end{aligned}$$

$$= \hat{z}$$

So $\boxed{\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}$

$$\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta} = \sin^2\theta\cos\phi\,\hat{x} + \sin^2\theta\sin\phi\,\hat{y} + \cancel{\sin\theta\cos\theta\,\hat{z}}$$
$$+ \cos^2\theta\cos\phi\,\hat{x} + \cos^2\theta\sin\phi\,\hat{y} - \cancel{\cos\theta\sin\theta\,\hat{z}}$$

$$\rightarrow \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

Thus $\cos\phi\sin\theta\,\hat{r} + \cos\phi\cos\theta\,\hat{\theta} = \cos^2\phi\,\hat{x} + \cancel{\cos\phi\sin\phi\,\hat{y}}$

$$-\sin\phi\,\hat{\phi} = \sin^2\phi\,\hat{x} - \cancel{\sin\phi\cos\phi\,\hat{y}}$$

$$\rightarrow \boxed{\hat{x} = \cos\phi\sin\theta\,\hat{r} + \cos\phi\cos\theta\,\hat{\theta} - \sin\phi\,\hat{\phi}}$$

Also $\sin\phi\sin\theta\,\hat{r} + \sin\phi\cos\theta\,\hat{\theta} = \cancel{\sin\phi\cos\phi\,\hat{x}} + \sin^2\phi\,\hat{y}$

$$\cos\phi\,\hat{\phi} = \cancel{-\sin\phi\cos\phi\,\hat{x}} + \cos^2\phi\,\hat{y}$$

$$\rightarrow \boxed{\hat{y} = \sin\phi\sin\theta\,\hat{r} + \sin\phi\cos\theta\,\hat{\theta} + \cos\phi\,\hat{\phi}}$$

$$\nabla_{\hat{r}}\,\hat{r} = \frac{\partial}{\partial r}\left[\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}\right]$$
$$= 0$$

$$\nabla_{\hat{\theta}}\,\hat{r} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}\right]$$
$$= \frac{1}{r}\left[\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}\right]$$
$$= \frac{1}{r}\hat{\theta}$$

$$\nabla_{\hat\phi}\,\hat r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\sin\theta\cos\phi\,\hat x + \sin\theta\sin\phi\,\hat y + \cos\theta\,\hat z\right]$$

$$= \frac{1}{r\sin\theta}\left[-\sin\theta\sin\phi\,\hat x + \sin\theta\cos\phi\,\hat y\right]$$

$$= \frac{1}{r}\left[-\sin\phi\,\hat x + \cos\phi\,\hat y\right]$$

$$= \frac{1}{r}\,\hat\phi$$

$$\nabla_{\hat r}\,\hat\theta = \frac{\partial}{\partial r}\left[\cos\theta\cos\phi\,\hat x + \cos\theta\sin\phi\,\hat y - \sin\theta\,\hat z\right]$$

$$= 0$$

$$\nabla_{\hat\theta}\,\hat\theta = \frac{1}{r}\frac{\partial}{\partial\theta}\left[\cos\theta\cos\phi\,\hat x + \cos\theta\sin\phi\,\hat y - \sin\theta\,\hat z\right]$$

$$= \frac{1}{r}\left[-\sin\theta\cos\phi\,\hat x - \sin\theta\sin\phi\,\hat y - \cos\theta\,\hat z\right]$$

$$= -\frac{1}{r}\,\hat r$$

$$\nabla_{\hat\phi}\,\hat\theta = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\cos\theta\cos\phi\,\hat x + \cos\theta\sin\phi\,\hat y - \sin\theta\,\hat z\right]$$

$$= \frac{1}{r\sin\theta}\left[-\cos\theta\sin\phi\,\hat x + \cos\theta\cos\phi\,\hat y\right]$$

$$= \frac{\cot\theta}{r}\,\hat\phi$$

$$\nabla_{\hat r}\,\hat\phi = \frac{\partial}{\partial r}\left[-\sin\phi\,\hat x + \cos\phi\,\hat y\right]$$

