Chapter 1
Elementary Newtonian Mechanics

Much of classical mechanics was developed to provide powerful mathematical tools for obtaining the equations of motion for systems of objects subject to external and internal forces. These include *Newton's laws*, the *principle of virtual work*, and *Hamilton's principle*, which we shall discuss, in turn, in the first three chapters. These tools let us choose coordinates that are most suitable for the solution of a given problem; they also allow us to describe motion when observed from non-inertial reference frames, such as the rotating surface of the Earth. A deeper study of these mathematical tools and how they respond to different transformations of the system (e.g., translations or rotations of the coordinates) leads to a better understanding of the nature of Newtonian mechanics, and points the way to the modern physics of quantum mechanics and special relativity.

For the greater part of this book, we will concentrate on Newton's formulation of mechanics, in which the universe exists in a flat, three-dimensional space described by Euclidean geometry. Changes in this Newtonian universe are measured using a standard clock that ticks at a uniform rate over all space. Adapting Newtonian mechanics to the non-Euclidean geometry of special relativity will be discussed at the end of the text in Chap. 11.

In this chapter, we review some of the basic methods familiar from introductory physics for obtaining and solving the equations of motion for *single particles* and then *systems of particles*, with and without constraints on their motion.

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1.1 Newton's Laws of Motion

From introductory physics, we are familiar with Newton's laws of motion. The first law describes the motion of an object with respect to an **inertial reference frame**:

> **Newton's 1st law:** Unless acted on by an outside force the natural motion of an object is constant velocity.

One way of thinking about this law is that it provides us with a procedure for determining if we are using an inertial reference frame. That is, if we can find a way to turn off (or shield) all external and internal forces from a system, and we find that all particles in the system are moving with constant velocities, then we will know that we are describing the system in an inertial frame of reference.

Once we have determined that we are in an inertial reference frame, Newton's 2nd law tells us how an applied force will alter this natural motion:

> **Newton's 2nd law:** The effect of an applied force $\mathbf{F}$ upon an object of mass m is to induce an acceleration $\mathbf{a}$ such that
>
> $$\mathbf{F} = m\mathbf{a}\,. \tag{1.1}$$

This simple form of Newton's 2nd law assumes that the mass is constant, but we can include the effect of a varying mass by writing Newton's 2nd law in terms of **momentum** $\mathbf{p} \equiv m\mathbf{v}$, so that

$$\mathbf{F} = \dot{\mathbf{p}} \equiv \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\,. \tag{1.2}$$

Note that unless specifically stated otherwise, we will assume throughout this text that the mass of an object is constant, for which $\mathbf{F} = m\mathbf{a}$ and $\mathbf{F} = \dot{\mathbf{p}}$ are equivalent statements of Newton's 2nd law.

When there are multiple objects exchanging forces between themselves within a system, Newton's 3rd law describes how the forces of interaction behave:

> **Newton's 3rd law:** If an object applies a force $\mathbf{F}$ on a second object, then the second object applies an equal and opposite force $-\mathbf{F}$ on the first object.

In its simplest form, the 3rd law insures that the internal forces between particles in a system do not provide an unbalanced force on the system as a whole, which would allow the system to spontaneously accelerate away in the absence of external forces. Note that not all forces obey Newton's 3rd law, but these involve a field which can carry away momentum.[1] There is also a **strong form** of Newton's 3rd

[1] A simple example of such a force is the *electromagnetic* force between two moving point charges; see, e.g., Sect. 8.2.1 of Griffiths (1999).

law, which requires that $\mathbf{F}_{IJ}$, the interparticle forces between particles I and J, not only satisfy $\mathbf{F}_{JI} = -\mathbf{F}_{IJ}$, but also point in the direction of the lines connecting pairs of particles—i.e., $\mathbf{F}_{JI} \propto \mathbf{r}_{IJ}$, where $\mathbf{r}_{IJ} \equiv \mathbf{r}_I - \mathbf{r}_J$ is the displacement vector joining particles I and J. Such forces are called **central** (or **radial**) forces. The strong form of Newton's 3rd law is needed for conservation of angular momentum, as we will explore in more detail in Sect. 1.4.

Example 1.1 Consider a rocket moving in interstellar space, free of all external forces, as shown in Fig. 1.1. We want to determine the velocity v of the rocket as a function of time, assuming that its mass decreases at a constant rate, $\mathrm{d}m/\mathrm{d}t \equiv -\alpha$ (where α is positive so $\mathrm{d}m/\mathrm{d}t$ is explicitly negative), as it expels exhaust gases through the nozzle of the rocket engine.

To do this calculation, we need to use Newton's 2nd law in the form $F = \mathrm{d}p/\mathrm{d}t$, since the mass of the rocket is not constant. (We have dropped the vector symbols in this equation since this is a 1-dimensional problem.) Let's assume that at time t the rocket has mass m, and that it is moving vertically upward with velocity v. At time $t + \mathrm{d}t$, the rocket will have lost mass $\mathrm{d}m' \equiv -\mathrm{d}m > 0$ (the exhaust gases), and will have changed its velocity to $v + \mathrm{d}v$. We will assume that the exhaust gases $\mathrm{d}m'$ exit the rocket with *constant* velocity $-u$ with respect to the rocket, so that with respect to the fixed inertial frame, the exhaust gases are moving with velocity $v - u$. The change in the total momentum of the system over the time interval t to $t + \mathrm{d}t$ is then

$$
\begin{aligned}
\mathrm{d}p &= p(t + \mathrm{d}t) - p(t) \\
&= \left[(m - \mathrm{d}m')(v + \mathrm{d}v) + \mathrm{d}m'(v - u)\right] - mv \\
&= m\,\mathrm{d}v - u\,\mathrm{d}m' \\
&= m\,\mathrm{d}v + u\,\mathrm{d}m\,,
\end{aligned}
\tag{1.3}
$$

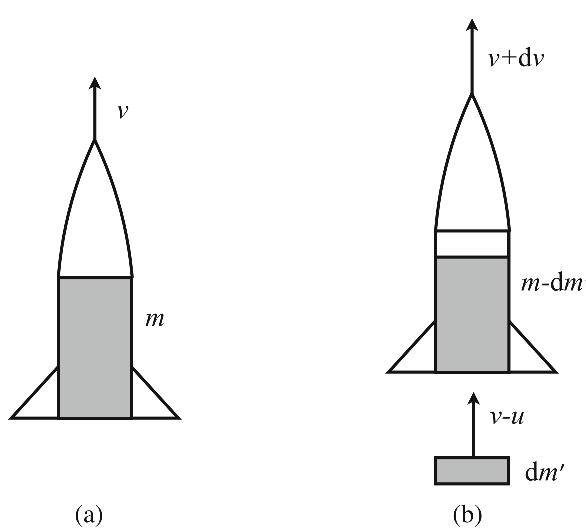


Fig. 1.1 A rocket moving in interstellar space, free of all external forces. Panel (a): Rocket at time t (mass m, velocity v). Panel (b): Rocket and exhaust at time $t + \mathrm{d}t$ (mass $m - \mathrm{d}m'$, velocity $v + \mathrm{d}v$; mass $\mathrm{d}m'$, velocity $v - u$)

where we ignored the $-\mathrm{d}m'\,\mathrm{d}v$ term (since it is 2nd-order small) to get the third line, and switched back to $\mathrm{d}m$ to get the last line. But note, however, that there are no external forces acting on the system, so $\mathrm{d}p/\mathrm{d}t = F = 0$, which implies

$$0 = m\,\mathrm{d}v + u\,\mathrm{d}m\,, \tag{1.4}$$

or, equivalently,

$$\mathrm{d}v = -u\,\frac{\mathrm{d}m}{m}\,. \tag{1.5}$$

This is a separable differential equation, which can be immediately integrated, subject to the initial condition that $v = v_0$ when $m = m_0$:

$$v - v_0 = -u\ln(m/m_0)\,. \tag{1.6}$$

To get the time dependence of v, we make use of the assumption that the mass-loss rate is constant,

$$\frac{\mathrm{d}m}{\mathrm{d}t} \equiv -\alpha = \text{const}\,, \tag{1.7}$$

which implies

$$m(t) = m_0 - \alpha t\,. \tag{1.8}$$

Making this substitution into (1.6), we have

$$v(t) = v_0 - u\ln\left(1 - \frac{\alpha t}{m_0}\right)\,. \tag{1.9}$$

Note that this equation is valid only up to time t_f, when all of the fuel has been exhausted, and the mass of the rocket is m_f (> 0). After that time, the rocket moves with constant velocity $v_\mathrm{f} = v_0 - u\ln(m_\mathrm{f}/m_0)$. □

Exercise 1.1 What fraction of the total initial mass m_0 of a rocket must be exhausted as fuel in order for a payload of mass m_f to be accelerated through a change in velocity Δv?

Exercise 1.2 Repeat the analysis of Example 1.1 for a rocket moving in a uniform gravitational field $\mathbf{g}$ pointing opposite to $\mathbf{v}$. You should find

$$v(t) = v_0 - gt - u\ln\left(1 - \frac{\alpha t}{m_0}\right)\,. \tag{1.10}$$

1.2 Single-Particle Mechanics

In this section, we will discuss the motion of a single object (a *particle*) that is subject to external forces. Our use of the term "particle" implies that the object has no internal structure and no physical extent (i.e., it is effectively a zero-dimensional point). This will allow us to focus simply on its motion without having to consider the influence that the external forces may have on the internal structure or orientation of the object. (We will treat real three-dimensional objects later in Chaps. 6 and 7, in the context of rotational motion.) Note that we can use the particle approximation even for extended objects provided the changes in internal energy or rotational state of the object are negligible. In these cases, we simply use a point within the object as a stand-in for the particle's position.

Let's first consider a particle viewed in an inertial frame of reference. Within this frame, the position of the particle is defined by a time-dependent vector $\mathbf{r}(t)$, and its linear momentum is $\mathbf{p} = m\dot{\mathbf{r}} = m\mathbf{v}$. Since we are in an inertial frame, any variation in $\mathbf{p}$ will be due to an impressed external force, so $\mathbf{F} = \dot{\mathbf{p}}$.

Exercise 1.3 Let a particle's position be given by $\mathbf{r}(t)$ in an inertial frame O, and let the mass be constant, so that $\mathbf{F} = \dot{\mathbf{p}} = m\mathbf{a} = m\ddot{\mathbf{r}}$. Transform to a new reference frame O' that is moving at constant velocity $\mathbf{u}$ with respect to the original one, so $\mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{u}\,t$. Show that $m\ddot{\mathbf{r}}' = m\ddot{\mathbf{r}} = \mathbf{F}$, so that Newton's 2nd law has the same form in this new reference frame. Thus, the new reference frame is also inertial.

In a single particle universe, if mass is conserved, then the mass of the particle must be constant. The impressed force $\mathbf{F} \equiv \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$ then governs the acceleration of the particle, and we obtain a 2nd-order differential equation, which must be solved in order to determine the motion $\mathbf{r}(t)$. In the remainder of this section, we will review the fundamentals of single-particle mechanics and recover some of the familiar conservation laws.

Example 1.2 Air resistance can be modeled as a velocity-dependent force with $\mathbf{F} = -b\mathbf{v}$, where b is a real, positive proportionality constant. If a particle starts with an initial velocity $\mathbf{v}_0$, how far does it go before coming to rest under the influence of air resistance alone?

We can obtain the equation of motion from $\mathbf{F} = m\mathbf{a}$ and solve for $\mathbf{r}(t)$, but we are more interested in $\mathbf{v}$ as a function of $\mathbf{r}$. Note that the problem is essentially one-dimensional, so let's choose a coordinate system with an x-axis that lies along the initial velocity, so we can dispense with the boldface vector notation. Then Newton's 2nd law reads:

$$F = -bv = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}x} = mv\frac{\mathrm{d}v}{\mathrm{d}x}\,. \tag{1.11}$$

This leaves us with the simple differential equation

$$-\frac{b}{m}=\frac{\mathrm{d}v}{\mathrm{d}x}\,, \tag{1.12}$$

which is solved by $v = v_0 - bx/m$. Consequently the distance traveled by the particle is the value of x for which $v = 0$. This is $x = mv_0/b$. □

1.2.1 Work

When a particle is subject to an external force, the force does **work** on the particle as it moves along a path $\mathbf{s}(t)$ according to the line integral

$$W_{12} \equiv \int_{\wp_1}^{\wp_2} \mathbf{F}\cdot \mathrm{d}\mathbf{s}\,, \tag{1.13}$$

where $\wp_1$ and $\wp_2$ are the endpoints of the path, corresponding to the times $t = t_1$ and $t = t_2$. (See Fig. 1.2.) The work can be thought of as the amount of energy deposited into the particle by the agent producing the force. Note that, in general, the work done by a force will be dependent upon the path taken by the particle.

