Searches for stochastic gravitational-wave backgrounds

Lecture 2
Les Houches Summer School
20 July 2018

Joe Romano
University of Texas Rio Grande Valley
Texas Tech University

Plan for lectures

Yesterday: Overview / Basics

1. Motivatio / context
2. Different types of stochastic backgrounds
3. Characterizing a stochastic GW background
4. Correlation methods
5. Some simple examples

Today: Details / Example

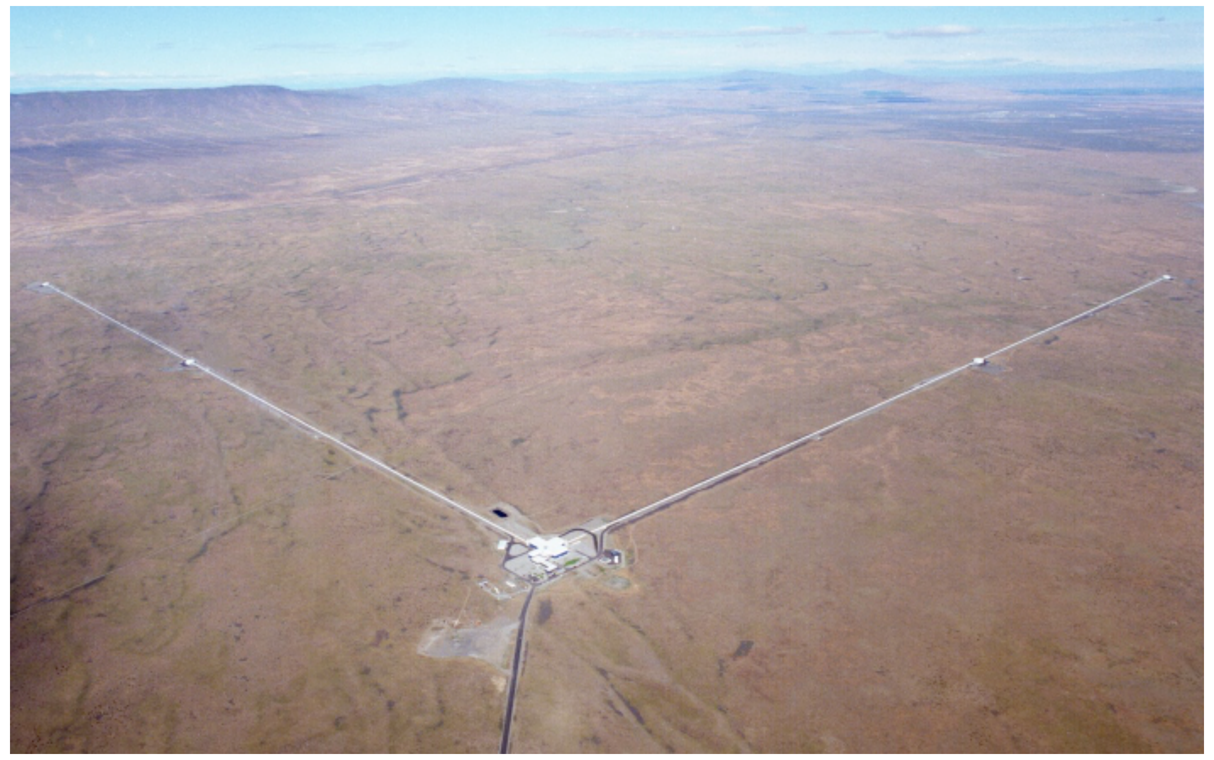
1. **Non-trivial response functions**
2. **Non-trivial overlap functions**
3. **What to do in the absence of correlations (e.g., for LISA)?**
4. **Frequentist and Bayesian methods**
5. **Example: searching for the background from BBH mergers**

1. Non-trivial response functions

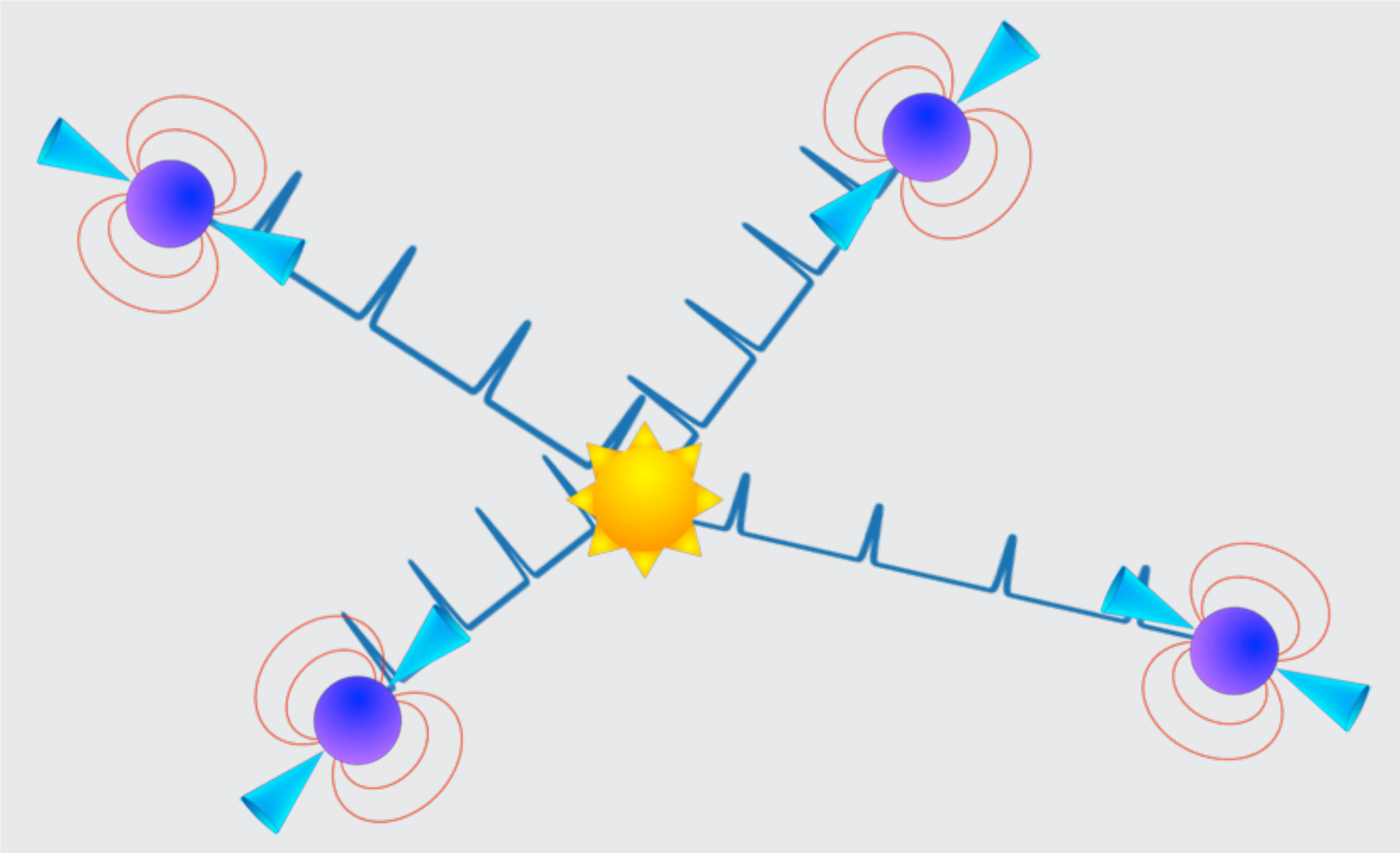
Beam detectors

(use electromagnetic radiation to monitor the separation of two or more test masses)