$$= 0$$

$$\nabla_{\hat\theta}\hat\phi = \frac{1}{r}\frac{\partial}{\partial\theta}\left[-\sin\phi\,\hat x + \cos\phi\,\hat y\right]$$

$$= 0$$

$$\nabla_{\hat\phi}\hat\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[-\sin\phi\,\hat x + \cos\phi\,\hat y\right]$$

$$= \frac{1}{r\sin\theta}\left[-\cos\phi\,\hat x - \sin\phi\,\hat y\right]$$

$$= \frac{1}{r\sin\theta}\Big[-\cos\phi\left(\cos\phi\sin\theta\,\hat r + \cos\phi\cos\theta\,\hat\theta \cancel{- \sin\phi\,\hat\phi}\right)$$

$$- \sin\phi\left(\sin\phi\sin\theta\,\hat r + \sin\phi\cos\theta\,\hat\theta \cancel{+ \cos\phi\,\hat\phi}\right)\Big]$$

$$= \frac{1}{r\sin\theta}\left[-\sin\theta\,\hat r - \cos\theta\,\hat\theta\right]$$

$$= -\frac{1}{r}\left[\hat r + \cot\theta\,\hat\theta\right]$$

$$= -\frac{1}{r}\hat r - \frac{1}{r}\cot\theta\,\hat\theta$$

Directional derivatives of $\hat{e}_i$ in cylindrical coords: A.6

$$x = \rho\cos\phi\,,\quad y = \rho\sin\phi\,,\quad z = z$$

$$\hat{\rho} = \overrightarrow{\left(\frac{\partial}{\partial\rho}\right)} = \frac{\partial x}{\partial\rho}\hat{x} + \frac{\partial y}{\partial\rho}\hat{y} + \frac{\partial z}{\partial\rho}\hat{z}$$

$$= \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = \frac{1}{\rho}\overrightarrow{\left(\frac{\partial}{\partial\phi}\right)} = \frac{1}{\rho}\left[\frac{\partial x}{\partial\phi}\hat{x} + \frac{\partial y}{\partial\phi}\hat{y} + \frac{\partial z}{\partial\phi}\hat{z}\right]$$

$$= \frac{1}{\rho}\left[-\rho\sin\phi\,\hat{x} + \rho\cos\phi\,\hat{y}\right]$$

$$= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\hat{z} = \overrightarrow{\left(\frac{\partial}{\partial z}\right)} = \frac{\partial x}{\partial z}\hat{x} + \frac{\partial y}{\partial z}\hat{z} + \frac{\partial z}{\partial z}\hat{z}$$

$$= \hat{z}$$

Thus,

$$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$
$$\hat{z} = \hat{z}$$

Inverse relations:

$$\hat{x} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}$$
$$\hat{y} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}$$
$$\hat{z} = \hat{z}$$

Thus,

$$\nabla_{\hat{\rho}}\,\hat{\rho} = \frac{\partial}{\partial\rho}\left[\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right]$$

$$= \boxed{0}$$

$$\nabla_{\hat{\phi}}\,\hat{\rho} = \frac{1}{\rho}\frac{\partial}{\partial\phi}\left[\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right]$$

$$= \frac{1}{\rho}\left[-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}\right]$$

$$= \boxed{\frac{1}{\rho}\hat{\phi}}$$

$$\nabla_{\hat{z}}\,\hat{\rho} = \frac{\partial}{\partial z}\left[\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right]$$

$$= \boxed{0}$$

$$\nabla_{\hat{\rho}}\,\hat{\phi} = \frac{\partial}{\partial\rho}\left[-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}\right]$$

$$= \boxed{0}$$

$$\nabla_{\hat{\phi}}\,\hat{\phi} = \frac{1}{\rho}\frac{\partial}{\partial\phi}\left[-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}\right]$$

$$= \frac{1}{\rho}\left[-\cos\phi\,\hat{x} - \sin\phi\,\hat{y}\right]$$

$$= \boxed{-\frac{1}{\rho}\hat{\rho}}$$

$$\nabla_{\hat{z}}\,\hat{\phi} = \boxed{0}$$

$$\left.\begin{aligned} \nabla_{\hat{\rho}}\,\hat{z} &= \boxed{0} \\ \nabla_{\hat{\phi}}\,\hat{z} &= \boxed{0} \\ \nabla_{\hat{z}}\,\hat{z} &= \boxed{0} \end{aligned}\right\} \quad \text{since } \hat{z} \text{ is const}$$