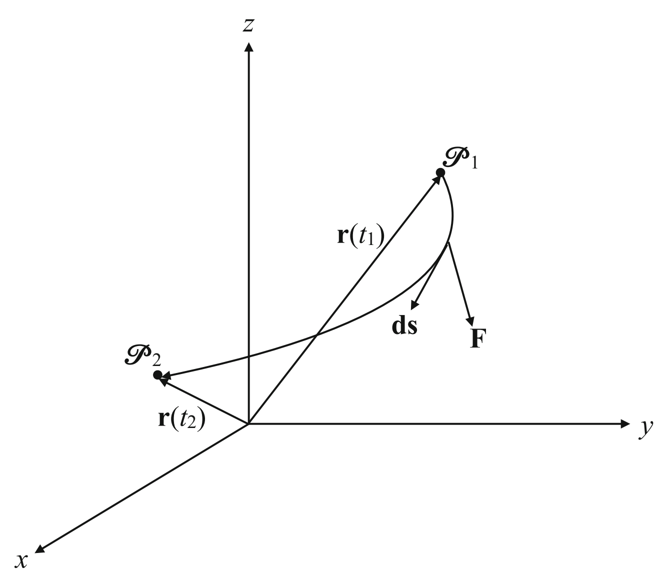


Fig. 1.2 The work W_{12} is calculated for the particular path that a particle takes in moving from point $\wp_1$ to point $\wp_2$

Exercise 1.4 A particle of mass m is subject to a force that is dependent upon its velocity, $\mathbf{F} = -b\mathbf{v}$, where b is a real, positive proportionality constant. (a) Calculate the work done by the force as the particle moves with constant velocity along the x-axis from $x = -a$ to $x = +a$. (b) Calculate the work done by the force if the particle moves with constant speed along a semicircle of radius a from $x = -a$ to $x = +a$. (c) Along which path does the force do the most work?

Exercise 1.5 A particle of mass m is subject to a force that is dependent upon its velocity, $\mathbf{F} = -bv^2\hat{\mathbf{v}}$, where b is a real, positive proportionality constant and $\hat{\mathbf{v}}$ is a unit vector in the direction of $\mathbf{v}$. Assuming that this is the *only* force acting on the particle, show that the work done by this force as the particle moves a distance a along a straight line is

$$W = \frac{1}{2}mv_0^2\left(e^{-2ba/m} - 1\right), \tag{1.14}$$

where v_0 is the initial velocity. (*Hint*: Treat this as a 1-dimensional problem and use $\mathbf{F} = m\mathbf{a}$ to solve for v as a function of x.)

1.2.2 Work-Energy Theorem

The expression for the **kinetic energy** of a particle,

$$T \equiv \frac{1}{2}mv^2, \tag{1.15}$$

arises naturally if one calculates the work done on the particle by the net force in moving it from one location to another. To see this, assume that the mass m of the particle is constant, so that the net force is given by $\mathbf{F} = m\mathbf{a} = m\mathrm{d}\mathbf{v}/\mathrm{d}t$. Then

$$\int_{\wp_1}^{\wp_2} \mathbf{F}\cdot \mathrm{d}\mathbf{s} = \int_{\wp_1}^{\wp_2} m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot \mathrm{d}\mathbf{s} = \int_{\wp_1}^{\wp_2} m\mathrm{d}\mathbf{v}\cdot\mathbf{v} = \int_{\wp_1}^{\wp_2} \mathrm{d}\left(\frac{1}{2}mv^2\right). \tag{1.16}$$

But this last integral is trivial to evaluate, so

$$W_{12} \equiv \int_{\wp_1}^{\wp_2} \mathbf{F}\cdot \mathrm{d}\mathbf{s} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \equiv T_2 - T_1, \tag{1.17}$$

where v_i is the velocity of the particle at point $\wp_i$. This is the **work-energy theorem** for a single particle, which relates the work done on a particle by the net force to its change in kinetic energy.

1.2.3 Conservative Forces

There is a certain class of forces for which the work done is *independent* of the path and depends only upon the endpoints. These forces are called **conservative forces**. In order for a line integral to be independent of the path, the integrand must be expressible as the gradient of a scalar function. Specifically, if $\mathbf{F} = -\nabla U(\mathbf{r})$ for some function U, then $\mathbf{F}$ is conservative and $U(\mathbf{r})$ is the **potential energy** for the force $\mathbf{F}$. For a conservative force, the work done is the difference between the values of the potential energy at the endpoints:

$$W_{12} = \int_{\wp_1}^{\wp_2} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = -\int_{\wp_1}^{\wp_2} \nabla U \cdot \mathrm{d}\mathbf{s} = -\left(U_2 - U_1\right) . \tag{1.18}$$

If we combine the above result for a conservative force with (1.17), which holds in general, we see that $U_1 - U_2 = T_2 - T_1$ or, equivalently, $T_1 + U_1 = T_2 + U_2$, so the quantity

$$E \equiv T + U, \tag{1.19}$$

called the **mechanical energy** of the particle, is constant throughout the motion. *Thus, the mechanical energy of a particle is conserved if the external forces are conservative*.

1.2.4 Angular Momentum

We can also define angular momentum about a preferred point, even in a single particle universe. If we place the origin of our coordinate system at this preferred point, then the **angular momentum** is defined as

$$\boldsymbol{\ell} \equiv \mathbf{r} \times \mathbf{p} . \tag{1.20}$$

The time derivative of $\boldsymbol{\ell}$ is

$$\dot{\boldsymbol{\ell}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{r} \times \mathbf{F}\,, \tag{1.21}$$

where we have used Newton's 2nd law and the fact that $\dot{\mathbf{r}} \times \mathbf{p} = \dot{\mathbf{r}} \times (m\dot{\mathbf{r}}) = 0$. *Thus, the angular momentum is conserved if the* **torque** $\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}$ *is zero.*

Exercise 1.6 In an inertial frame with a Cartesian coordinate system, a particle of mass m starts at rest with an initial position of $\mathbf{r}_0 = x_0\hat{\mathbf{x}} + y_0\hat{\mathbf{y}}$. At $t = 0$ the particle experiences a force $\mathbf{F} = F\hat{\mathbf{x}}$. (a) Using Newton's 2nd law $\mathbf{F} = \dot{\mathbf{p}}$, solve the equation of motion to obtain $\mathbf{r}(t)$ and $\mathbf{p}(t)$. (b) Determine the angular momentum about the origin and show that it satisfies $\boldsymbol{\tau} = \dot{\boldsymbol{\ell}}$. (c) Now choose a new coordinate system that is translated in the y direction by y_0, so that $\mathbf{r}'_0 = x_0\hat{\mathbf{x}}$. Repeat part (a) and calculate the new torque $\boldsymbol{\tau}$. Is angular momentum conserved in this coordinate system?

1.3 Systems of Particles

When we expand our scope to include systems with multiple particles, we must take into account **interparticle forces** and the apparent bulk motion of the entire system. For a system of N particles, the momentum $\mathbf{p}_I$ of the Ith particle can change due to interactions with other particles as well as to impressed external forces. Thus, Newton's 2nd law reads

$$\frac{\mathrm{d}\mathbf{p}_I}{\mathrm{d}t} = \mathbf{F}_I^{(\mathrm{e})} + \sum_{J \neq I} \mathbf{F}_{JI}\,, \qquad I = 1, 2, \cdots, N\,, \tag{1.22}$$

where the sum is over all other particles in the system (J runs from 1 to N, excluding I), and $\mathbf{F}_{JI}$ is the force that particle J exerts on particle I. (To simplify the notation in what follows, we will define $\mathbf{F}_{II} = 0$ so that such sums can run over *all* indices, including $J = I$.) The **total linear momentum** of the system is then the sum of the individual particle momenta,

$$\mathbf{P} \equiv \sum_I \mathbf{p}_I\,. \tag{1.23}$$

The change in the total linear momentum is then

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_I \mathbf{p}_I = \sum_I \frac{\mathrm{d}\mathbf{p}_I}{\mathrm{d}t} = \sum_I \mathbf{F}_I^{(\mathrm{e})} + \sum_{I,J} \mathbf{F}_{JI}\,, \tag{1.24}$$

where the double summation $\sum_{I,J} \equiv \sum_I \sum_J$ counts each particle twice (once as I and once as J). But from Newton's 3rd law, $\mathbf{F}_{JI} = -\mathbf{F}_{IJ}$, so the interparticle forces sum to zero. Defining the net external force to be $\mathbf{F}^{(e)} \equiv \sum_I \mathbf{F}_I^{(e)}$, we have

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}^{(e)}\,, \tag{1.25}$$

which shows that *the total linear momentum of a system is conserved if the net external force on the system is zero.*

1.3.1 Center of Mass

The total momentum of a system of particles acts as if the system were a single particle under the influence of the net external applied force. Thus, it is possible to define a single position for the system. This position is known as the **center of mass**, which is defined by

$$\mathbf{R} \equiv \frac{1}{M} \sum_I m_I \mathbf{r}_I\,, \tag{1.26}$$

where $M \equiv \sum m_I$ is the total mass of the system.

Exercise 1.7 Show that the total momentum can be expressed as $\mathbf{P} = M\dot{\mathbf{R}}$. (Note that we assume that the masses of the individual particles are constant.)

1.3.2 Angular Momentum

In a similar fashion to the definition of the total (linear) momentum, we can define the **total angular momentum** of a system of particles to be the sum of the individual angular momenta,

$$\mathbf{L} \equiv \sum_I \boldsymbol{\ell}_I\,. \tag{1.27}$$

If the interparticle forces are *central* (i.e., they are all directed along the line segments joining pairs of particles), then the total angular momentum responds to the action of the net external torque in the same way that a single particle does, i.e.,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\tau}^{(\mathrm{e})}\,, \tag{1.28}$$

where $\boldsymbol{\tau}^{(\mathrm{e})} \equiv \sum_I \boldsymbol{\tau}_I^{(\mathrm{e})}$ is the sum of the external torques on the individual particles. *Thus, we see that the total angular momentum of a system is conserved if the interparticle forces are central and the net external torque on the system is zero.*

Exercise 1.8 Verify (1.28). (*Hint*: You will need to assume that the interparticle forces are central (i.e., $\mathbf{F}_{JI} \propto \mathbf{r}_{IJ} \equiv \mathbf{r}_I - \mathbf{r}_J$) in order to have only the *external* torques $\boldsymbol{\tau}_I^{(\mathrm{e})} \equiv \mathbf{r}_I \times \mathbf{F}_I^{(\mathrm{e})}$ contribute to the final sum.)

Exercise 1.9 For a system of particles, we can write the position of particle I as $\mathbf{r}_I = \mathbf{R} + \mathbf{r}'_I$, where $\mathbf{r}'_I$ is the position of the particle relative to $\mathbf{R}$—the location of the center of mass. Show that

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} + \sum_I \mathbf{r}'_I \times \mathbf{p}'_I\,, \tag{1.29}$$

where $\mathbf{p}'_I \equiv m_I \dot{\mathbf{r}}'_I$.

1.3.3 Work

The time evolution of a single particle is described by the path traced-out in three dimensions by its position vector $\mathbf{r}(t)$. For a system of N particles, each particle traces out a different path $\mathbf{r}_I(t)$, where $I = 1, 2, \cdots, N$, so time evolution of a system corresponds to motion of a point in an abstract $3N$-dimensional space, called the **configuration space** of the system. Thus, the instantaneous positions of all the particles of the system correspond to a *single point* in configuration space. As the system evolves, this point traces out a (1-dimensional) curve in configuration space. The work done on a system of particles as it goes from configuration 1 to configuration 2 is the sum of the work done on each individual particle in the system. Thus,

$$W_{12} = \sum_I \int_1^2 \mathbf{F}_I \cdot \mathrm{d}\mathbf{s}_I\,. \tag{1.30}$$

Defining the total kinetic energy of the system of particles to be

$$T \equiv \sum_I \frac{1}{2} m_I v_I^2\,, \tag{1.31}$$

we find that the total work done on a system of particles is equal to the change in the total kinetic energy, so that

$$W_{12} = T_2 - T_1 \,. \tag{1.32}$$

This is the **work-energy theorem** in the context of a system of particles.

1.3.4 Conservative Forces

For a single particle, a force is conservative if it is the gradient of a potential. This can be simply expressed as $\mathbf{F} = -\nabla U(\mathbf{r})$, where the independent variable $\mathbf{r}$ is the position of the particle. In multi-particle systems, there are coordinates $\mathbf{r}_I$ for each particle in the system. Thus, net external forces are conservative if and only if the external force on *each* particle is conservative, i.e.,

$$\mathbf{F}_I^{(\mathrm{e})} = -\nabla_I U_I^{(\mathrm{e})}(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) \,, \tag{1.33}$$

where ∇_I means the gradient of the potential with respect to the coordinate position of particle I. Note that the potential itself carries a subscript I and may depend on the properties (e.g., position, mass, charge, $\cdots$) of each individual particle, but it does not explicitly depend on the velocities $\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \cdots$ or the time t.