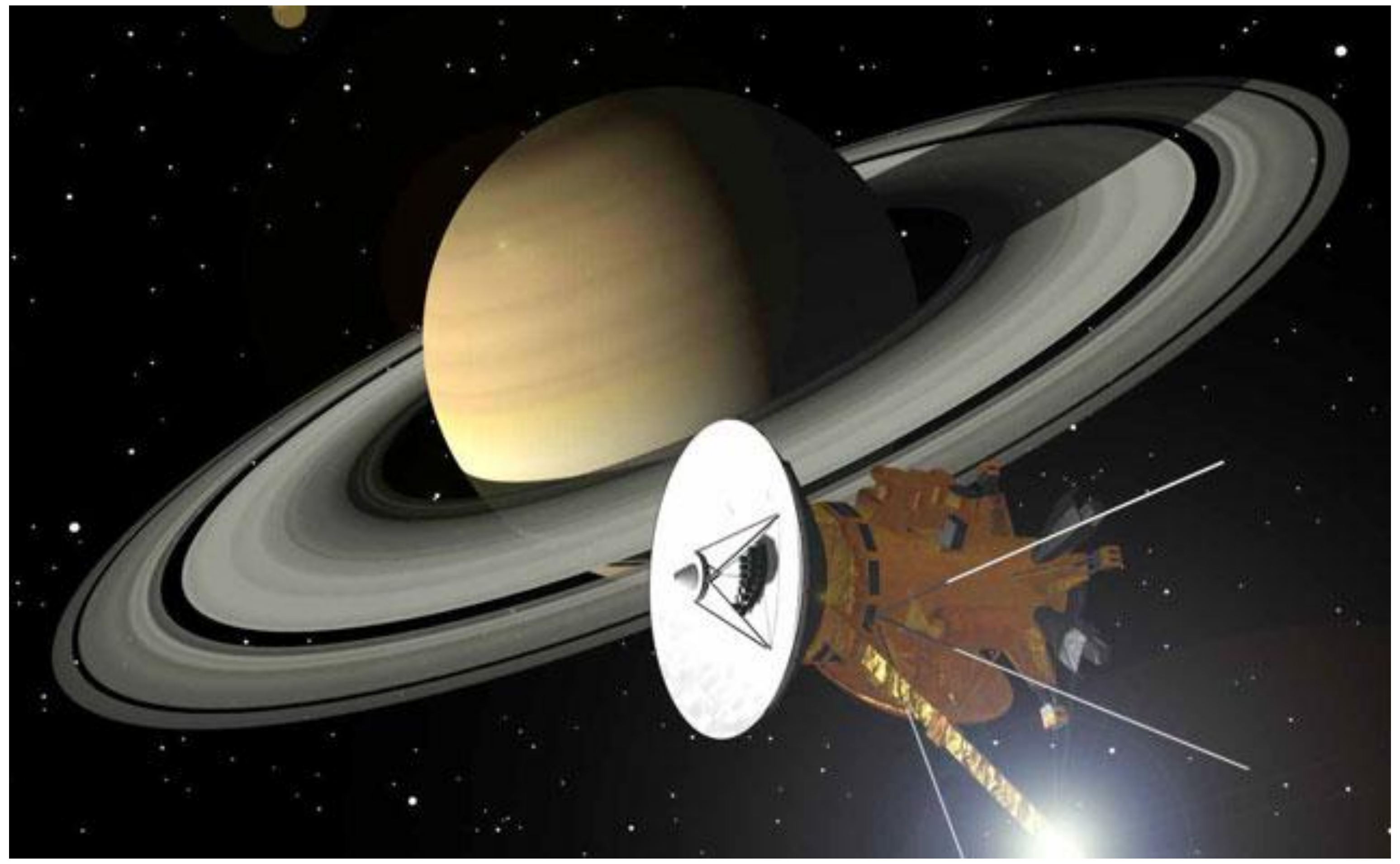
laser interferometers

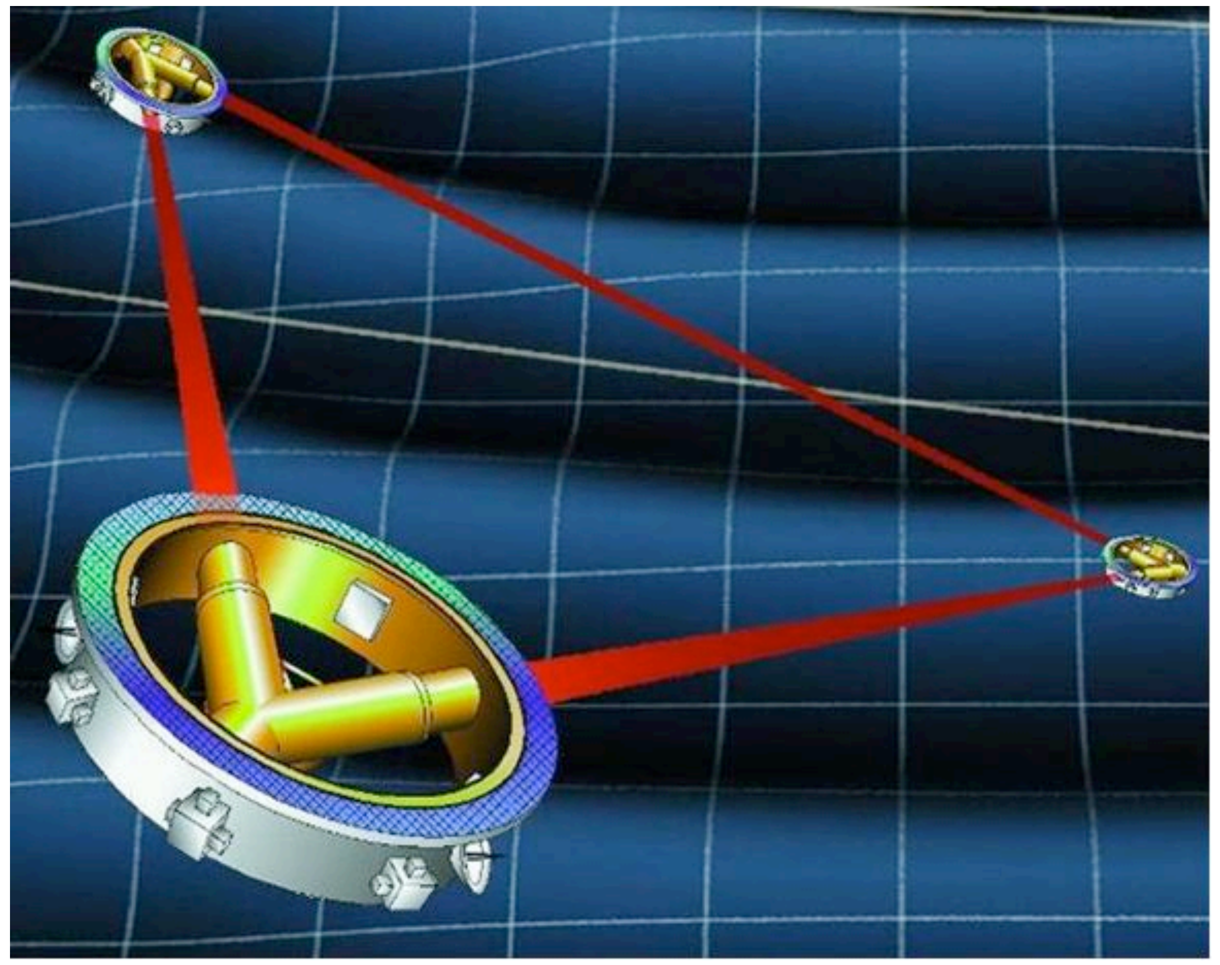


pulsar timing

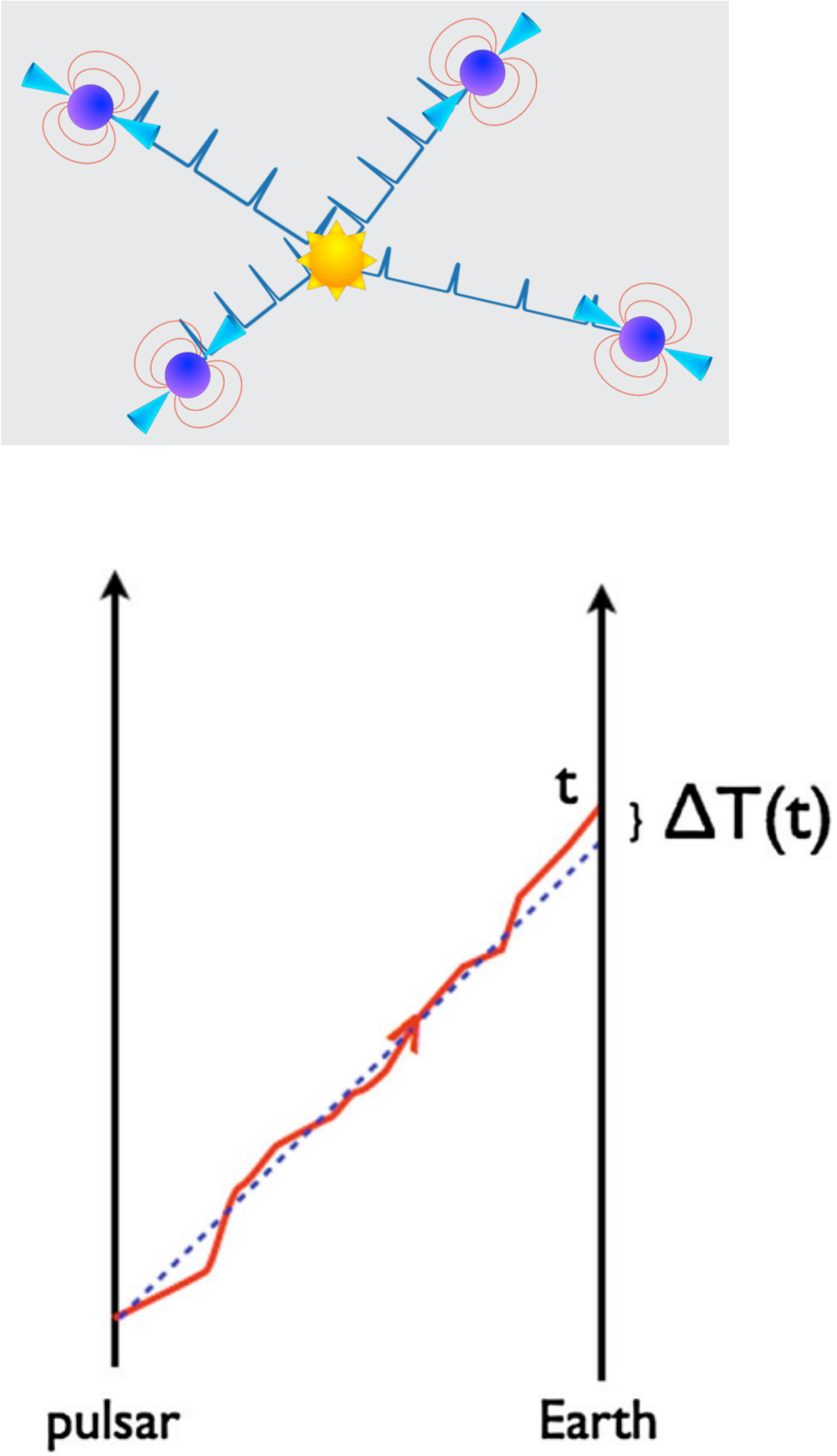


spacecraft Doppler tracking

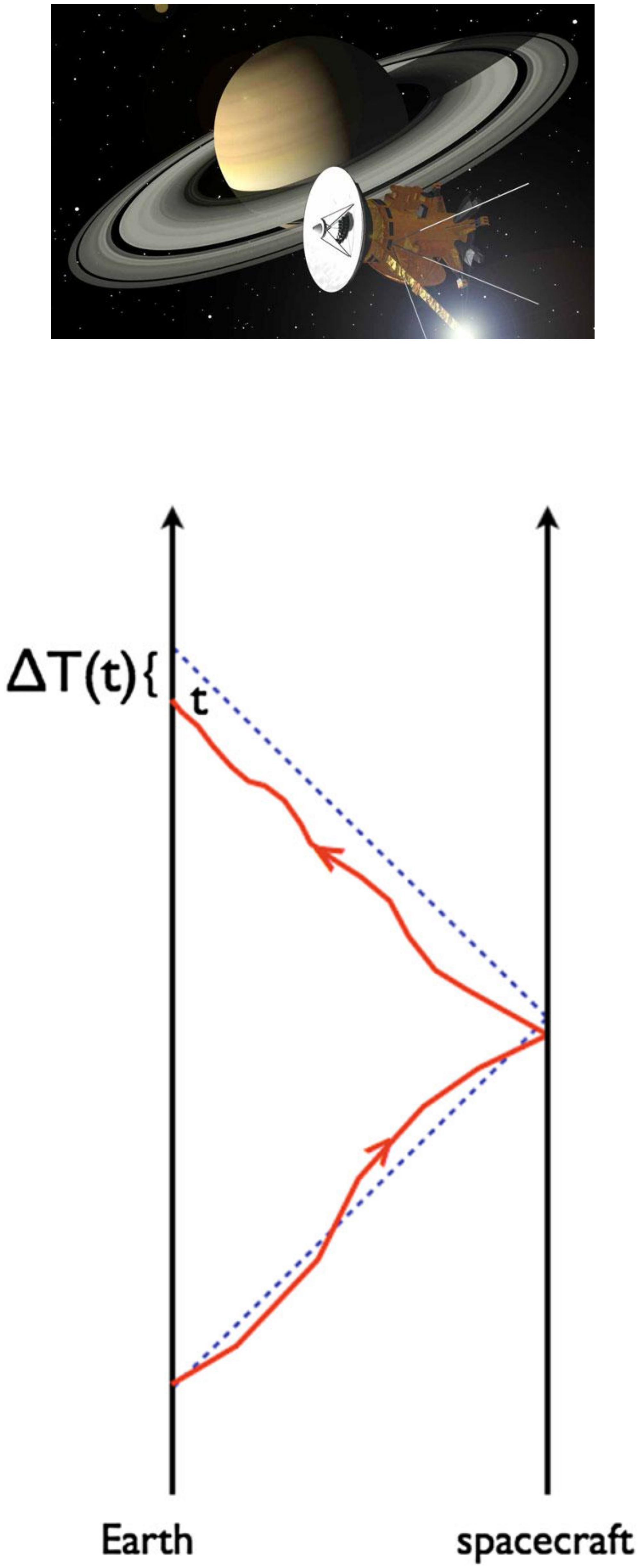




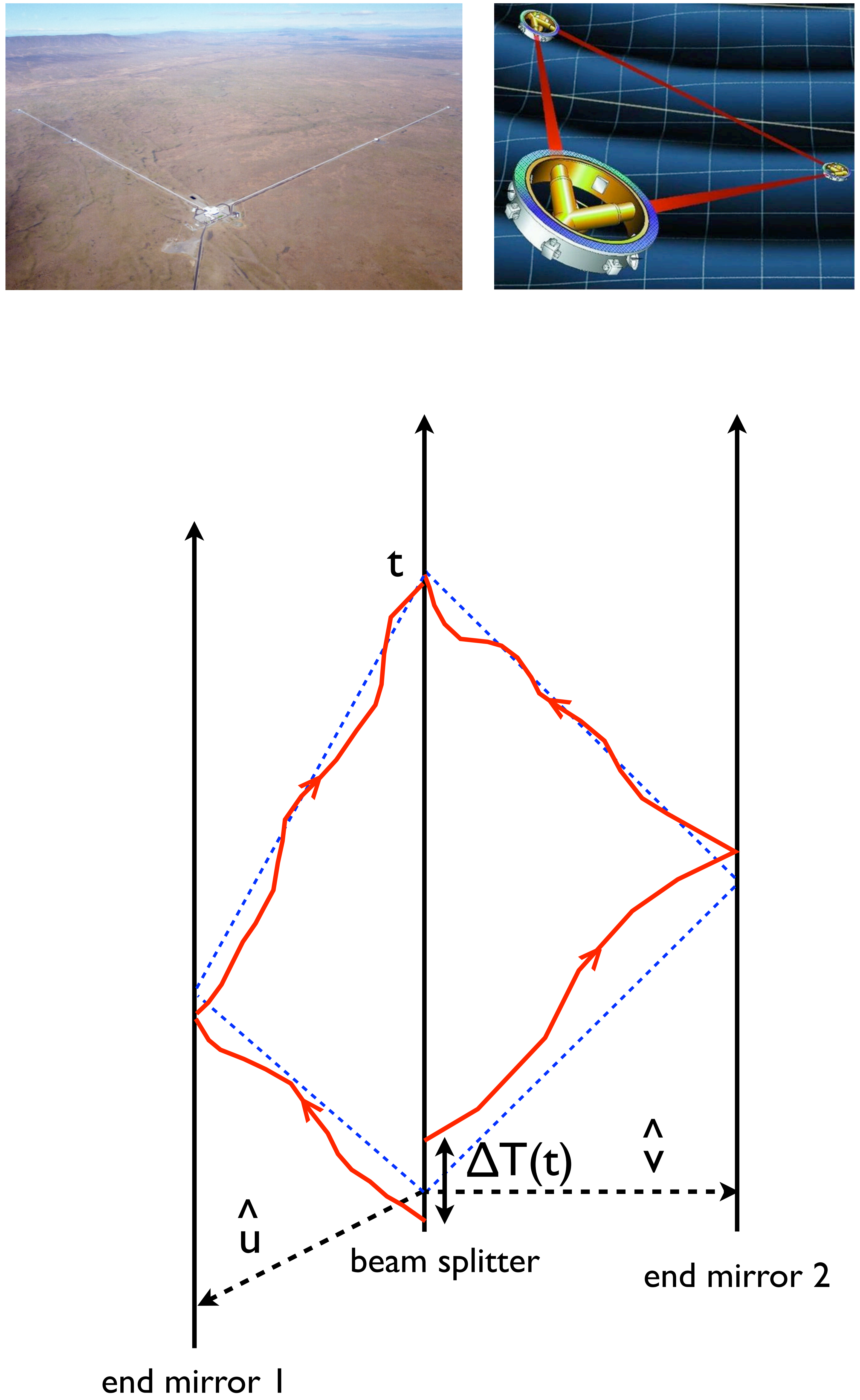
GW perturbs the photon propagation time between the test masses



(1-arm, 1-way)



(1-arm, 2-way)



(2-arm, 2-way)

Different types of response

timing: $h(t) \equiv \Delta T(t)$ (pulsar timing)

strain: $h(t) \equiv \dfrac{\Delta L(t)}{L} = \dfrac{\Delta T(t)}{T}$ (LIGO, Virgo, …)

Doppler frequency: $h(t) \equiv \dfrac{\Delta \nu(t)}{\nu_0} = \dfrac{d\Delta T(t)}{dt}$ (pulsar timing, spacecraft Doppler tracking)

phase: $h(t) \equiv \Delta \Phi(t) = 2\pi\nu_0 \, \Delta T(t)$ (LISA)

All these responses are derivable from the change in light travel time ΔT(t)