A.7

$$ds^2 = \sum_{i,j} g_{ij}\, dx^i dx^j$$

In Cartesian coords $dV = dx\,dy\,dz = d^3x'$ $(x^{i'} = (x, y, z))$

$$g'_{ij} \to \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \sqrt{g'} = 1$$

Under a coord transformation $(x, y, z) \to x^i = (x^1, x^2, x^3)$

$$g_{ij} = \cancel{} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j} \delta_{i'j'}$$

$$\det g = \det\left(\frac{\partial x'}{\partial x} \frac{\partial x'}{\partial x} \delta\right)$$

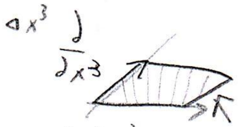
$$= \det\left(\frac{\partial x'}{\partial x}\right)^2 \cdot 1$$

so $\dfrac{\partial x'}{\partial x} = \sqrt{g}$

Thus, $dV = dx\,dy\,dz = \dfrac{\partial x'}{\partial x} dx^1 dx^2 dx^3$

$$= \sqrt{g}\, dx^1 dx^2 dx^3$$

Area element:



$$\hat{n} = \frac{\vec{\partial}_2 \times \vec{\partial}_3}{|\vec{\partial}_2 \times \vec{\partial}_3|} = \cancel{\frac{\vec{\partial}_2 \times \vec{\partial}_3}{\sqrt{g_{22}}\sqrt{g_{33}}}}$$

$$|\vec{\partial}_2| = \sqrt{\vec{\partial}_2 \cdot \vec{\partial}_2} = \sqrt{g_{22}}$$

$$|\vec{\partial}_3| = \sqrt{g_{33}}$$

$$da = \Delta x^2 \Delta x^3 |\vec{\partial}_2| |\vec{\partial}_3| \sin\theta_{23}$$

$$= \Delta x^2 \Delta x^3 \sqrt{g_{22}} \sqrt{g_{33}} \sqrt{1 - \cos^2\theta_{23}}$$

$$= \Delta x^2 \Delta x^3 \sqrt{g_{22}} \sqrt{g_{33}} \sqrt{1 - \left(\frac{\vec{\partial}_2 \cdot \vec{\partial}_3}{|\vec{\partial}_2||\vec{\partial}_3|}\right)^2}$$

$$d\,a = \Delta x^2 \Delta x^3 \sqrt{g_{22}} \sqrt{g_{33}} \sqrt{1 - \frac{g_{23}^2}{g_{22}\, g_{33}}}$$

$$= \Delta x^2 \Delta x^3 \sqrt{g_{22}\, g_{33} - g_{23}^2}$$

and similarly for $x^2 = \text{const}$, $x^3 = \text{const}$.

~~Another way of getting~~ dV :

$$\cancel{dV = \Delta x^1 \left(\frac{\vec{\partial}}{\partial x^1}\right) \cdot \hat{n}\, da}$$

$$\cancel{= \Delta x^1 \left(\vec{\partial}_1\right) \cdot \frac{\vec{\partial}_2 \times \vec{\partial}_3}{\sqrt{g_{22}}\sqrt{g_{33}}}\, \Delta x^2 \Delta x^3 \sqrt{g_{22}\, g_{33} - g_{23}^2}}$$

$$\cancel{= \Delta x^1 \Delta x^2 \Delta x^3 \sqrt{1 - \frac{g_{23}^2}{g_{22}\, g_{33}}}\; \vec{\partial}_1 \cdot \left(\vec{\partial}_2 \times \vec{\partial}_3\right)}$$

$$\cancel{= \Delta x^1 \Delta x^2 \Delta x^3 \sqrt{1 - \frac{g_{23}^2}{g_{22} g_{33}}}\; \left(\vec{\partial}_1\right)^i \epsilon_{ijk} \left(\vec{\partial}_2\right)^j \left(\vec{\partial}_3\right)^k}$$

Exercise A.8

Find the component of $\vec{F} = -\vec{\nabla} U$ for
$U(x,y,z) = \frac{1}{2} k (x^2+y^2) + mgz$
in spherical polar coordinates.