Let's look at the work done and the conditions that are placed on the forces in order for us to be able to describe a well-defined potential energy for a system of particles. In general, the work done is

$$W_{12} = \sum_I \int_1^2 \mathbf{F}_I \cdot \mathrm{d}\mathbf{s}_I = \sum_I \int_1^2 \left(\mathbf{F}_I^{(\mathrm{e})} + \sum_J \mathbf{F}_{JI} \right) \cdot \mathrm{d}\mathbf{s}_I \,, \tag{1.34}$$

where the sum over all particles J describes the work done by the interparticle forces. (Recall that we have defined $\mathbf{F}_{II} = 0$.) Thus, the work done on the system splits into two parts—the work done by external forces and the work done by interparticle forces. If the external forces are conservative, then the work done by them is simply minus the change in the external potential from configuration 1 to configuration 2—i.e., $-\Delta U^{(\mathrm{e})} \equiv U_1^{(\mathrm{e})} - U_2^{(\mathrm{e})}$, where $U^{(\mathrm{e})} \equiv \sum_I U_I^{(\mathrm{e})}$. The interparticle forces that appear in the second term of (1.34) may depend on the position of particle J, and may contribute to the work in a path-dependent way. We can make this dependence on the position of particle J explicit by noticing that the double sum over I and J counts each pair of particles twice—once as experiencing a force and once as exerting a force. Thus, we can write the sum as

$$\sum_{I,J} \int_1^2 \mathbf{F}_{JI} \cdot \mathrm{d}\mathbf{s}_I = \frac{1}{2} \sum_{I,J} \left[\int_1^2 \mathbf{F}_{JI} \cdot \mathrm{d}\mathbf{s}_I + \int_1^2 \mathbf{F}_{IJ} \cdot \mathrm{d}\mathbf{s}_J \right]. \tag{1.35}$$

Because of Newton's third law, $\mathbf{F}_{IJ} = -\mathbf{F}_{JI}$, we then have

$$\sum_{I,J} \int_1^2 \mathbf{F}_{JI} \cdot \mathrm{d}\mathbf{s}_I = \frac{1}{2} \sum_{I,J} \int_1^2 \mathbf{F}_{JI} \cdot (\mathrm{d}\mathbf{s}_I - \mathrm{d}\mathbf{s}_J) = \frac{1}{2} \sum_{I,J} \int_1^2 \mathbf{F}_{JI} \cdot \mathrm{d}\mathbf{r}_{IJ}, \tag{1.36}$$

where $\mathrm{d}\mathbf{r}_{IJ}$ is the change in the relative separation between particles I and J, which we denote by $\mathbf{r}_{IJ} \equiv \mathbf{r}_I - \mathbf{r}_J$. If the interparticle forces can also be described as a gradient of a potential, so that $\mathbf{F}_{JI} = -\nabla_{IJ} U_{IJ}(\mathbf{r}_{IJ})$, then the integral becomes path-independent and the total work done is

$$W_{12} = U_1 - U_2 \,, \tag{1.37}$$

where

$$U \equiv \sum_I U_I^{(\mathrm{e})}(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) + \frac{1}{2} \sum_{I,J} U_{IJ}(\mathbf{r}_{IJ}) \,. \tag{1.38}$$

This total potential is the sum of the external potential energies of each particle as well as the internal potential energies due to interparticle interactions. *Thus, if all the forces (both external and internal) are conservative, then the total mechanical energy $E = T + U$ is conserved for the system.*

Example 1.3 Let's consider the effects of an interparticle force that is not directed along the line joining the two particles. Let two particles, with $m_1 = m_2 \equiv m$, lie at rest in the xy-plane, separated by an initial distance $2a$. These two particles feel no external force, but are subject to an interparticle force given by $\mathbf{F}_{21} = k\hat{\mathbf{z}} \times \mathbf{r}_{12}$, where $\hat{\mathbf{z}}$ is the usual unit vector in the z-direction in cylindrical coordinates and k is a constant (units of N/m). This force will still obey the weak form of Newton's 3rd law, so that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. Let's choose a reference frame in which the center of mass lies at the origin, as shown in Fig. 1.3. Since the net force on the two particles is zero, the total momentum is conserved and the center of mass will remain at the origin. The force on particle 1 is then $\mathbf{F}_{21} = 2kr\hat{\boldsymbol{\phi}}$, where $r \equiv |\mathbf{r}_1| = |\mathbf{r}_2|$. Recalling that $\hat{\boldsymbol{\phi}}$ changes direction as we move from point to point, this problem is easier to solve using Cartesian coordinates. Newton's 2nd law gives the following coupled equations:

$$\begin{aligned} m\ddot{x} &= -2ky \,, \\ m\ddot{y} &= +2kx \,. \end{aligned} \tag{1.39}$$

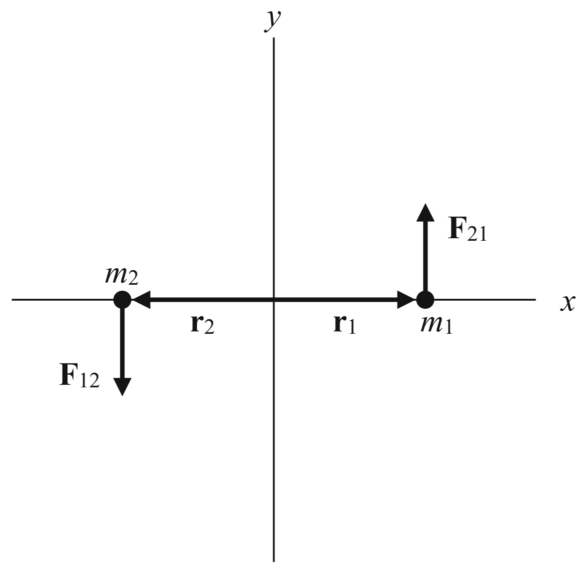


Fig. 1.3 Initial positions of the particles in Example 1.3. The interparticle forces obey $\mathbf{F}_{21} = k\hat{\mathbf{z}} \times \mathbf{r}_{12}$

We can combine these two equations by defining the complex function $\zeta \equiv x + \mathrm{i}y$, giving the single complex differential equation

$$m\ddot{\zeta} = 2\mathrm{i}k\zeta \, . \tag{1.40}$$

The solution to this equation is simply

$$\zeta(t) = A\mathrm{e}^{t\sqrt{2\mathrm{i}k/m}} + B\mathrm{e}^{-t\sqrt{2\mathrm{i}k/m}} \, . \tag{1.41}$$

The initial conditions for this problem are that $\zeta(0) = a$ and $\dot{\zeta}(0) = 0$. Imposing these conditions requires $A = B = a/2$. Defining $\omega \equiv \sqrt{k/m}$ and noting that $\sqrt{2\mathrm{i}k/m} = \sqrt{k/m}\,(1+\mathrm{i}) = \omega\,(1+\mathrm{i})$, we find

$$\zeta(t) = \frac{a}{2}\left(\mathrm{e}^{\omega t}\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\omega t}\mathrm{e}^{-\mathrm{i}\omega t}\right) \, . \tag{1.42}$$

Taking its real and imaginary parts:

$$\begin{aligned} x(t) &= \mathrm{Re}\,\zeta(t) = \frac{a}{2}\left(\mathrm{e}^{\omega t}\cos\omega t + \mathrm{e}^{-\omega t}\cos\omega t\right) = a\cos\omega t\cosh\omega t \, , \\ y(t) &= \mathrm{Im}\,\zeta(t) = \frac{a}{2}\left(\mathrm{e}^{\omega t}\sin\omega t - \mathrm{e}^{-\omega t}\sin\omega t\right) = a\sin\omega t\sinh\omega t \, . \end{aligned} \tag{1.43}$$

Thus, these particles spiral away from each other, gaining angular momentum and kinetic energy as shown in Fig. 1.4. The total angular momentum of the system is

$$\mathbf{L} = 2\mathbf{r} \times \mathbf{p} = 2m\,(x\dot{y} - y\dot{x})\,\hat{\mathbf{z}} = ma^2\omega\,[\sin(2\omega t) + \sinh(2\omega t)]\,\hat{\mathbf{z}} \, . \tag{1.44}$$

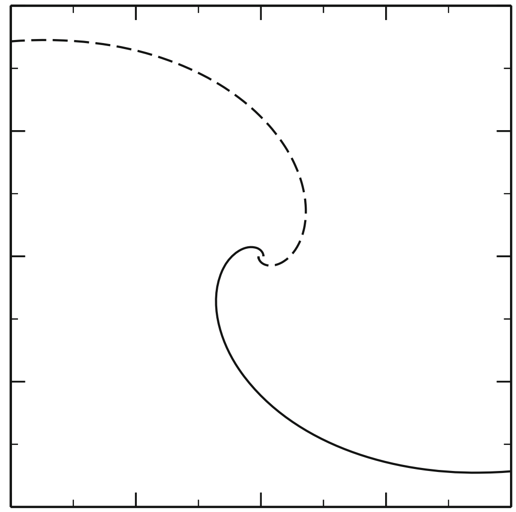


Fig. 1.4 The trajectories of the particles under the influence of the interparticle force $\mathbf{F}_{JI} = k\hat{\mathbf{z}} \times \mathbf{r}_{IJ}$

This increase is the direct result of the fact that the interparticle forces do not point along the interparticle separation. Although there are no external forces on this system, the net torque on the system is

$$\sum_I \boldsymbol{\tau}_I = 4\mathbf{r} \times \left(k\hat{\mathbf{z}} \times \mathbf{r}\right) = 4kr^2\hat{\mathbf{z}}\,. \tag{1.45}$$

> **Exercise 1.10** Calculate the net torque from (1.45) and show that it is equal to the time derivative of the total angular momentum given in (1.44).

The changing angular momentum in this problem indicates that there is an increase in the kinetic energy of the system. This increase comes from the work done by the interparticle forces. Consider the infinitesimal work $\mathrm{d}W$ done by the forces,

$$\mathrm{d}W = \mathbf{F}_{21} \cdot \mathrm{d}\mathbf{r}_1 + \mathbf{F}_{12} \cdot \mathrm{d}\mathbf{r}_2\,. \tag{1.46}$$

Since $\mathbf{r}_1 = -\mathbf{r}_2$, the rate of work done is then

$$\frac{\mathrm{d}W}{\mathrm{d}t} = 4k\left(\hat{\mathbf{z}} \times \mathbf{r}_1\right) \cdot \mathbf{v}_1\,. \tag{1.47}$$

> **Exercise 1.11** (a) Using the scalar triple product identity (A.9), show that the rate at which work is done can be written as
>
> $$\frac{\mathrm{d}W}{\mathrm{d}t} = 2\omega^2 L\,, \tag{1.48}$$

where L is the magnitude of the total angular momentum vector. (b) Integrate this equation to show that the work done by the forces as a function of time is

$$W = ka^2 \left[\cosh(2\omega t) - \cos(2\omega t)\right] . \tag{1.49}$$

(c) Finally, show that this is equal to the total kinetic energy calculated using (1.31).

One may be tempted to explain this increase in kinetic energy by invoking some sort of potential energy that was stored in the system in the process of bringing these two particles in from infinity to their initial positions. Then the apparent increase in kinetic energy is simply the release of this potential energy as the particles spiral back to infinity. However, we can always set up the initial conditions by bringing the particles together from $\pm\infty$ along the x-axis. In this way, the interparticle forces are always *perpendicular* to the motion of the particles, and so no work is done. The interparticle forces that are invoked in this example are not conservative, and it is not possible to define a potential energy associated with these forces. The real solution to this apparent conundrum is that we are using nonsensical forces in this example. These forces are the equivalent of frictional forces that point in the direction of motion (as opposed to against the motion).

Exercise 1.12 We know that a necessary and sufficient condition for a force to be described as the gradient of a potential is that the integral $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{s}$ vanish. Choose a circle of radius r_0 centered on one particle and show that this integral is non-zero, thus proving that $\mathbf{F}$ is non-conservative.

□

1.4 Conservation Laws

Newton's laws provide us with 2nd-order differential equations for the motion of a system of particles $\mathbf{r}_I(t)$ by relating the accelerations to the known forces acting on the particles. These are known as the equations of motion for the system. If certain combinations of the positions and velocities of the particles can be shown to be time-independent, then these quantities are conserved. Each conserved quantity can reduce the order of the equations of motion by one, so they are also called **integrals of the motion**. The common conserved quantities are the total linear momentum, total angular momentum, and total mechanical energy of the system. Certain conditions are placed on the forces acting on the system in order for these quantities to be conserved. From our analyses in the previous sections, we have seen the following conservation laws:

I. **Conservation of Linear Momentum:** If the net external force on a system is zero, then the total linear momentum is conserved:

$$\sum_I \mathbf{F}_I^{(\mathrm{e})} = 0 \quad \Rightarrow \quad \mathbf{P} \equiv \sum_I m_I \mathbf{v}_I = \mathrm{const}\,. \tag{1.50}$$

II. **Conservation of Angular Momentum:** If the net external torque on a system is zero and the *strong form* of Newton's 3rd law holds (so that $\mathbf{F}_{JI}$ is directed along the line connecting particles I and J), then the total angular momentum is conserved:

$$\sum_I \boldsymbol{\tau}_I^{(\mathrm{e})} = 0\,, \quad \mathbf{F}_{JI} \propto \mathbf{r}_{IJ} \quad \Rightarrow \quad \mathbf{L} \equiv \sum_I \mathbf{r}_I \times \mathbf{p}_I = \mathrm{const}\,. \tag{1.51}$$

III. **Conservation of Mechanical Energy:** If both the external forces and interparticle forces are expressible as gradients of scalar potentials,

$$\mathbf{F}_I^{(\mathrm{e})} = -\nabla_I U_I^{(\mathrm{e})}(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N)\,, \quad \mathbf{F}_{JI} = -\nabla_{IJ} U_{IJ}(\mathbf{r}_{IJ})\,, \tag{1.52}$$

then the total mechanical energy of the system $E \equiv T + U$ is conserved:

$$E = \frac{1}{2}\sum_I m_I v_I^2 + \sum_I U_I^{(\mathrm{e})}(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) + \frac{1}{2}\sum_{I,J} U_{IJ}(\mathbf{r}_{IJ}) = \mathrm{const}\,. \tag{1.53}$$

We will return to these conservation laws in Sects. 3.3 and 3.6.2, after we have developed the Lagrangian and Hamiltonian formulations of mechanics.