Detector response

GWs are weak => detector is a linear system which converts metric perturbations to detector output

detector

$$h_{ab}(t, \vec{x}) \longrightarrow \boxed{R^{ab}(\tau, \vec{y})} \longrightarrow h(t)$$

convolution

$$h(t) = (\mathbf{R} * \mathbf{h})(t, \vec{x}) \equiv \int_{-\infty}^{\infty} d\tau \int d^3y\, R^{ab}(\tau, \vec{y}) h_{ab}(t - \tau, \vec{x} - \vec{y})$$

detector output

detector
location

impulse response

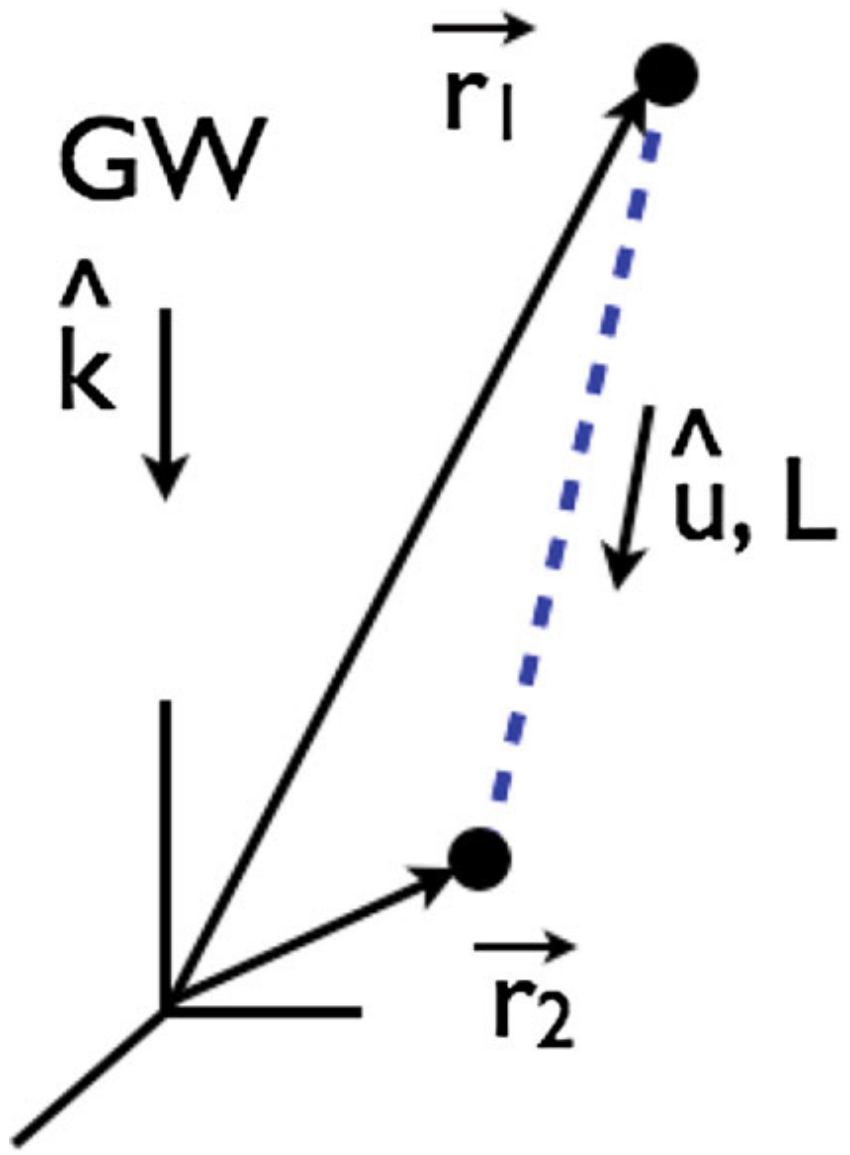
$$\Longrightarrow \quad \tilde{h}(f) = \int \mathrm{d}^2\Omega_{\hat{n}} \sum_A \boxed{R^A(f, \hat{n})}\, h_A(f, \hat{n})$$

detector response for a plane-wave
with frequency f, direction n, polarization A

$$R^A(f, \hat{n}) \equiv R^{ab}(f, \hat{n}) e^A_{ab}(\hat{n})$$

$$R^{ab}(f, \hat{n}) \equiv e^{i2\pi f \hat{n}\cdot\vec{x}/c} \int_{-\infty}^{\infty} \mathrm{d}\tau \int \mathrm{d}^3y\, R^{ab}(\tau, \vec{y})\, e^{-i2\pi f(\tau + \hat{n}\cdot\vec{y}/c)}$$

Example: 1-arm, 1-way timing response function (e.g., pulsar timing)



$$h(t) \equiv \Delta T(t) = \frac{1}{2c} u^a u^b \int_0^L \mathrm{d}s\, h_{ab}(t(s), \vec{x}(s))$$

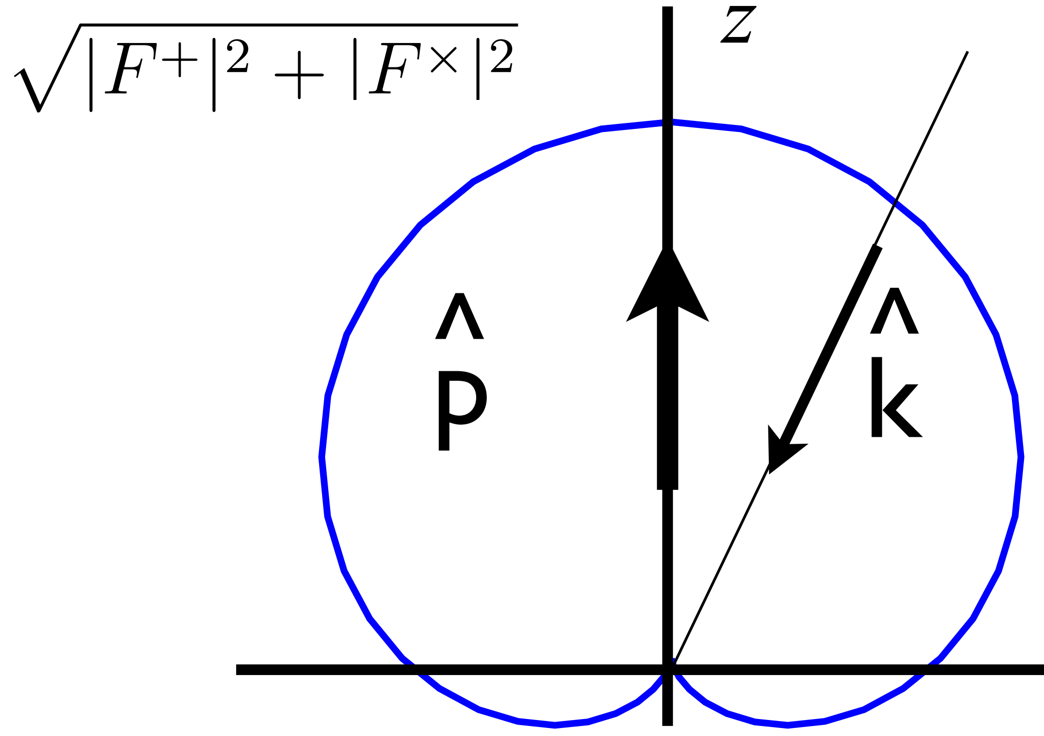
where

$$t(s) = (t - L/c) + s/c\,, \qquad \vec{x}(s) = \vec{r}_1 + s\hat{u}$$

$$R^A(f,\hat{n}) = \underbrace{\frac{1}{i2\pi f}}_{=1 \text{ for Doppler freq measurement}} \underbrace{\boxed{\frac{1}{2} u^a u^b e^A_{ab}(\hat{n}) \frac{1}{1+\hat{n}\cdot\hat{u}}}}_{F^A(\hat{n})} \Big[\overbrace{1}^{\text{earth term}} - \overbrace{e^{-\frac{i2\pi f L}{c}(1+\hat{n}\cdot\hat{u})}}^{\text{pulsar term}} \Big]\, e^{i2\pi f \hat{n}\cdot\vec{r}_2/c}$$