Transformation: $(x,y,z) \rightarrow (r,\theta,\phi)$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$U(r,\theta,\phi) \equiv U(x,y,z)\Big|_{x=r\sin\theta\cos\phi,\ \ldots}$$

$$= \frac{1}{2} k (r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi) + mgr\cos\theta$$

$$= \frac{1}{2} k r^2 \sin^2\theta + mgr\cos\theta$$

$$\vec{F} = -\vec{\nabla} U$$
$$= -\frac{\partial U}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\phi}$$

Now:
$$\frac{\partial U}{\partial r} = k r \sin^2\theta + mg\cos\theta$$
$$\frac{\partial U}{\partial \theta} = k r^2 \sin\theta\cos\theta - mgr\sin\theta$$
$$\frac{\partial U}{\partial \phi} = 0$$

so
$$F_r = -kr\sin^2\theta - mg\cos\theta$$
$$F_\theta = -kr\sin\theta\cos\theta + mg\sin\theta$$
$$F_\phi = 0$$

NOTE: These are component w.r.t. orthonormal basis $(\hat{r},\hat{\theta},\hat{\phi})$ not coordinate basis $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}$

In cylindrical coords

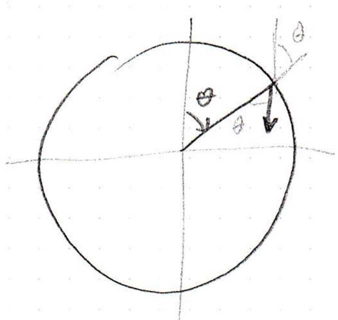
$(x, y, z) \rightarrow (\rho, \phi, z)$

$$x = \rho \cos\phi \qquad x^2 + y^2 = \rho^2$$
$$y = \rho \sin\phi$$
$$z = z$$

$$U(\rho, \phi, z) = \tfrac{1}{2} k \rho^2 + mgz$$

$$\vec{F} = -\vec{\nabla} U$$
$$= -\frac{\partial U}{\partial \rho}\hat{\rho} - \frac{1}{\rho}\frac{\partial U}{\partial \phi}\hat{\phi} - \frac{\partial U}{\partial z}\hat{z}$$
$$= -k\rho\,\hat{\rho} - mg\,\hat{z}$$

so
$$F_\rho = -k\rho$$
$$F_\phi = 0$$
$$F_z = -mg$$



Div, grad, curl, etc in sph. polar coords: A.9

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$$f = 1, \quad g = r, \quad h = r\sin\theta; \quad u = r, \quad v = \theta, \quad w = \phi$$

Thus

$$\vec{\nabla} U = \frac{1}{f}\frac{\partial U}{\partial u}\hat{u} + \frac{1}{g}\frac{\partial U}{\partial v}\hat{v} + \frac{1}{h}\frac{\partial U}{\partial w}\hat{w}$$

$$= \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\phi}$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{gh}\left[\frac{\partial}{\partial v}(A_w h) - \frac{\partial}{\partial w}(A_v g)\right]\hat{u}$$

$$+ \frac{1}{hf}\left[\frac{\partial}{\partial w}(A_u f) - \frac{\partial}{\partial u}(A_w h)\right]\hat{v}$$

$$+ \frac{1}{fg}\left[\frac{\partial}{\partial u}(A_v g) - \frac{\partial}{\partial v}(A_u f)\right]\hat{w}$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi r\sin\theta) - \frac{\partial}{\partial\phi}(A_\theta r)\right]\hat{r}$$