1.5 Non-inertial Reference Frames

So far we have restricted our attention to studying the motion of a particle (or a system of particles) as seen from an *inertial* frame of reference. We saw in Exercise 1.3 that inertial reference frames move at constant velocity with respect to one another. We can formalize this relationship as a coordinate transformation (known as a **Galilean transformation**) between the two frames as

$$\mathbf{r} = \mathbf{r}' + \mathbf{u}t\,, \tag{1.54}$$

where the origin of the primed coordinate system is moving with constant velocity $\mathbf{u}$ within the unprimed coordinate system. Recall that, in an inertial frame, a particle moves with constant velocity (i.e., has zero acceleration) if there are no forces acting on it. When we are not in an inertial frame, there will be spurious (or **fictitious**) accelerations arising from the acceleration of the reference frame. These effects can be seen in simple every-day situations such as sitting in a vehicle that is accelerating or rounding a corner. In these situations, loose objects will appear to accelerate relative to the observer or vehicle.

We perceive these accelerations because we effectively carry around with us an origin O' and a set of orthonormal basis vectors $\hat{\mathbf{e}}_{i'}$ that are fixed with respect to us, as shown in Fig. 1.5. The motion of the origin O' is described by the position vector $\mathbf{R}(t)$ as seen in the frame of an inertial observer O, with corresponding orthonormal basis vectors $\hat{\mathbf{e}}_i$. The position of a particle located at $\wp$ is described in the inertial and non-inertial reference frames by the displacement vectors $\mathbf{r}(t)$ and $\mathbf{r}'(t)$, respectively, which are defined with respect to the observers O and O'. These two displacement vectors are related by the vector $\mathbf{R}(t)$ joining O and O', so that

$$\mathbf{r} = \mathbf{r}' + \mathbf{R}\,. \tag{1.55}$$

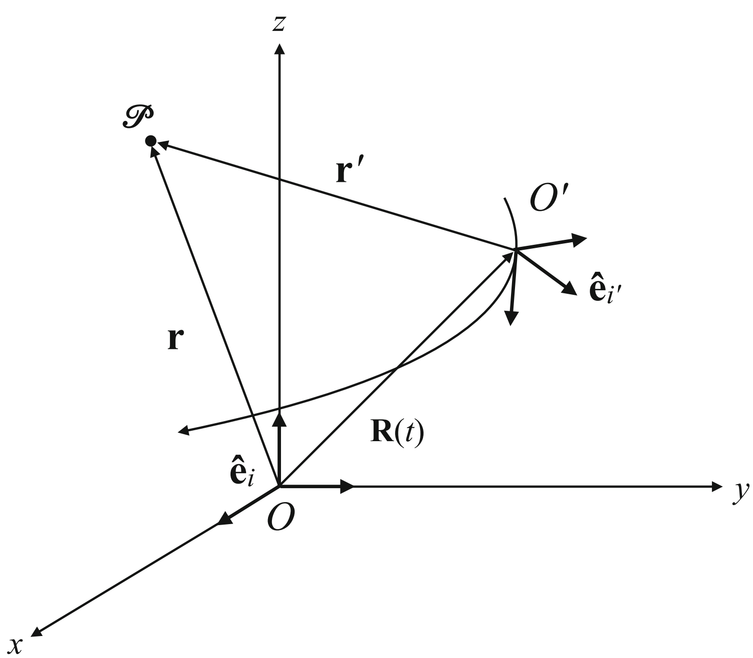


Fig. 1.5 The motion of a non-inertial observer O' described by $\mathbf{R}(t)$ in the reference frame of inertial observer O. O' carries a set of orthonormal basis vectors $\hat{\mathbf{e}}_{i'}$. The position $\wp$ of a particle is described in the inertial and non-inertial reference frames by the displacement vectors $\mathbf{r} = \mathbf{r}(t)$ from O to $\wp$, and $\mathbf{r}' = \mathbf{r}'(t)$ from O' to $\wp$, respectively

Note that both the translational motion of the origin O' and the rotational motion of the basis vectors $\hat{\mathbf{e}}_{i'}$ relative to the fixed (inertial) frame lead to differences in the velocity and acceleration of the particle as seen in these two frames. To determine what these differences are, it is simplest to first *separate* the effects of the translational and rotational motion, and then combine the results at the end to handle the more general case of translational-plus-rotational motion. We do this in the following three subsections.

1.5.1 Translational Motion

Let's begin with the simplest scenario, which is to allow O' to move with respect O, but to keep the basis vectors of the non-inertial reference frame fixed with respect to the inertial frame—i.e., $\hat{\mathbf{e}}_{i'} = \hat{\mathbf{e}}_i$ for $i = 1, 2, 3$. To relate the accelerations of the particle as measured with respect to both O and O', we simply differentiate (1.55) twice with respect to time—i.e., $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}' + \ddot{\mathbf{R}}$, or, equivalently,

$$\mathbf{a} = \mathbf{a}' + \ddot{\mathbf{R}}\,. \tag{1.56}$$

Thus, Newton's 2nd law, $\mathbf{F} = m\mathbf{a}$, which is valid in the inertial frame O, becomes

$$m\mathbf{a}' = \mathbf{F} - m\ddot{\mathbf{R}} \tag{1.57}$$

with respect to O'. Note the presence of the *fictitious force* $\mathbf{F}_{\text{accel}} \equiv -m\ddot{\mathbf{R}}$, which is non-zero if O' is accelerating with respect to O, and which points in the direction *opposite* to the acceleration $\ddot{\mathbf{R}}$.

Example 1.4 Consider a reference frame O' that is accelerating with constant linear acceleration—e.g., a car starting up from a stop. Since the basis vectors in the accelerated and inertial frames are identical, the observed acceleration of loose objects in the car is simply $-\ddot{\mathbf{R}}$. We perceive this acceleration to be caused by the fictitious force $\mathbf{F}_{\text{accel}} = -m\ddot{\mathbf{R}}$, which points in the direction opposite to the car's acceleration. From the perspective of the passengers in the car, they perceive that they have *zero* acceleration relative to their reference frame, i.e., $\mathbf{a}' = 0$, but this is due to the exact cancellation of two forces. One is the fictitious force $\mathbf{F}_{\text{accel}} = -m\ddot{\mathbf{R}}$, which they feel pushing them back in their seats, and the other is the true force $\mathbf{F} = m\mathbf{a}$, which is accelerating them along with the car, but which they perceive as a *normal* force from the seat acting in response to the backward-directed fictitious force. □

1.5.2 Rotational Motion

Now let's consider the case where the origins O and O' of the two reference frames occupy the same position in space, but the basis vectors $\hat{\mathbf{e}}_{i'}$ of the non-inertial frame are *rotating* with respect to the basis vectors $\hat{\mathbf{e}}_i$ of the inertial frame. Then

$$\hat{\mathbf{e}}_{i'} = \sum_j R_{i'j} \hat{\mathbf{e}}_j \,, \tag{1.58}$$

where $R_{i'j}$ are the component of a **rotation matrix** R. (Note that $R_{i'j} = \hat{\mathbf{e}}_{i'} \cdot \hat{\mathbf{e}}_j \equiv \cos\theta_{i'j}$, where $\theta_{i'j}$ is the angle between $\hat{\mathbf{e}}_{i'}$ and $\hat{\mathbf{e}}_j$. These are just the **direction cosines** relating the basis vectors of the two frames.) Since rotations preserve the length of vectors, R is an **orthogonal** matrix, which means that $\mathsf{R}^{-1} = \mathsf{R}^T$ (the transpose of R), or, equivalently,[2]

$$\sum_{i'} R_{i'j} R_{i'k} = \delta_{jk} \,, \qquad \sum_i R_{j'i} R_{k'i} = \delta_{j'k'} \,. \tag{1.59}$$

Using this result, it follows that the components A_i and $A_{i'}$ of a vector $\mathbf{A}$ with respect to the two reference frames are related by

$$A_{i'} = \sum_j R_{i'j} A_j \,, \tag{1.60}$$

which has the same form as the transformation equation (1.58) for the basis vectors.

To calculate the time derivative of $\mathbf{A}$, we will expand $\mathbf{A}$ in the two different reference frames. If we first expand with respect to the inertial frame, we find

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i A_i \hat{\mathbf{e}}_i\right) = \sum_i \left(\frac{\mathrm{d}A_i}{\mathrm{d}t}\hat{\mathbf{e}}_i + A_i \frac{\mathrm{d}\hat{\mathbf{e}}_i}{\mathrm{d}t}\right) = \sum_i \frac{\mathrm{d}A_i}{\mathrm{d}t}\hat{\mathbf{e}}_i \,, \tag{1.61}$$

where the last equality follows from the basis vectors $\hat{\mathbf{e}}_i$ being at rest in the inertial frame. If we expand with respect to the rotating frame, we find

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i'} A_{i'} \hat{\mathbf{e}}_{i'}\right) = \sum_{i'} \left(\frac{\mathrm{d}A_{i'}}{\mathrm{d}t}\hat{\mathbf{e}}_{i'} + A_{i'} \frac{\mathrm{d}\hat{\mathbf{e}}_{i'}}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right)_{\mathrm{r}} + \sum_{i'} A_{i'} \frac{\mathrm{d}\hat{\mathbf{e}}_{i'}}{\mathrm{d}t} \,, \tag{1.62}$$

[2] These concepts are described in more detail in Chap. 6 and Appendix D.

where

$$\left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right)_{\mathrm{r}} \equiv \sum_{i'} \frac{\mathrm{d}A_{i'}}{\mathrm{d}t}\hat{\mathbf{e}}_{i'} \tag{1.63}$$

is the time derivative of $\mathbf{A}$ as seen in the rotating frame of reference (hence the subscript 'r'). To evaluate the last term in (1.62), we use (1.58) and (1.60) to expand $A_{i'}$ and $\hat{\mathbf{e}}_{i'}$ in terms of A_i and $\hat{\mathbf{e}}_i$. This yields

$$\sum_{i'} A_{i'}\frac{\mathrm{d}\hat{\mathbf{e}}_{i'}}{\mathrm{d}t} = \sum_{i'}\sum_{j} R_{i'j}A_j\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_k R_{i'k}\hat{\mathbf{e}}_k\right) = \sum_j A_j\sum_k\left(\sum_{i'} R_{i'j}\frac{\mathrm{d}R_{i'k}}{\mathrm{d}t}\right)\hat{\mathbf{e}}_k\,. \tag{1.64}$$

But note that the matrix defined as

$$M_{jk} \equiv \sum_{i'} R_{i'j}\frac{\mathrm{d}R_{i'k}}{\mathrm{d}t} \tag{1.65}$$

is *anti-symmetric* (i.e., $M_{jk} = -M_{kj}$) as a consequence of (1.59). Since an anti-symmetric 3×3 matrix has three independent components, we can define the components ω_i of a vector $\boldsymbol{\omega}$ in terms of M_{jk} and the (totally anti-symmetric) Levi-Civita symbol ε_{ijk}, defined in (A.7):

$$M_{jk} \equiv \sum_i \omega_i\varepsilon_{ijk} \quad\Leftrightarrow\quad \omega_i = \frac{1}{2}\sum_{j,k}\varepsilon_{ijk}M_{jk}\,. \tag{1.66}$$

Thus,

$$\sum_j A_j\sum_k\left(\sum_{i'} R_{i'j}\frac{\mathrm{d}R_{i'k}}{\mathrm{d}t}\right)\hat{\mathbf{e}}_k = \sum_j A_j\sum_k M_{jk}\hat{\mathbf{e}}_k = \sum_{i,j,k}\omega_i A_j\varepsilon_{ijk}\hat{\mathbf{e}}_k = \boldsymbol{\omega}\times\mathbf{A}\,. \tag{1.67}$$

Putting all these results together,

$$\left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right)_{\mathrm{f}} = \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right)_{\mathrm{r}} + \boldsymbol{\omega}\times\mathbf{A}\,, \tag{1.68}$$

where we have written $\mathrm{d}\mathbf{A}/\mathrm{d}t = (\mathrm{d}\mathbf{A}/\mathrm{d}t)_{\mathrm{f}}$, which follows from (1.61); the subscript 'f' indicates the fixed (inertial) frame.

Exercise 1.13 It turns out that $\boldsymbol{\omega}$ defined by (1.66) and (1.65) is the **instantaneous angular velocity vector** of the rotating reference frame relative to the inertial reference frame. Verify that this is indeed the case by calculating $\boldsymbol{\omega}$ for the simple case of a rotation about the z-axis with constant angular velocity ω:

$$R_{i'j} = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.69}$$

You should find that $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$.