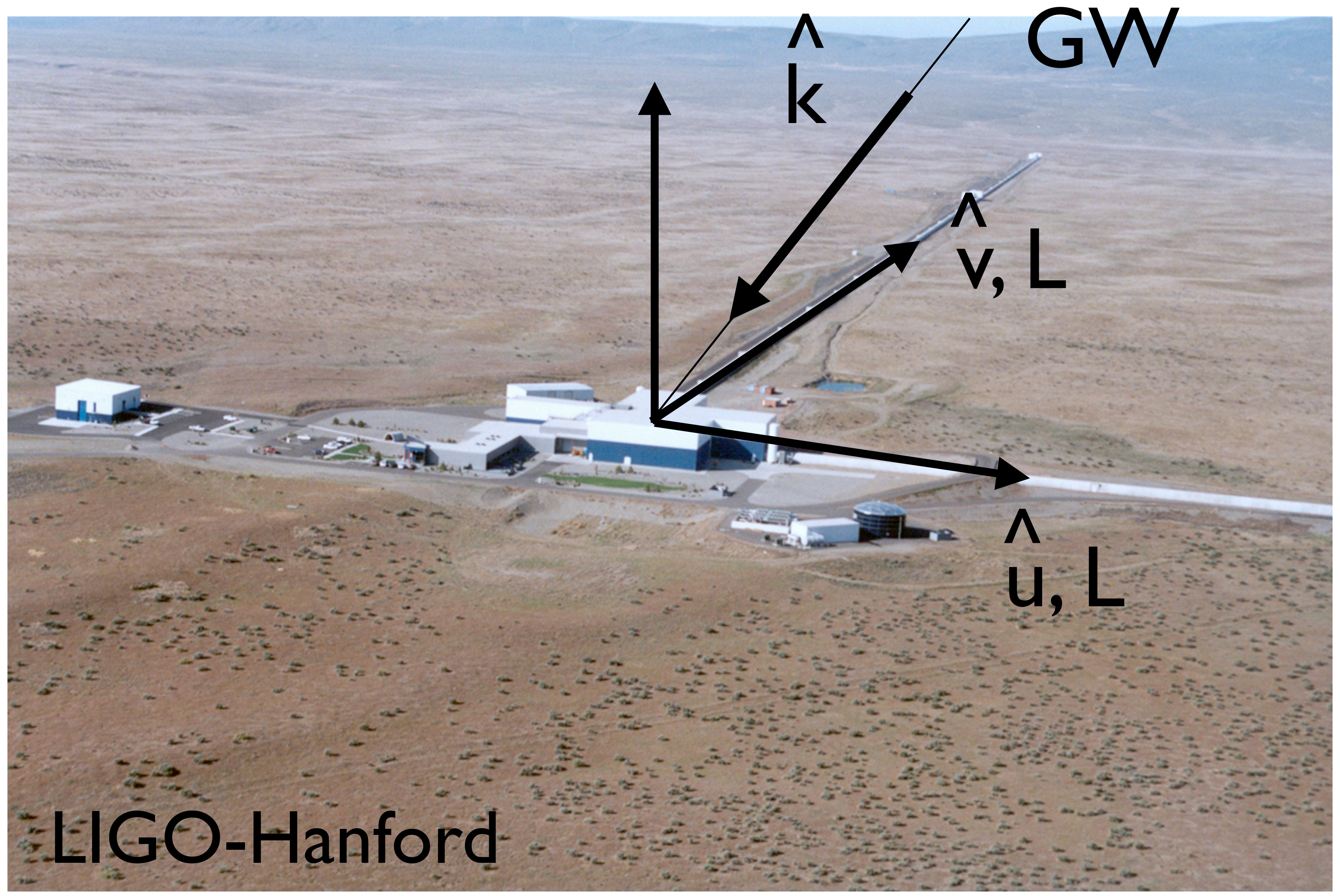
(typically ignored for pulsar timing)

Exercise 6: Derive this expression for the response function



$\hat{p} = -\hat{u}$
$\hat{k} = -\hat{n}$

Example: LIGO response (equal-arm, short-antenna limit)