$$+ \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\phi}(A_r) - \frac{\partial}{\partial r}(A_\phi r\sin\theta)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(A_\theta r) - \frac{\partial}{\partial\theta}(A_r)\right]\hat{\phi}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(A_\phi r)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(A_\theta r) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(A_u gh) + \frac{\partial}{\partial v}(A_v hf) + \frac{\partial}{\partial w}(A_w fg)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(A_r r^2\sin\theta) + \frac{\partial}{\partial\theta}(A_\theta r\sin\theta) + \frac{\partial}{\partial\phi}(A_\phi r)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(A_r r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla^2 U = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial U}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{hf}{g}\frac{\partial U}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial U}{\partial w}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta\frac{1}{r}\frac{\partial U}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(r\frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}$$

Div, grad, curl, etc. in cylindrical coords: A.10

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

$$f = 1, \quad g = \rho, \quad h = 1 \quad ; \quad u = \rho, \quad v = \phi, \quad w = z$$

Thus,

$$\vec{\nabla} U = \frac{1}{f}\frac{\partial U}{\partial u}\hat{u} + \frac{1}{g}\frac{\partial U}{\partial v}\hat{v} + \frac{1}{h}\frac{\partial U}{\partial w}\hat{w}$$

$$= \frac{\partial U}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\hat{\phi} + \frac{\partial U}{\partial z}\hat{z}$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{gh}\left[\frac{\partial}{\partial v}(A_w h) - \frac{\partial}{\partial w}(A_v g)\right]\hat{u}$$

$$+ \frac{1}{hf}\left[\frac{\partial}{\partial w}(A_u f) - \frac{\partial}{\partial u}(A_w h)\right]\hat{v}$$

$$+ \frac{1}{fg}\left[\frac{\partial}{\partial u}(A_v g) - \frac{\partial}{\partial v}(A_u f)\right]\hat{w}$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \phi}(A_z) - \frac{\partial}{\partial z}(A_\phi \rho)\right]\hat{\rho}$$

$$+ \left[\frac{\partial}{\partial z}(A_\rho) - \frac{\partial}{\partial \rho}(A_z)\right]\hat{\phi}$$

$$+ \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(A_\phi \rho) - \frac{\partial}{\partial \phi}(A_\rho)\right]\hat{z}$$

$$= \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\rho}$$

$$+ \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\hat{\phi}$$

$$+ \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(A_\phi \rho) - \frac{\partial A_\rho}{\partial \phi}\right]\hat{z}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(A_u g h) + \frac{\partial}{\partial v}(A_v h f) + \frac{\partial}{\partial w}(A_w f g)\right]$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(A_\rho \rho) + \frac{\partial}{\partial \phi}(A_\phi) + \frac{\partial}{\partial z}(A_z \rho)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(A_\rho \rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \rho} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 U = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial U}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{hf}{g}\frac{\partial U}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial U}{\partial w}\right)\right]$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial U}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial U}{\partial z}\right)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

Div, grad, curl, and shear of 2-d vector fields: A.11

$$\vec{A} = x\hat{x} + y\hat{y}$$
$$\vec{B} = -y\hat{x} + x\hat{y}$$
$$\vec{C} = y\hat{x} + xy\hat{y}$$