Equation (1.68) is a general result, so we can apply it to *any* vector $\mathbf{A}$. In particular, if we take $\mathbf{A}$ to be the position vector $\mathbf{r}$ of a particle relative to the shared origin of O and O', then

$$\mathbf{v}_\mathrm{f} = \mathbf{v}_\mathrm{r} + \boldsymbol{\omega} \times \mathbf{r}\,, \tag{1.70}$$

where $\mathbf{v}_\mathrm{f}$ and $\mathbf{v}_\mathrm{r}$ are shorthand for $(\mathrm{d}\mathbf{r}/\mathrm{d}t)_\mathrm{f}$ and $(\mathrm{d}\mathbf{r}/\mathrm{d}t)_\mathrm{r}$. Similarly, if we apply (1.68) to the angular velocity vector $\boldsymbol{\omega}$, we find

$$\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_\mathrm{f} = \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_\mathrm{r} \tag{1.71}$$

since $\boldsymbol{\omega} \times \boldsymbol{\omega} = 0$. Thus, we can write $\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t \equiv \dot{\boldsymbol{\omega}}$ without ambiguity. Finally, if we take $\mathbf{A}$ to equal $\mathbf{v}_\mathrm{f}$ from (1.70), we find

$$\begin{aligned}
\left(\frac{\mathrm{d}\mathbf{v}_\mathrm{f}}{\mathrm{d}t}\right)_\mathrm{f} &= \left(\frac{\mathrm{d}\mathbf{v}_\mathrm{f}}{\mathrm{d}t}\right)_\mathrm{r} + \boldsymbol{\omega} \times \mathbf{v}_\mathrm{f} \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}_\mathrm{r} + \boldsymbol{\omega} \times \mathbf{r})\right)_\mathrm{r} + \boldsymbol{\omega} \times (\mathbf{v}_\mathrm{r} + \boldsymbol{\omega} \times \mathbf{r}) \\
&= \left(\frac{\mathrm{d}\mathbf{v}_\mathrm{r}}{\mathrm{d}t}\right)_\mathrm{r} + \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_\mathrm{r} \times \mathbf{r} + \boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_\mathrm{r} + \boldsymbol{\omega} \times \mathbf{v}_\mathrm{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\
&= \left(\frac{\mathrm{d}\mathbf{v}_\mathrm{r}}{\mathrm{d}t}\right)_\mathrm{r} + \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_\mathrm{r} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v}_\mathrm{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\,,
\end{aligned} \tag{1.72}$$

or, more compactly,

$$\mathbf{a}_\mathrm{f} = \mathbf{a}_\mathrm{r} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v}_\mathrm{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\,, \tag{1.73}$$

where $\mathbf{a}_\mathrm{f}$ and $\mathbf{a}_\mathrm{r}$ are shorthand for $(\mathrm{d}\mathbf{v}_\mathrm{f}/\mathrm{d}t)_\mathrm{f}$ and $(\mathrm{d}\mathbf{v}_\mathrm{r}/\mathrm{d}t)_\mathrm{r}$.

Thus, Newton's 2nd law $\mathbf{F} = m\mathbf{a}_\mathrm{f}$, which is valid in an inertial frame, can be written in a rotating reference frame as

$$m\mathbf{a}_\mathrm{r} = \mathbf{F} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2m\boldsymbol{\omega} \times \mathbf{v}_\mathrm{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\,. \tag{1.74}$$

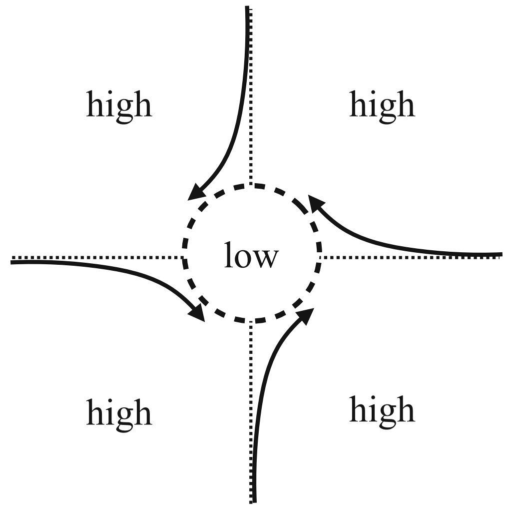


Fig. 1.6 Cyclonic motion of air currents as seen in the Northern hemisphere, driven by pressure gradients and the Coriolis force associated with Earth's rotational motion

If we interpret these last three terms as additional (fictitious) forces, then Newton's 2nd law in the rotating frame has the more standard looking form, $m\mathbf{a}_{\rm r} = \mathbf{F}_{\rm eff}$. The first fictitious force term, $\mathbf{F}_{\rm ang\ accel} \equiv -m\dot{\boldsymbol{\omega}} \times \mathbf{r}$, is related to the angular acceleration of the rotating reference frame. For a *uniformly* rotating frame, like a lab attached to the surface of the Earth, $\dot{\boldsymbol{\omega}} = 0$, so this fictitious force vanishes. The last two fictitious force terms are the **Coriolis** and **centrifugal** forces, respectively:

$$\mathbf{F}_{\rm coriolis} \equiv -2m\boldsymbol{\omega} \times \mathbf{v}_{\rm r}\,, \qquad \mathbf{F}_{\rm centrifugal} \equiv -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\,. \tag{1.75}$$

The centrifugal force is directed radially away from the axis of rotation and has magnitude $m\omega^2 r \sin\theta$ where θ is the angle between $\boldsymbol{\omega}$ and $\mathbf{r}$. The Coriolis force is non-zero only if $\mathbf{v}_{\rm r} \neq 0$, and is directed perpendicular to both $\mathbf{v}_{\rm r}$ and $\boldsymbol{\omega}$. As viewed along the direction of $\mathbf{v}_{\rm r}$, the Coriolis force associated with counter-clockwise rotational motion produces a deflection to the right; for clockwise rotational motion, it produces a deflection to the left. The Coriolis force associated with Earth's rotational motion is responsible for the circulating or **cyclonic** weather patterns associated with hurricanes and cyclones, as illustrated in Fig. 1.6. Basically, a pressure gradient gives rise to air currents that tend to flow from high pressure to low pressure regions. But as the air flows toward the low pressure region, the Coriolis force deflects the air currents away from their straight line paths. Since the projection of $\boldsymbol{\omega}$ perpendicular to the local tangent plane changes sign as one crosses the equator, the direction of the cyclonic motion (either counter-clockwise or clockwise) is different in the Northern and Southern hemispheres.

1.5.3 Combined Translational and Rotational Motion

Given (1.57) and (1.74), it is now a simple manner to write down the equivalent of Newton's 2nd law in a general non-inertial reference frame having both translational and rotational motion. The result is

$$m\mathbf{a}' = \mathbf{F} - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2m\boldsymbol{\omega} \times \mathbf{v}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - m\ddot{\mathbf{R}}\,, \qquad (1.76)$$

where the primes $'$ denote quantities calculated with respect to the non-inertial frame.

Example 1.5 Consider a carnival ride that spins a cylinder about its central axis with angular velocity ω, causing all riders to feel that they are pressed against the walls of the cylinder. In this case, the non-inertial observer feels a net acceleration as he/she rotates around the central axis, with his/her basis vectors rotating with respect to the inertial frame at the same rate (See Fig. 1.7). We'd like to know what fictitious forces the rider feels, and if the rider threw a ball in toward the center of the ride, where would it land?

To do this problem, we first note that the basis vectors in O' are related to those in O by the rotation matrix

$$R_{i'j} = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (1.77)$$

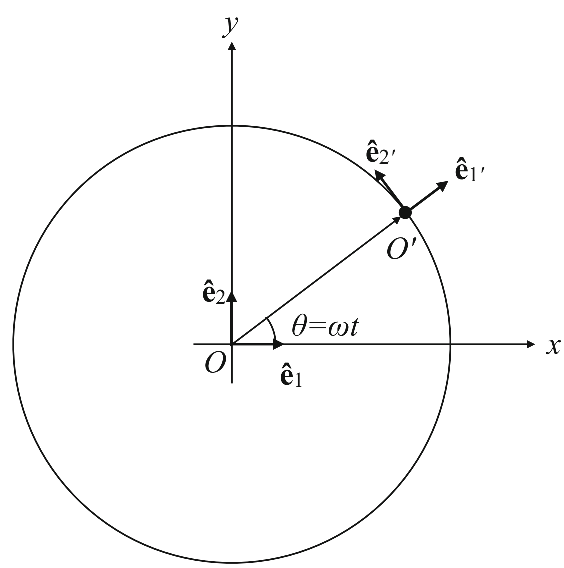


Fig. 1.7 Basis vectors and reference frames for an inertial observer O at the center of a carnival ride and for a non-inertial observer O' on the ride. The basis vectors of the non-inertial observer, $\hat{\mathbf{e}}_{1'}$, $\hat{\mathbf{e}}_{2'}$, rotate along with the rider at the same rate. Note that $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_{3'} = \hat{\mathbf{z}}$ for both observers, which points out of the page

As shown in Exercise 1.13, the associated angular velocity vector $\boldsymbol{\omega}$ is simply $\omega\hat{\mathbf{z}}$, consistent with the motion of the carnival ride. Also, since the position O' of the rider with respect to the inertial frame is

$$\mathbf{R} = R\cos\omega t\,\hat{\mathbf{e}}_1 + R\sin\omega t\,\hat{\mathbf{e}}_2\,, \tag{1.78}$$

then $\ddot{\mathbf{R}} = -\omega^2\mathbf{R}$. Thus, Newton's 2nd law in this non-inertial frame reduces to

$$m\mathbf{a}' = \mathbf{F} - 2m\boldsymbol{\omega}\times\mathbf{v}' + m\omega^2\mathbf{r}' + m\omega^2\mathbf{R}\,, \tag{1.79}$$

where we have also assumed that the position vector $\mathbf{r}'$ of a particle as seen in the non-inertial frame has no z-component in order to write the second-to-last term in that form. The last two terms in the above expression can be thought of as centrifugal force terms associated with (i) the rotational motion of the basis vectors $\hat{\mathbf{e}}_{i'}$, and (ii) the rotational motion of the origin O' (i.e., the rider) with respect to O. The "origin" centrifugal force is the fictitious force that appears to drive objects out toward the wall from the center of the ride. The "basis" centrifugal force is the fictitious force that appears to drive objects away from the rider. Finally, the Coriolis force $-2m\boldsymbol{\omega}\times\mathbf{v}'$ is the result of the fact that the rider is moving with tangential velocity $\boldsymbol{\omega}\times\mathbf{R}$. Any additional velocity of an observed particle will add to this tangential velocity. Thus, if the rider throws a ball in toward the center, the Coriolis force will cause it to appear to accelerate *in the direction of motion of the rider*. We can also understand this by noting that the ball has an initial velocity of $\mathbf{v}' + \boldsymbol{\omega}\times\mathbf{R}$ with respect to the inertial frame. As the ball moves in toward the center, its tangential velocity is now greater than the comoving tangential velocity, so it will appear to move in the direction of rotation. This is similar to what we saw in Fig. 1.6 for the deflection of air currents due to the Earth's rotational motion. □

1.5.4 Foucault's Pendulum

A simple way to demonstrate the Earth's rotational motion is to show that the plane of a swinging pendulum precesses with time, with a precessional period equal to (1 day)/ $\sin\lambda$, where $\lambda = \pi/2 - \theta$ is the latitude of the pendulum's location.[3] In this subsection, we solve the equations of motion for the swinging pendulum as seen in a rotating reference frame attached to the surface of the Earth, and derive the above expression for the precessional period. Such a demonstration is called **Foucault's pendulum** in honor of the French physicist, Jean Léon Foucault who first exhibited this demonstration in Paris in 1851.

[3] θ is the usual spherical coordinate angle measured from the z-axis (the North pole), and is called the *co-latitude*.

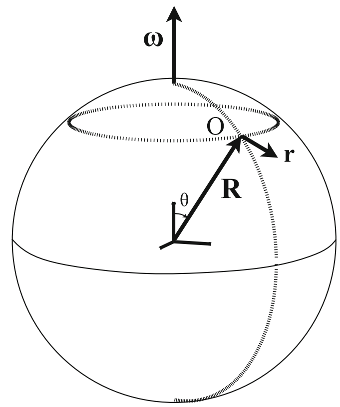


Fig. 1.8 Definitions of $\mathbf{R}$ and $\mathbf{r}$ for a non-inertial reference frame with origin O attached to the surface of the Earth. The size of the pendulum bob displacement $\mathbf{r}$ relative to the Earth's radius $\mathbf{R}$ has been exaggerated in this figure for ease of visualization

To simplify the notation in what follows, we will drop the primes on quantities calculated in the non-inertial frame attached to the surface of the Earth, so that (1.76) becomes

$$m\mathbf{a} = \mathbf{F} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2m\boldsymbol{\omega} \times \mathbf{v} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - m\ddot{\mathbf{R}}_{\mathrm{f}}\,, \tag{1.80}$$

where $\mathbf{R}$ is the radius vector pointing from the center of the Earth to the origin O of the local reference frame, and $\mathbf{r}$ is the displacement of the pendulum bob away from equilibrium, as shown in Fig. 1.8. (The subscript "f" on $\ddot{\mathbf{R}}_{\mathrm{f}}$ is to indicate that this acceleration is calculated with respect to the *fixed* (i.e., inertial) frame.) In addition, $\mathbf{F} = \mathbf{T} + m\mathbf{g}_0$, where $\mathbf{T}$ is the tension in the string attached to the pendulum bob and $\mathbf{g}_0$ points towards the center of the Earth (in the direction of $-\mathbf{R}$); $\dot{\boldsymbol{\omega}} = 0$, since the angular velocity of the Earth is constant; and $\ddot{\mathbf{R}}_{\mathrm{f}} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$, as a consequence of (1.68). Thus,

$$m\mathbf{a} = \mathbf{T} + m\left[\mathbf{g}_0 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{R}))\right] - 2m\boldsymbol{\omega} \times \mathbf{v}\,. \tag{1.81}$$

The term in square brackets

$$\mathbf{g} \equiv \mathbf{g}_0 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{R}))\,, \tag{1.82}$$

defines the effective local direction of the Earth's gravitational field, which differs from $\mathbf{g}_0$ by the centrifugal acceleration associated with the Earth's rotational motion. Since the displacement r of the pendulum bob away from equilibrium is small compared to the Earth's radius R, the centrifugal acceleration is dominated by $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$. Note that a *plumb line* (i.e., a mass suspended from the end of a string) points in the direction of $\mathbf{g}$ (and not $\mathbf{g}_0$), as shown in Fig. 1.9. Thus, for our analysis, we will define our local coordinate system so that $\hat{\mathbf{z}}$ points along $-\mathbf{g}$. We then choose $\hat{\mathbf{x}}$ perpendicular to $\hat{\mathbf{z}}$, pointing South; and $\hat{\mathbf{y}}$ perpendicular to both $\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}$, pointing East (along the line of constant latitude), as shown in Fig. 1.10.