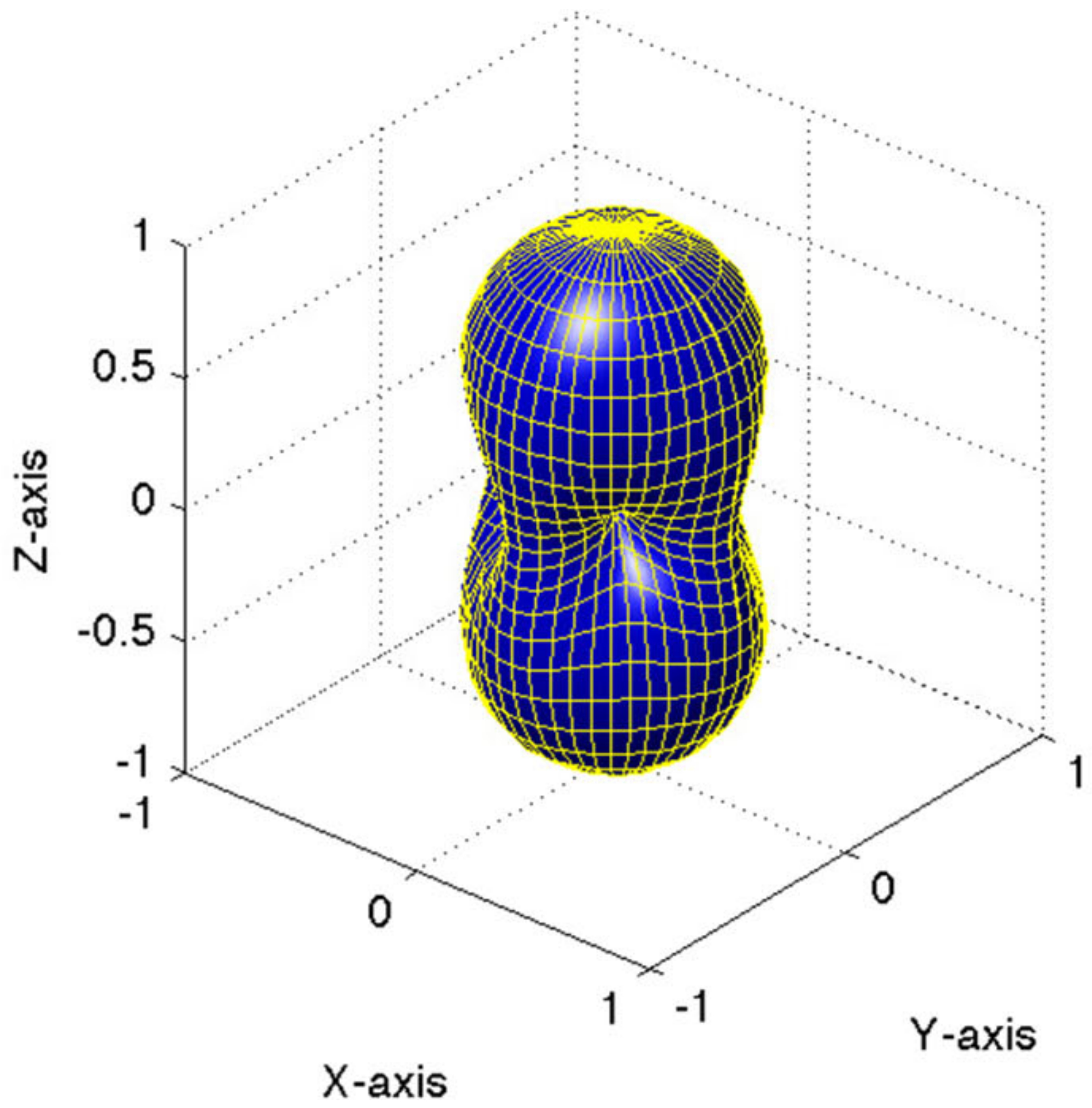


$$h(t) = \frac{1}{2}\left(\frac{\Delta T_{\vec{u},\,\mathrm{roundtrip}}(t)}{T} - \frac{\Delta T_{\vec{v},\,\mathrm{roundtrip}}(t)}{T}\right)$$

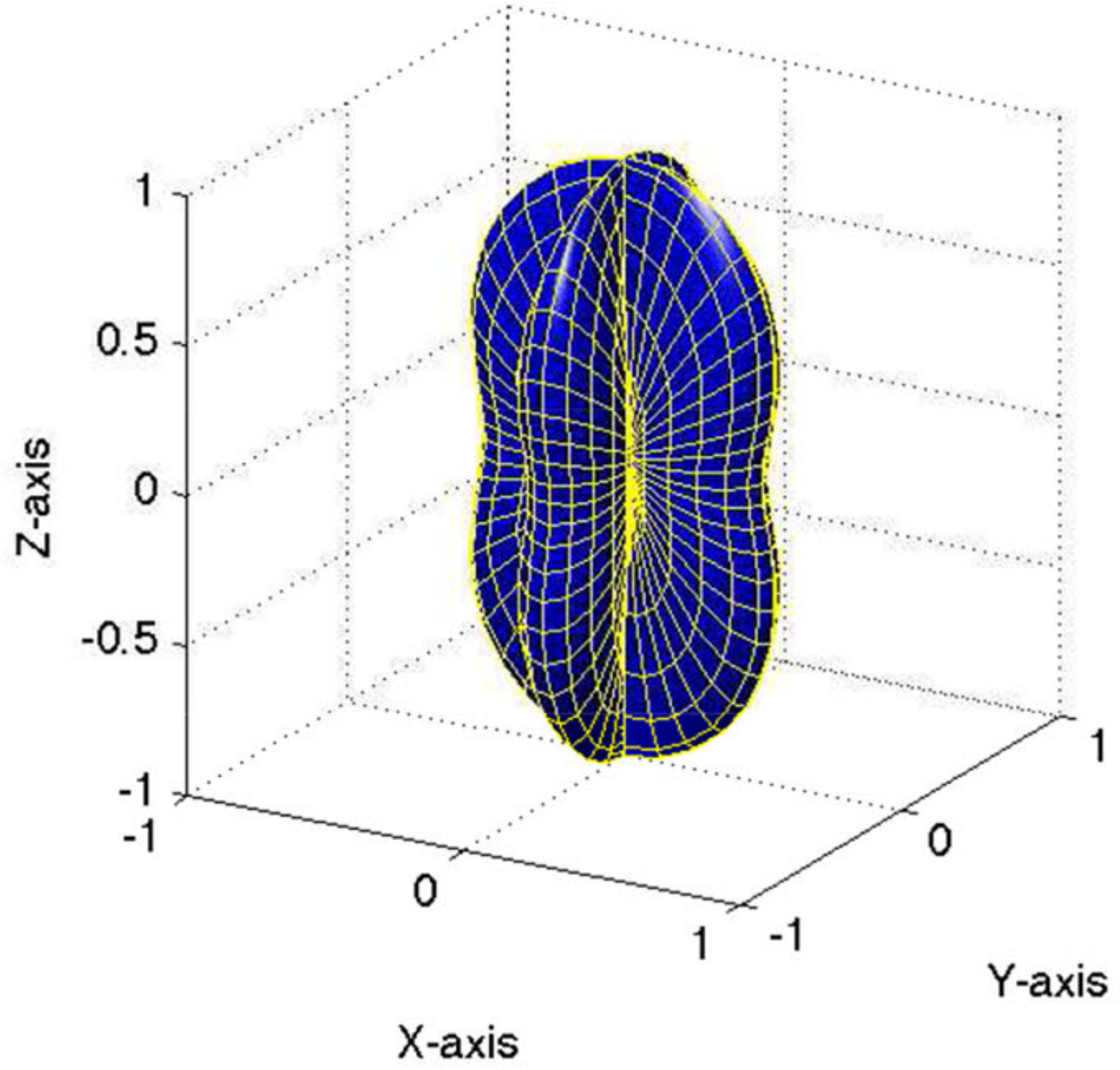
$$R^A(f,\hat{n}) \simeq \boxed{\frac{1}{2}\left(u^a u^b - v^a v^b\right)}\, e^A_{ab}(\hat{n})$$