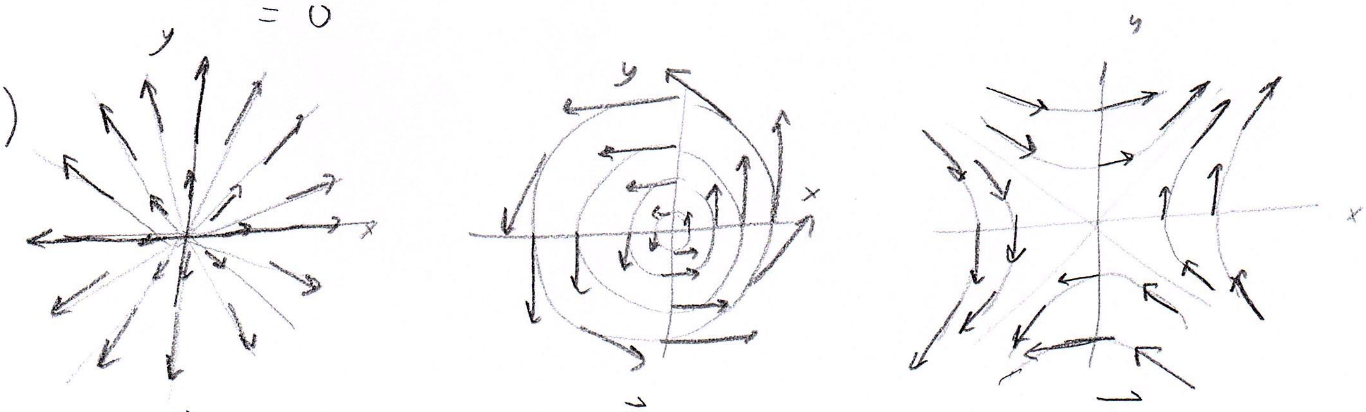
(a)
$$\vec{\nabla}\times\vec{A} = \hat{x}(\partial_y \cancelto{0}{A_z} - \cancelto{0}{\partial_z} A_y) + \hat{y}(\cancelto{0}{\partial_z} A_x - \partial_x \cancelto{0}{A_z}) + \hat{z}(\partial_x A_y - \partial_y A_x)$$
$$= \hat{z}(\partial_x y - \partial_y x)$$
$$= 0$$

$$\vec{\nabla}\cdot\vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \cancelto{0}{\frac{\partial B_z}{\partial z}}$$
$$= \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)$$
$$= 0$$

$$\vec{\nabla}\times\vec{C} = \hat{x}(\partial_y \cancelto{0}{C_z} - \cancelto{0}{\partial_z} C_y) + \hat{y}(\cancelto{0}{\partial_z} C_x - \partial_x \cancelto{0}{C_z}) + \hat{z}(\partial_x C_y - \partial_y C_x)$$
$$= \hat{z}(\partial_x x - \partial_y y)$$
$$= \hat{z}(1-1)$$
$$= 0$$

$$\vec{\nabla}\cdot\vec{C} = \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \cancelto{0}{\frac{\partial C_z}{\partial z}}$$
$$= \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y}$$
$$= 0$$

(b)



$\vec{A}$ $\vec{B}$ $\vec{C}$

(c) $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

$$\hat{\rho} = \left(\frac{\partial \vec{}}{\partial \rho}\right) = \left(\frac{\partial x}{\partial \rho}\right)\hat{x} + \left(\frac{\partial y}{\partial \rho}\right)\hat{y} + \cancel{\left(\frac{\partial z}{\partial \rho}\right)}^{0}\hat{z}$$

$$= \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$= \frac{x}{\rho}\hat{x} + \frac{y}{\rho}\hat{y}$$

Thus, $\rho\hat{\rho} = x\hat{x} + y\hat{y} = \vec{A}$

$$\hat{\phi} = \frac{1}{\rho}\left(\frac{\partial \vec{}}{\partial \phi}\right) = \frac{1}{\rho}\left[\left(\frac{\partial x}{\partial \phi}\right)\hat{x} + \left(\frac{\partial y}{\partial \phi}\right)\hat{y} + \cancel{\left(\frac{\partial z}{\partial \phi}\right)}^{0}\hat{z}\right]$$

$$= \frac{1}{\rho}\left[-\rho\sin\phi\,\hat{x} + \rho\cos\phi\,\hat{y}\right]$$

$$= \frac{1}{\rho}\left[-y\hat{x} + x\hat{y}\right]$$

Thus, $\rho\hat{\phi} = -y\hat{x} + x\hat{y} = \vec{B}$

(d) Define $U = x^2 - y^2$, $V = 2xy$

$$\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \cancel{\frac{\partial V}{\partial z}}^{0}\hat{z}$$