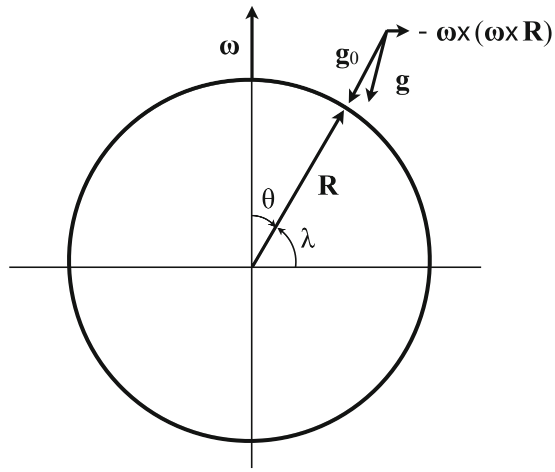


Fig. 1.9 Change in the direction of the local gravitational field due to Earth's rotational motion. The deviation from $\mathbf{g}_0$ has been exaggerated for ease of visualization

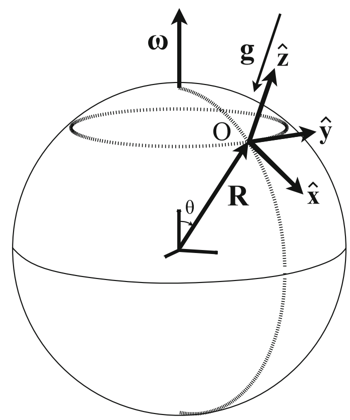


Fig. 1.10 Definition of a non-inertial reference frame attached to the surface of the Earth. Note that $\hat{\mathbf{z}}$ points along $-\mathbf{g}$, which is opposite the direction of the effective local gravitational field. The direction of $\mathbf{g}$ differs slightly from $\mathbf{g}_0$ (which points in the direction of $-\mathbf{R}$) due to the centrifugal acceleration associated with the Earth's rotational motion, as shown previously in Fig. 1.9

Exercise 1.14 Show that the angle δ between $\mathbf{g}$ (the direction of a plumb line at the surface of the Earth at latitude $\lambda = \pi/2 - \theta$) and $\mathbf{g}_0$ (the direction pointing toward the center of the Earth) is given to leading order by

$$\delta \approx \frac{R\omega^2}{g_0} \sin\theta \cos\theta \,, \tag{1.83}$$

with maximum value $\delta = 0.0017$ rad $\approx 0.1°$ at $\theta = \pi/4$. Show also that the centrifugal acceleration vector $\delta\mathbf{g} \equiv -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$ has maximum magnitude $|\delta\mathbf{g}|/g_0 = 0.003$ at the equator, $\theta = \pi/2$.
Hint: First show that the maximum value of $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$ is small compared to $g_0 = 9.8$ m/s^2, where $\omega = 2\pi/(1 \text{ day})$ and $R = 6400$ km. Then use the law of sines and the small-angle approximation to obtain (1.83).

So we need to solve

$$m\mathbf{a} = \mathbf{T} + m\mathbf{g} - 2m\boldsymbol{\omega} \times \mathbf{v} \,. \tag{1.84}$$

We will consider small-angle oscillations of the pendulum bob in the xy-plane, so that v_z can be ignored relative to v_x and v_y, and $a_z \approx 0$. Given these approximations, we can write

$$\begin{aligned}
\mathbf{a} &\approx \ddot{x}\,\hat{\mathbf{x}} + \ddot{y}\,\hat{\mathbf{y}} \,, \\
\mathbf{T} &\approx -T(x/L)\,\hat{\mathbf{x}} - T(y/L)\,\hat{\mathbf{y}} + T\,\hat{\mathbf{z}} \,, \\
\mathbf{g} &= -g\,\hat{\mathbf{z}} \,, \\
\boldsymbol{\omega} \times \mathbf{v} &\approx -\omega_z\dot{y}\,\hat{\mathbf{x}} + \omega_z\dot{x}\,\hat{\mathbf{y}} + \omega_x\dot{y}\,\hat{\mathbf{z}} \,,
\end{aligned} \tag{1.85}$$

where L is the length of the pendulum and

$$\omega_x \approx -\omega\sin\theta \,, \qquad \omega_y = 0 \,, \qquad \omega_z \approx \omega\cos\theta \,. \tag{1.86}$$

The three equations of motion are thus

$$\begin{aligned}
m\ddot{x} &\approx -Tx/L + 2m\omega_z\dot{y} \,, \\
m\ddot{y} &\approx -Ty/L - 2m\omega_z\dot{x} \,, \\
0 &\approx T - mg - 2m\omega_x\dot{y} \,.
\end{aligned} \tag{1.87}$$

Now, one can show (Exercise 1.15) that

$$|\omega_x\dot{y}| \ll g \,, \tag{1.88}$$

for a typical pendulum with period $P \ll 1$ day $= 2\pi/\omega$. Thus, we can ignore the last term in the $\ddot{z} \approx 0$ equation in (1.87) and solve it for the tension,

$$T = mg \,, \tag{1.89}$$

giving the expected result. Using this value for T, the $\ddot{x}$ and $\ddot{y}$ equations reduce to:

$$\begin{aligned}
\ddot{x} &\approx -\Omega^2 x + 2\omega_z\dot{y} \,, \\
\ddot{y} &\approx -\Omega^2 y - 2\omega_z\dot{x} \,,
\end{aligned} \tag{1.90}$$

where $\Omega \equiv \sqrt{g/L}$ is the unperturbed frequency of oscillation that we expect for a pendulum of length L. Since $\omega \ll \Omega$, the terms proportional to ω_z in the above equations act as perturbations to the standard simple harmonic oscillator equations $\ddot{x} = -\Omega^2 x$ and $\ddot{y} = -\Omega^2 y$, which describe simple harmonic motion with angular frequency Ω.

Exercise 1.15 Verify (1.88). *Hint*: If the maximum displacement of the pendulum bob away from equilibrium is D, then you can show that $\dot{y}$ is bounded by $D\Omega$, where $\Omega \equiv \sqrt{g/L}$ is the unperturbed (angular) frequency of oscillation. Note that $D \ll L$ to be consistent with the small-angle approximation for the pendulum bob.

To solve the coupled differential equations in (1.90), we perform the same "trick" that we used in Example 1.3 and form the complex combination

$$\zeta \equiv x + \mathrm{i}y\,, \tag{1.91}$$

allowing us to recast the two equations in (1.90) as a *single* complex differential equation

$$\ddot{\zeta} + 2\mathrm{i}\omega_z\dot{\zeta} + \Omega^2\zeta = 0\,. \tag{1.92}$$

This is a 2nd-order ordinary differential equation with constant coefficients, which can be solved in the usual way (See, e.g., Chap. 8 in Boas (2006)). Substituting the trial solution $\zeta(t) = \mathrm{e}^{\lambda t}$, with complex λ, we obtain a quadratic equation for λ:

$$\lambda^2 + 2\mathrm{i}\omega_z\lambda + \Omega^2 = 0\,. \tag{1.93}$$

This equation has two complex solutions

$$\lambda_\pm = -\mathrm{i}\left(\omega_z \mp \sqrt{\Omega^2 + \omega_z^2}\right) \approx -\mathrm{i}(\omega_z \mp \Omega)\,, \tag{1.94}$$

where we've used $\omega_z \ll \Omega$ to get the last (approximate) equality. Thus, the general solution to (1.92) is

$$\zeta(t) = A\mathrm{e}^{\lambda_+ t} + B\mathrm{e}^{\lambda_- t}\,, \tag{1.95}$$

where A and B are complex coefficients, to be determined by the initial conditions.

If we assume that the pendulum bob is pulled out a distance D in the x-direction and released from rest, then

$$x(0) = D\,, \quad y(0) = 0\,, \quad \dot{x}(0) = 0\,, \quad \dot{y}(0) = 0\,, \tag{1.96}$$

or, equivalently,

$$\zeta(0) = D\,, \qquad \dot{\zeta}(0) = 0\,. \tag{1.97}$$

Imposing these conditions on $\zeta(t)$ determines A and B, leading to (Exercise 1.16):

$$\begin{aligned} x(t) &= D\left[\cos\omega_z t\cos\Omega t + \frac{\omega_z}{\Omega}\sin\omega_z t\sin\Omega t\right], \\ y(t) &= D\left[-\sin\omega_z t\cos\Omega t + \frac{\omega_z}{\Omega}\cos\omega_z t\sin\Omega t\right]. \end{aligned} \tag{1.98}$$

Note that these equations can be written in matrix form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos\omega_z t & \sin\omega_z t \\ -\sin\omega_z t & \cos\omega_z t \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{y}(t) \end{bmatrix}, \tag{1.99}$$

where

$$\bar{x}(t) = D\cos\Omega t, \qquad \bar{y}(t) = D\frac{\omega_z}{\Omega}\sin\Omega t. \tag{1.100}$$

The matrix

$$R_z \equiv \begin{bmatrix} \cos\omega_z t & \sin\omega_z t \\ -\sin\omega_z t & \cos\omega_z t \end{bmatrix}, \tag{1.101}$$

which appears in (1.99), represents a uniform rotation in the xy-plane with angular velocity $\omega_z = \omega\cos\theta$. This is just the precessional frequency of the plane of oscillation of the pendulum. The period of the precession is then

$$P_{\text{precession}} = \frac{2\pi}{\omega_z} = \frac{1\text{ day}}{\cos\theta} = \frac{1\text{ day}}{\sin\lambda}, \tag{1.102}$$

where $\lambda = \pi/2 - \theta$ is the latitude. For example, if we take $\lambda = 49°$, which is the latitude of Paris (where Foucault first did this demonstration), we have a precessional period of 31 hours and 48 minutes. At the equator, the pendulum does *not* precess.

Exercise 1.16 Verify the solution given in (1.98).

Plots of $(x(t), y(t))$ and $(\bar{x}(t), \bar{y}(t))$ are shown in Figs. 1.11 and 1.12. For these plots, we decreased the angular frequency of oscillation Ω by a factor of 200 compared to typical values, so as to easily see the precession of the plane of oscillation of the pendulum after only a few oscillations. Typical Foucault pendulum demonstrations have suspensions of order $L = 30$ m (roughly 100 ft). For such an L, the angular oscillation frequency $\Omega = \sqrt{g/L} \approx 0.57$ rad/s, which corresponds to an oscillation period of $2\pi/\Omega \approx 11$ s. For these figures, we have oscillation periods of roughly 2200 s $\approx$ 36 min, so only ~ 50 back-and-forth motions of the pendulum bob would be needed for a complete 360° precession at the latitude of Paris.

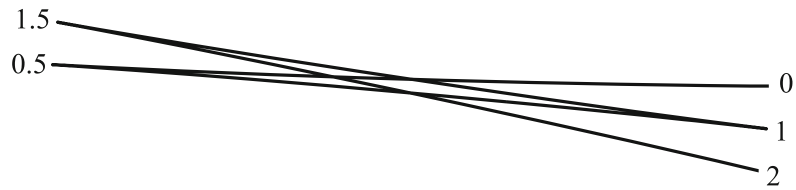


Fig. 1.11 Motion of the pendulum bob as seen in the non-inertial reference frame. Note that the plane of oscillation of the pendulum precesses. (The numbers correspond to back-and-forth oscillations of the pendulum bob.) As described in the text, the angular frequency of oscillation Ω has been reduced considerably for this figure so as to easily visualize the precession of the plane of oscillation in just two oscillation periods

Fig. 1.12 Motion of the pendulum bob as seen in a "corotating frame", which rotates relative to the non-inertial frame with the precession frequency ω_z of the plane of oscillation of the pendulum. As described in the text, the angular frequency of oscillation Ω has been reduced considerably for this figure so as to easily visualize the elliptical nature of the motion in this reference frame

1.6 Constrained Systems

For some systems, the motion of a particle (or particles) is restricted to a prescribed surface or path. The constraints on the motion are often the result of additional forces acting on the particle (such as normal forces or tension forces) that *adjust their values* in order to maintain the motion on the prescribed surface or path. These forces then become additional unknowns that must be solved for while obtaining the equations of motion for the system.

There are a variety of techniques for dealing with these forces of constraint. We will look at a few examples in order to see how the constraints are imposed on solutions obtained through Newton's laws. In many cases, this involves reducing the number of degrees of freedom in the system by finding an equation relating the coordinates to one another and using it to solve for one or more of the degrees of freedom in terms of the remaining variables. In Chap. 2, we will examine more powerful mathematical tools for handling constrained systems.