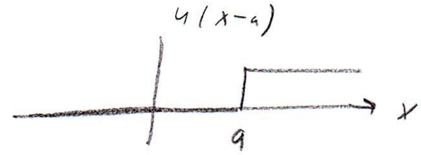
$$= 2y\hat{x} + 2x\hat{y}$$

$$= 2\left[y\hat{x} + x\hat{y}\right]$$

Thus, $\frac{1}{2}\vec{\nabla} V = y\hat{x} + x\hat{y} = \vec{C}$

Properties of 1-d Dirac delta functions

(i) $\delta(x-a) = \frac{d}{dx}[u(x-a)]$



Proof:

$$\int_{-\infty}^{\infty} dx\, f(x)\, \frac{d}{dx}[u(x-a)]$$

$$= \int_{-\infty}^{\infty} dx \left\{ \frac{d}{dx}[f(x)\, u(x-a)] - f'(x)\, u(x-a) \right\}$$

$$= \underbrace{f(x)\, u(x-a)\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} dx\, f'(x)\, u(x-a)$$

$$= -\int_{a}^{\infty} dx\, f'(x)$$

$$= -f(x)\Big|_{a}^{\infty}$$

$$= -\underbrace{f(\infty)}_{0} + f(a)$$

$$= f(a)$$

(ii) $\delta'(-x) = -\delta'(x)$

Proof:

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta'(-x) = \int_{\infty}^{-\infty} d(-u)\, f(-u)\, \delta'(u) \qquad [u = -x]$$

$$= \int_{-\infty}^{\infty} du\, f(-u)\, \delta'(u) = \int_{-\infty}^{\infty} du \left\{ \frac{d}{du}[f(-u)\, \delta(u)] - \frac{d}{du}(f(-u))\, \delta(u) \right\}$$

$$= \underbrace{f(-u)\, \delta(u)\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} du\, f'(-u)(-1)\, \delta(u) = \int_{-\infty}^{\infty} du\, f'(u)\, \delta(u) = \boxed{f'(0)}$$

$$\int_{-\infty}^{\infty} dx\, f(x)[-\delta'(x)] = -\int_{-\infty}^{\infty} dx \left\{ \frac{d}{dx}[f(x)\, \delta(x)] - f'(x)\, \delta(x) \right\}$$

$$= \underbrace{f(x)\, \delta(x)\Big|_{-\infty}^{\infty}}_{0} + \int_{-\infty}^{\infty} dx\, f'(x)\, \delta(x)$$

$$= \boxed{f'(0)}$$

(iii) $\delta(-x) = \delta(x)$

Proof: $$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(-x) = \int_{\infty}^{-\infty} d(-u)\, f(-u)\, \delta(u)$$

$$= \int_{-\infty}^{\infty} du\, f(-u)\, \delta(u) = f(-0) = \boxed{f(0)}$$

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(x) = \boxed{f(0)}$$

(iv) $\delta(ax) = \frac{1}{|a|}\delta(x)$

Proof: $$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(ax) = \int_{-a\infty}^{a\infty} \frac{du}{a}\, f\left(\frac{u}{a}\right) \delta(u)$$

Let: $u = ax$
$du = a\,dx$

$$= \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} du\, f\left(\frac{u}{a}\right)\delta(u) = \frac{1}{a} f(0), \text{ if } a > 0 \\ \frac{1}{a}\int_{\infty}^{-\infty} du\, f\left(\frac{u}{a}\right)\delta(u) = -\frac{1}{a}\int_{-\infty}^{\infty} du\, f\left(\frac{u}{a}\right)\delta(u) \end{cases}$$

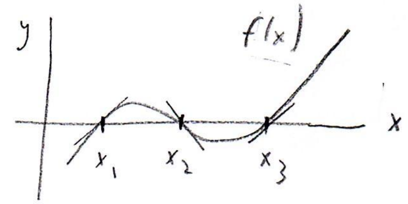
$$= -\frac{1}{a} f(0) \quad \text{if } a < 0$$