Example 1.6 A **spherical pendulum** consists of a mass m at the end of a massless rigid rod of length ℓ. The rod is free to pivot around the other end, so the particle is constrained to move under the influence of gravity on the surface of a sphere of radius ℓ. Spherical coordinates allow us to easily impose the constraint and reduce the number of degrees of freedom from three to two by requiring that $r = \ell$. In order to allow our solution to be easily compared with the well-known results of the simple pendulum, we orient our coordinates with the z-axis is pointing downward, so that the polar angle θ is measured as a displacement angle from the equilibrium position of the pendulum hanging vertically. The orientation of the coordinates and system are shown in Fig. 1.13

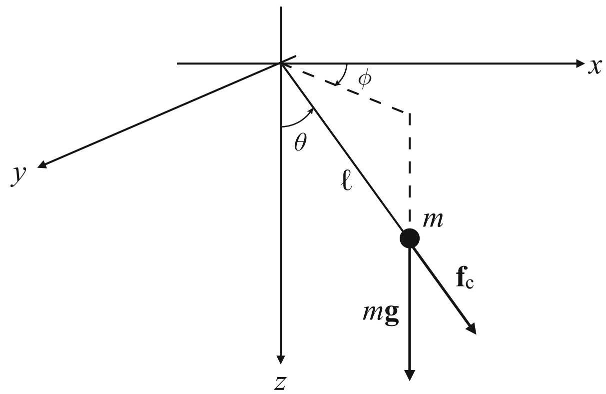


Fig. 1.13 The spherical pendulum with a mass m on the end of a massless rigid rod of length ℓ. The force of gravity points down in the positive z direction, and the force of constraint $\mathbf{f}_c$ points along the rod in the direction of $\hat{\mathbf{r}}$. If the constraint force is a tension force, it will point radially inward; if it is a normal force, it will point radially outward. The relevant polar and azimuthal angles are the usual spherical coordinates θ and ϕ

In these coordinates, the forces are $\mathbf{f}_c = f_c\hat{\mathbf{r}}$ and $m\mathbf{g} = mg\cos\theta\,\hat{\mathbf{r}} - mg\sin\theta\,\hat{\boldsymbol{\theta}}$, so the net force acting on the particle is

$$\sum \mathbf{F} = (f_c + mg\cos\theta)\,\hat{\mathbf{r}} - mg\sin\theta\,\hat{\boldsymbol{\theta}}\,. \tag{1.103}$$

The position of the particle is simply $\mathbf{r} = \ell\hat{\mathbf{r}}$, but since $\hat{\mathbf{r}}$ points in different directions as the particle moves, we must also include the time dependence of the spherical basis vectors when finding the acceleration. Referring to the definitions in (A.46) and taking the time derivatives explicitly, we find

$$\begin{aligned}\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} =& \dot{\phi}\sin\theta\left(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}\right)\\ &+ \dot{\theta}\left(\cos\phi\cos\theta\hat{\mathbf{x}} + \sin\phi\cos\theta\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}\right)\,,\end{aligned} \tag{1.104}$$

which can be written in terms of the spherical basis vectors as

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = \dot{\phi}\sin\theta\hat{\boldsymbol{\phi}} + \dot{\theta}\hat{\boldsymbol{\theta}}\,. \tag{1.105}$$

When we take the second time derivative, we will also need the time derivatives of $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$, which can be found using the same procedure. Thus, we additionally have

$$\begin{aligned}\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} &= -\dot{\theta}\hat{\mathbf{r}} + \dot{\phi}\cos\theta\hat{\boldsymbol{\phi}}\,,\\ \frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t} &= -\dot{\phi}\sin\theta\hat{\mathbf{r}} - \dot{\phi}\cos\theta\hat{\boldsymbol{\theta}}\,.\end{aligned} \tag{1.106}$$

Exercise 1.17 We can also obtain the above expressions for the time derivatives of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ using directional derivatives, which are discussed in Appendix A.4.2. Note that for any basis vector $\hat{\mathbf{e}}_i$, its time derivative is given by

$$\frac{\mathrm{d}\hat{\mathbf{e}}_i}{\mathrm{d}t} = \frac{\partial \hat{\mathbf{e}}_i}{\partial r}\dot{r} + \frac{\partial \hat{\mathbf{e}}_i}{\partial \theta}\dot{\theta} + \frac{\partial \hat{\mathbf{e}}_i}{\partial \phi}\dot{\phi}\,, \tag{1.107}$$

where

$$\frac{\partial \hat{\mathbf{e}}_i}{\partial r} = \nabla_{\hat{\mathbf{r}}}\hat{\mathbf{e}}_i\,, \qquad \frac{\partial \hat{\mathbf{e}}_i}{\partial \theta} = r\nabla_{\hat{\boldsymbol{\theta}}}\hat{\mathbf{e}}_i\,, \qquad \frac{\partial \hat{\mathbf{e}}_i}{\partial \phi} = r\sin\theta\nabla_{\hat{\boldsymbol{\phi}}}\hat{\mathbf{e}}_i\,. \tag{1.108}$$

Use (A.49) from Example A.2 to recover (1.105) and (1.106).

Taking two time derivatives of $\mathbf{r} = \ell\hat{\mathbf{r}}$, we find that the acceleration is

$$\begin{aligned}\mathbf{a} = &-\ell\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)\hat{\mathbf{r}} + \ell\left(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta\right)\hat{\boldsymbol{\theta}}\\ &+\ell\left(\ddot{\phi}\sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta\right)\hat{\boldsymbol{\phi}}\,.\end{aligned} \tag{1.109}$$

Newton's 2nd law then gives the three equations of motion

$$f_c + mg\cos\theta = -m\ell\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)\,, \tag{1.110a}$$

$$-mg\sin\theta = m\ell\left(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta\right)\,, \tag{1.110b}$$

$$0 = m\ell\left(\ddot{\phi}\sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta\right)\,. \tag{1.110c}$$

The first equation, (1.110a), gives us the constraint force

$$f_c = -mg\cos\theta - m\ell\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)\,, \tag{1.111}$$

and can easily be seen to be the tension (or normal) force needed to counteract the weight plus the centripetal force needed to keep the particle moving in a circle with speed v, where $v^2 = \ell^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$. The second equation, (1.110b), gives us the 2nd-order differential equation governing θ,

$$m\ell\ddot{\theta} = m\ell\dot{\phi}^2\sin\theta\cos\theta - mg\sin\theta\,. \tag{1.112}$$

The third equation, (1.110c), can be shown to be related to the conserved component of the angular momentum. First, note that we can multiply (1.110c) by $\ell\sin\theta$, for which the right-hand side becomes a total time derivative of $m\ell^2\dot{\phi}\sin^2\theta$, which then must be a conserved quantity. However, the net torque on this system is not zero, so we cannot say that the full angular momentum vector is conserved. The angular momentum is $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, which can be expanded as

$$\mathbf{L} = m\ell^2 \left[-\left(\dot{\theta}\sin\phi + \dot{\phi}\cos\phi\sin\theta\cos\theta\right)\hat{\mathbf{x}} \right.$$
$$\left. + \left(\dot{\theta}\cos\phi - \dot{\phi}\sin\phi\sin\theta\cos\theta\right)\hat{\mathbf{y}} + \dot{\phi}\sin^2\theta\hat{\mathbf{z}} \right] . \tag{1.113}$$

Since the net torque is $\boldsymbol{\tau} = \mathbf{r} \times mg\hat{\mathbf{z}}$, the $\hat{\mathbf{z}}$ component of the torque is zero. This means that $L_z = m\ell^2\dot{\phi}\sin^2\theta$ is a conserved quantity, as demonstrated by (1.110c). We can use this expression for L_z to eliminate $\dot{\phi}$ from (1.112), leaving a 2nd-order differential equation for θ alone. Unfortunately, (1.112) is non-linear, so it can be solved exactly only for special cases. □

In the spherical pendulum example, the constraints were expressed as an equation relating the three degrees of freedom to the restriction that $r = \ell$. In spherical coordinates, this constraining equation is so simple that it is hard to see it as a non-trivial equation. In Cartesian coordinates, it is more obvious as $\ell^2 = x^2 + y^2 + z^2$. When combined with Newton's laws, the constraint equations give the unknown forces of constraint. In some situations, the constraints are valid only when the forces of constraint lie within a restricted range. Using similar techniques to determine the forces of constraint, we can then determine under which conditions the constraints hold and when the system is no longer constrained.

Example 1.7 Consider a skier going down a hemispherical hill. At first the skier is constrained to follow the surface of the hill. As the skier descends and speeds up, she will eventually leave the surface and begin to follow a ballistic trajectory. We want to determine at what angle the skier leaves the slope.

This problem is effectively two-dimensional and cylindrical coordinates are the most appropriate. We will measure the angle ϕ off of the (vertical) y-axis instead of the x-axis, but otherwise these are the standard cylindrical coordinates. The radius of the hill is R, so the constraint equation is $R^2 = x^2 + y^2$. The forces and coordinate choices are shown in Fig. 1.14. The constraint force is the normal force that the hill exerts on the skier. When the combined radial force on the skier drops to zero, then there is no centripetal force left to constrain the skier to the surface of the hill. In the cylindrical basis, the forces acting on the body are

$$\begin{aligned} \mathbf{F}_n &= F_n\hat{\boldsymbol{\rho}} , \\ m\mathbf{g} &= -mg\cos\phi\hat{\boldsymbol{\rho}} + mg\sin\phi\hat{\boldsymbol{\phi}} . \end{aligned} \tag{1.114}$$

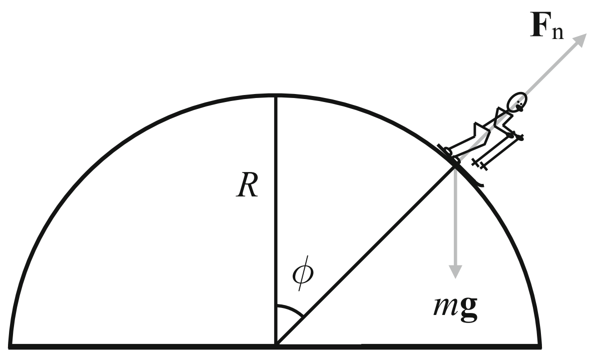
Remembering that it is easiest to use the Cartesian basis when computing the acceleration, we find

$$\mathbf{a} = -R\dot{\phi}^2\hat{\boldsymbol{\rho}} + R\ddot{\phi}\hat{\boldsymbol{\phi}}. \tag{1.115}$$

Thus, the equations of motion are obtained from Newton's 2nd law, giving

$$F_n - mg\cos\phi = -mR\dot{\phi}^2 \tag{1.116a}$$
$$mg\sin\phi = mR\ddot{\phi} . \tag{1.116b}$$

Fig. 1.14 A skier going down a spherical hill feels a weight force pointing downward ($-\hat{\mathbf{y}}$ direction) and a normal force pointing along the $\hat{\boldsymbol{\rho}}$ direction. The motion is constrained to the surface of the hill



Now, the skier leaves the surface when $F_n = 0$. If we knew $\dot{\phi}$, then we could solve (1.116a) for the value of ϕ at which the skier leaves the surface.

Equation (1.116b) provides the differential equation to solve for ϕ. There is a nice trick for solving this differential equation that involves converting it to a 1st-order differential equation. First, we multiply both sides by $\dot{\phi}$, to find

$$\dot{\phi}\ddot{\phi} = \frac{g}{R}\dot{\phi}\sin\phi\,. \tag{1.117}$$

The left-hand side is the total time derivative of $\frac{1}{2}\dot{\phi}^2$, while the right-hand side is the total time derivative of $-(g/R)\cos\phi$. We can then integrate both sides to find

$$\frac{1}{2}\dot{\phi}^2 = -\frac{g}{R}\cos\phi + C\,, \tag{1.118}$$

where C is an arbitrary constant that is fixed by the initial conditions. The skier starts from rest at $\phi = 0$, so $\dot{\phi} = 0$ when $\cos\phi = 1$. Thus, the arbitrary constant is $C = g/R$. We now have the solution for $\dot{\phi}^2$ that we need for (1.116a),

$$\dot{\phi}^2 = \frac{2g}{R}\left(1 - \cos\phi\right)\,. \tag{1.119}$$

Substituting this expression into (1.116a) and setting $F_n = 0$ gives the solution for the angle at which the skier leaves the surface,

$$\cos\phi = \frac{2}{3}\,. \tag{1.120}$$

□

In the previous example, the equation for the angular velocity (1.119) could also be obtained using conservation of mechanical energy, and that is frequently how this problem is solved in introductory physics courses. This should be somewhat surprising, however, since the force of constraint is not conservative and one of the requirements for conservation of mechanical energy is that all forces be conservative. This requirement allows the work done by the forces to be expressed as the difference between the values of the potential at the initial and final states. In this example, the constraint force is the normal force, and so it is always perpendicular to the constrained motion. Thus, the constraint force does no work on the system and does not contribute a change in mechanical energy of the system. Therefore, we can equate the initial mechanical energy to the final mechanical energy,

$$E_{\mathrm{i}} = mgR = mgR\cos\phi + \frac{1}{2}mR^2\dot{\phi}^2 = E_{\mathrm{f}}\,. \tag{1.121}$$

Solving for $\dot{\phi}^2$, we recover (1.119).