Thus, $$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(ax) = \frac{1}{|a|} f(0) = \int_{-\infty}^{\infty} dx\, f(x) \frac{1}{|a|}\delta(x)$$

so $$\boxed{\delta(ax) = \frac{1}{|a|}\delta(x)}$$

(v) $\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}$ where $f(x_i) = 0$, $f'(x_i) \neq 0$

$$\int_{-\infty}^{\infty} dx\, g(x)\, \delta[f(x)] = \sum_{i=1}^{n} \int_{x_i - \epsilon}^{x_i + \epsilon} dx\, g(x)\, \delta[f(x)]$$



$f'(x_1) > 0$, $f'(x_2) < 0$, $f'(x_3) > 0$

Around each zero x_i, let $y = f(x)$.
Can invert around x_i, provided $f'(x_i) \neq 0$
Then $x = f^{-1}(y)$ allows change of variables from x to y.

$$\int_{x_i-\epsilon}^{x_i+\epsilon} dx\, g(x)\, \delta[f(x)] = \int_{f(x_i-\epsilon)}^{f(x_i+\epsilon)} \frac{dy}{f'(x_i)}\, g(f^{-1}(y))\, \delta(y)$$

Now:

If $f'(x_i) > 0$

$$RHS = \int_{-\delta_1}^{+\delta_2} \frac{dy}{f'(x_i)}\, g(f^{-1}(y))\, \delta(y)$$

$$= \frac{1}{f'(x_i)}\, g(f^{-1}(0))$$

$$= \frac{1}{f'(x_i)}\, g(x_i)$$

If $f'(x_i) < 0$

$$RHS = \int_{+\delta_1}^{-\delta_2} \frac{dy}{f'(x_i)}\, g(f^{-1}(y))\, \delta(y)$$

$$= -\int_{-\delta_2}^{\delta_1} \frac{dy}{f'(x_i)}\, g(f^{-1}(y))\, \delta(y)$$

$$= -\frac{1}{f'(x_i)}\, g(f^{-1}(0))$$

$$= \frac{-1}{f'(x_i)}\, g(x_i)$$

For both cases:

$$RHS = \frac{1}{|f'(x_i)|}\, g(x_i)$$

Thus,

$$\int_{-\infty}^{\infty} dx\, g(x)\, \delta[f(x)] = \sum_{i=1}^{n} \frac{1}{|f'(x_i)|}\, g(x_i)$$

which is the same as doing

$$\int_{-\infty}^{\infty} dx\, g(x) \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|f'(x_i)|},$$

so

$$\boxed{\delta[f(x)] = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|f'(x_i)|}}$$

Curl and divergence of $r^n \hat{r}$: (A.13)

$$\vec{\nabla} \times (r^n \hat{r}) = 0 \qquad \forall n$$

$$\vec{\nabla} \cdot (r^n \hat{r}) = (n+2)\, r^{n-1} \qquad \text{for } n \neq -2$$

Use expression for curl and divergence in sph. polar coordinates noting that the vector field $\vec{A} \equiv r^n \hat{r}$ has only an r component.

$$\vec{\nabla} \times (r^n \hat{r}) = \frac{1}{r} \frac{1}{\sin\theta} \left(\cancel{\frac{\partial r^n}{\partial \phi}} \right) \hat{\theta} - \frac{1}{r} \left(\cancel{\frac{\partial r^n}{\partial \theta}} \right) \hat{\phi}$$

$$0 \qquad\qquad 0$$

$$= 0 \qquad \forall n.$$

$$\vec{\nabla} \cdot (r^n \hat{r}) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r^n)$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} (r^{n+2})$$

$$= \frac{1}{r^2} (n+2)\, r^{n+1}$$

$$= (n+2)\, r^{n-1} \qquad \text{for } n \neq -2$$

For $n = -2$, $\vec{\nabla} \cdot (r^n \hat{r}) = \frac{1}{r^2} \frac{\partial}{\partial r} (\overbrace{r^{n+2}}^{=1 \text{ for } n=-2})$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} (1)$$

$$= 0$$