Once we have the equation for $\dot{\phi}^2$, we can also integrate it to find the time-dependent solution $\phi(t)$:

$$\int_0^{\phi} \frac{\mathrm{d}\phi}{A\sqrt{1-\cos\phi}} = \int_0^t \mathrm{d}t\,, \tag{1.122}$$

where $A \equiv \sqrt{2g/R}$. Although it is not trivial, this integral can be done (or looked up in a handbook of integrals), and we get

$$\sqrt{\frac{R}{g}}\left[\ln\left(\tan\left(\frac{\phi}{4}\right)\right) - \ln\left(\tan\left(0\right)\right)\right] = t\,. \tag{1.123}$$

This all looks fine until we try to solve this equation for $\phi(t)$ and notice that $\ln(\tan(0)) \rightarrow -\infty$. What does this mean? A direct interpretation is that it will take an infinite amount of time for the skier to reach any angle ϕ. Upon further reflection, we see that the math is telling us something that we have been overlooking. Namely, the skier starts with an initial velocity of $v = R\dot{\phi} = 0$ at the top of the hill where the forces are in equilibrium, so the acceleration is zero. The skier isn't going anywhere until someone pushes her!

Exercise 1.18 Redo the problem for the skier in Example 1.7, but allowing for a non-zero initial velocity v_0, and determine how the angle at which the skier leaves the slope depends on the initial velocity. What is the maximum value of v_0 allowed for the skier to be on the ground at the top of the hill?

You may have noticed that direct application of Newton's laws to constrained systems frequently involves a lot of algebra and the careful solution of multiple equations with multiple unknowns to obtain the equations of motion and the equations for the constraint forces. A great deal of mathematical machinery has been developed to streamline and simplify the analysis of constrained systems with general coordinate systems. We will explore these in the next chapter.

Suggested References

Full references are given in the bibliography at the end of the book.

Fetter and Walecka (1980): Although more advanced, the first two chapters provide a thorough review of mechanics and non-inertial reference frames.

Marion and Thornton (1995): An excellent introductory text on classical mechanics, particularly suited for undergraduates.

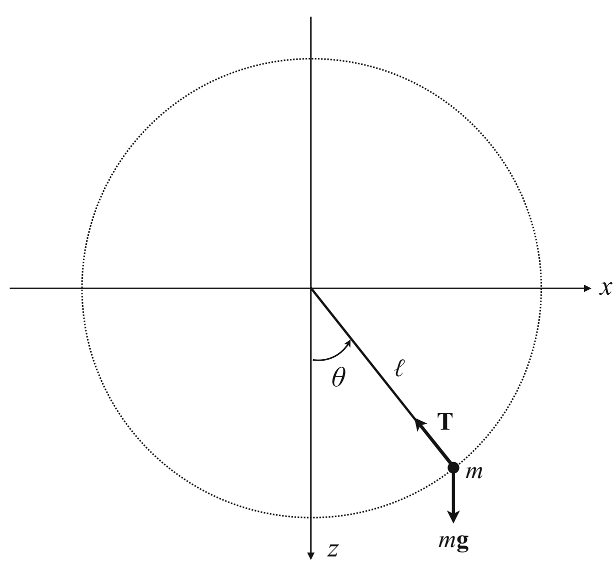
Additional Problems

Problem 1.1 Extend the calculation of Exercise 1.17 to obtain the acceleration vector in spherical coordinates (r, θ, ϕ) for *unconstrained* motion in three dimensions—that is, allowing the radial coordinate r to also change with time.

Problem 1.2 Calculate the acceleration vector for unconstrained motion in three dimensions in cylindrical coordinates (ρ, ϕ, z).

Problem 1.3 Consider a simple planar pendulum consisting of a mass m suspended from a (massless) string of length ℓ in a uniform gravitational field $\mathbf{g}$. (See Fig. 1.15.) Let $\mathbf{T}$ denote the tension in the string and v_0 denote the initial velocity of the pendulum bob—i.e., the tangential velocity at its lowest point $\theta = 0$.

Fig. 1.15 A pendulum bob of mass m is suspended from a (massless) string of length ℓ in a uniform gravitational field $\mathbf{g}$. The gravitational force $m\mathbf{g}$ and the tension $\mathbf{T}$ exerted by the string are shown in the figure



(a) Obtain an equation for the tension T as a function of θ and $\dot{\theta}$.
(b) Integrate the $\ddot{\theta}$ equation to obtain an equation relating θ to $\dot{\theta}$. This equation involves an integration constant that can solved for in terms of the initial velocity v_0. Interpret the equation in terms of the total energy of the pendulum.
(c) Determine the maximum value of θ having $\dot{\theta} = 0$ and $T \geq 0$.
(d) Determine the minimum initial velocity v_0 needed for the pendulum bob to make a complete loop-the-loop—i.e., to reach the top of the circle ($\theta = \pi$) with $T \geq 0$. What does $\dot{\theta}$ equal at the top of the circle for this minimum-initial-velocity case?

Problem 1.4 The planar **double pendulum** consists of two point masses (m_1 and m_2) at the end of two massless rigid rods of lengths ℓ_1 and ℓ_2 as shown in Fig. 1.16. If we choose Cartesian coordinates with the x-axis pointing down and the y-axis pointing to the right, then the constraints can be incorporated into the positions of the particles with

$$\begin{aligned}\mathbf{r}_1 &= \ell_1 \cos\phi_1 \hat{\mathbf{x}} + \ell_1 \sin\phi_1 \hat{\mathbf{y}}\,,\\ \mathbf{r}_2 &= (\ell_1 \cos\phi_1 + \ell_2 \cos\phi_2)\,\hat{\mathbf{x}} + (\ell_1 \sin\phi_1 + \ell_2 \sin\phi_2)\,\hat{\mathbf{y}}\,,\end{aligned} \tag{1.124}$$

which reduces the number of degrees of freedom from four (x_1, y_1, x_2, y_2) to two (ϕ_1, ϕ_2).

(a) Apply Newton's 2nd law to each mass and show that the magnitudes of the constraint forces $\mathbf{T}_1$ and $\mathbf{T}_2$ obey

$$\begin{aligned}T_1 \sin\phi_1 &= -(m_1+m_2)\ell_1\left(\ddot{\phi}_1\cos\phi_1 - \dot{\phi}_1^2\sin\phi_1\right) - m_2\ell_2\left(\ddot{\phi}_2\cos\phi_2 - \dot{\phi}_2^2\sin\phi_2\right)\,,\\ T_2 \sin\phi_2 &= -m_2\ell_1\left(\ddot{\phi}_1\cos\phi_1 - \dot{\phi}_1^2\sin\phi_1\right) - m_2\ell_2\left(\ddot{\phi}_2\cos\phi_2 - \dot{\phi}_2^2\sin\phi_2\right)\,.\end{aligned} \tag{1.125}$$

(b) Use the result from part (a) to obtain the following equations of motion

$$\begin{aligned}g\sin\phi_1 &= -\ell_1\ddot{\phi}_1 - \frac{m_2}{m_1+m_2}\ell_2\left(\ddot{\phi}_2\cos(\phi_1-\phi_2) + \dot{\phi}_2^2\sin(\phi_1-\phi_2)\right)\,,\\ g\sin\phi_2 &= -\ell_1\ddot{\phi}_1\cos(\phi_1-\phi_2) + \ell_1\dot{\phi}_1^2\sin(\phi_1-\phi_2) - \ell_2\ddot{\phi}_2\,.\end{aligned} \tag{1.126}$$

Note that the equations for $\phi_1(t)$, $\phi_2(t)$ must be solved numerically.

Problem 1.5 (*Adapted from Kuchăr (1995).*) Consider a cylindrical bucket of radius R, with water filled to height h (significantly less than the height of the bucket). The bucket is then rotated uniformly around its axis with angular velocity ω. Determine the shape $z = f(r)$ of the surface of water in the rotating bucket, as a function of the perpendicular distance r from the axis. You should find

$$z = h + \frac{\omega^2}{2g}\left(r^2 - \frac{R^2}{2}\right)\,. \tag{1.127}$$

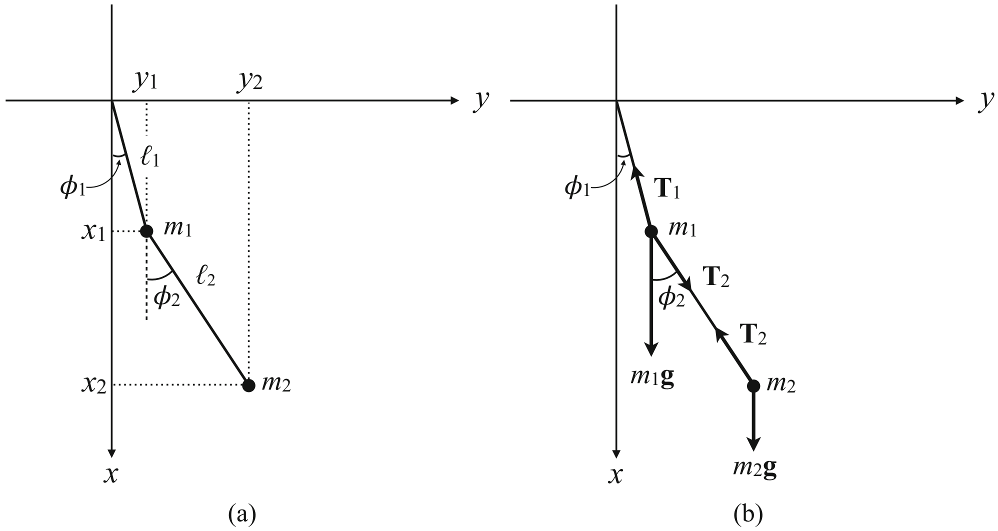


Fig. 1.16 The double pendulum. Panel (a): The two masses, m_1 and m_2, are constrained by the two massless rigid rods. The four degrees of freedom can be reduced to two (ϕ_1 and ϕ_2) due to these constraints. Panel (b) shows the two constraint forces $\mathbf{T}_1$ and $\mathbf{T}_2$, and the gravitational forces $m_1\mathbf{g}$ and $m_2\mathbf{g}$

Hint: Minimize the sum of the gravitational and centrifugal potential energies of a cylindrical shell of water of mass $\mathrm{d}m = \rho 2\pi r z(r)\mathrm{d}r$ in the non-inertial reference frame rotating with the water, subject to the constraint

$$\int_0^R 2\pi r z(r)\mathrm{d}r = \pi R^2 h\,. \tag{1.128}$$

See Appendix C.8 if you need a refresher on variational problems subject to constraints.

Problem 1.6 A projectile is launched vertically from the equator with an initial speed v_0. We want to find out where it will land, assuming that we can approximate the gravitational force as uniform, with $\mathbf{F} = m\mathbf{g}_0$.

(a) Starting with (1.80) and using the coordinates shown in Fig. 1.10 (with $\theta = \pi/2$), show that the equations of motion for the projectile are

$$\begin{aligned} \ddot{x} &= 0\,, \\ \ddot{y} &= \omega^2 y - 2\omega\dot{z}\,, \\ \ddot{z} &= \left(\omega^2 R - g_0\right) + \omega^2 z + 2\omega\dot{y}\,. \end{aligned} \tag{1.129}$$

(b) In the absence of the Earth's rotation, $\omega = 0$ and the unperturbed trajectory follows $\ddot{y}_0 = 0$ and $\ddot{z}_0 = -g_0$, so $y_0(t) = 0$ and $z_0(t) = v_0 t - \frac{1}{2} g_0 t^2$. Write the perturbed trajectory as

$$y(t) = \psi(t)\,, \qquad z(t) = z_0(t) + \zeta(t)\,, \tag{1.130}$$

where the perturbations $\psi(t)$ and $\zeta(t)$ are kept only to 1st-order in ω. Show that to 1st-order in ω, the perturbations obey

$$\ddot{\psi} = -2\omega\dot{z}_0\,, \qquad \ddot{\zeta} = 0\,. \tag{1.131}$$

(c) Show that

$$\psi(t) = \frac{1}{3}\omega g_0 t^3 - \omega v_0 t^2\,. \tag{1.132}$$

(d) If the projectile is launched to the edge of space (an altitude of 100 km), how far away from the launch site does it land? Does it land to the east or to the west?

Problem 1.7 An object is dropped from a point above the equator at an altitude of 100 km. Following the procedure outlined in Problem 1.6, determine where the object lands relative to the point directly below the release point.

Problem 1.8 Generalize the procedure outlined in Problem 1.6 for arbitrary co-latitude θ. In so doing, define

$$\mathbf{g} \equiv \mathbf{g}_0 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})\,, \tag{1.133}$$

which points in the direction of a plumb line located at the origin of the non-inertial reference frame, and which defines the direction of the local vertical, i.e., $\hat{\mathbf{z}} \propto -\mathbf{g}$. For a particle launched in the $\hat{\mathbf{z}}$ direction to an altitude of h, show that:

(a) to 1st-order in ω, the displacement of the projectile in the y-direction is

$$\Delta y = -\frac{8}{3}\sqrt{\frac{2h^3}{g}}\,\omega\sin\theta\,, \tag{1.134}$$

with the minus sign indicating that the projectile lands *west* of the launch site.

(b) to 2nd-order in ω, the displacment of the projectile in the x-direction is

$$\Delta x = -\frac{8h^2}{g}\omega^2\sin\theta\cos\theta\,, \tag{1.135}$$

with the minus sign indicating that the projectile lands *north* of the launch site in the Northern hemisphere and south of launch site in the Southern hemisphere.

(c) Calculate the displacements Δx and Δy for a projectile launched from 26° north latitude and that reaches an altitude of 100 km.