detector tensor

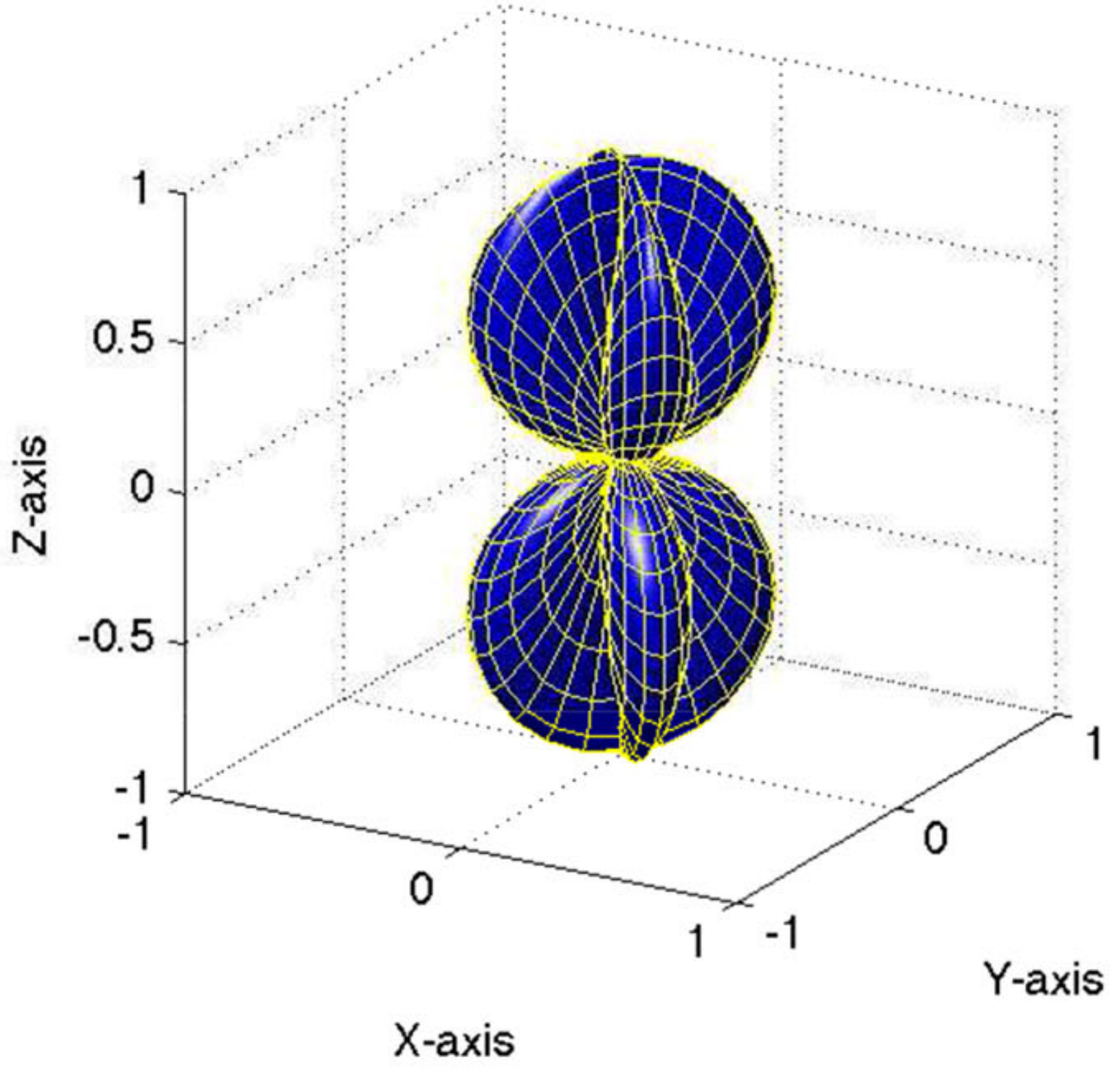
$$\sqrt{|R^+|^2 + |R^\times|^2}$$



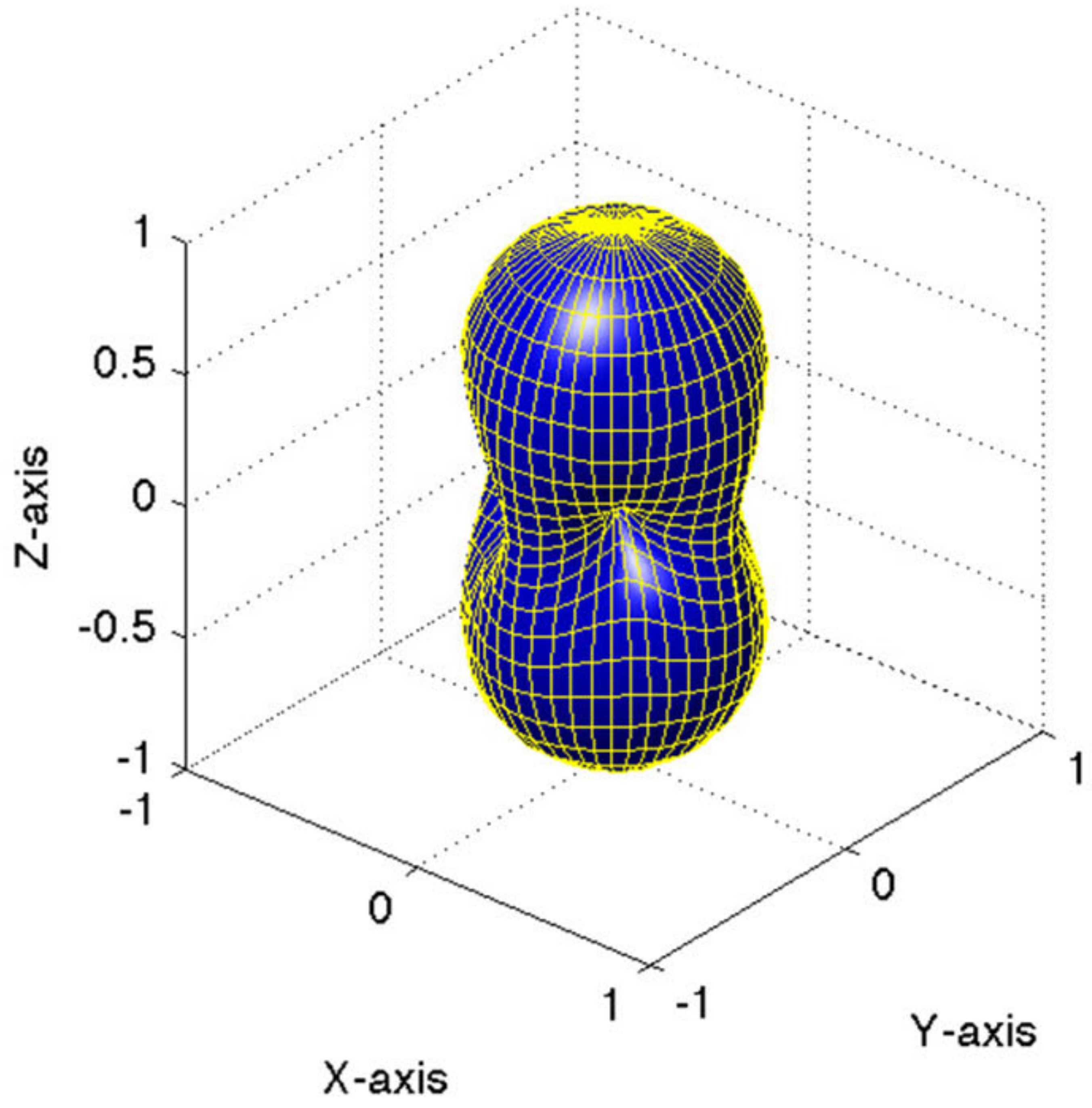
Beam pattern functions



$|R^{+}(f,\hat{n})|$



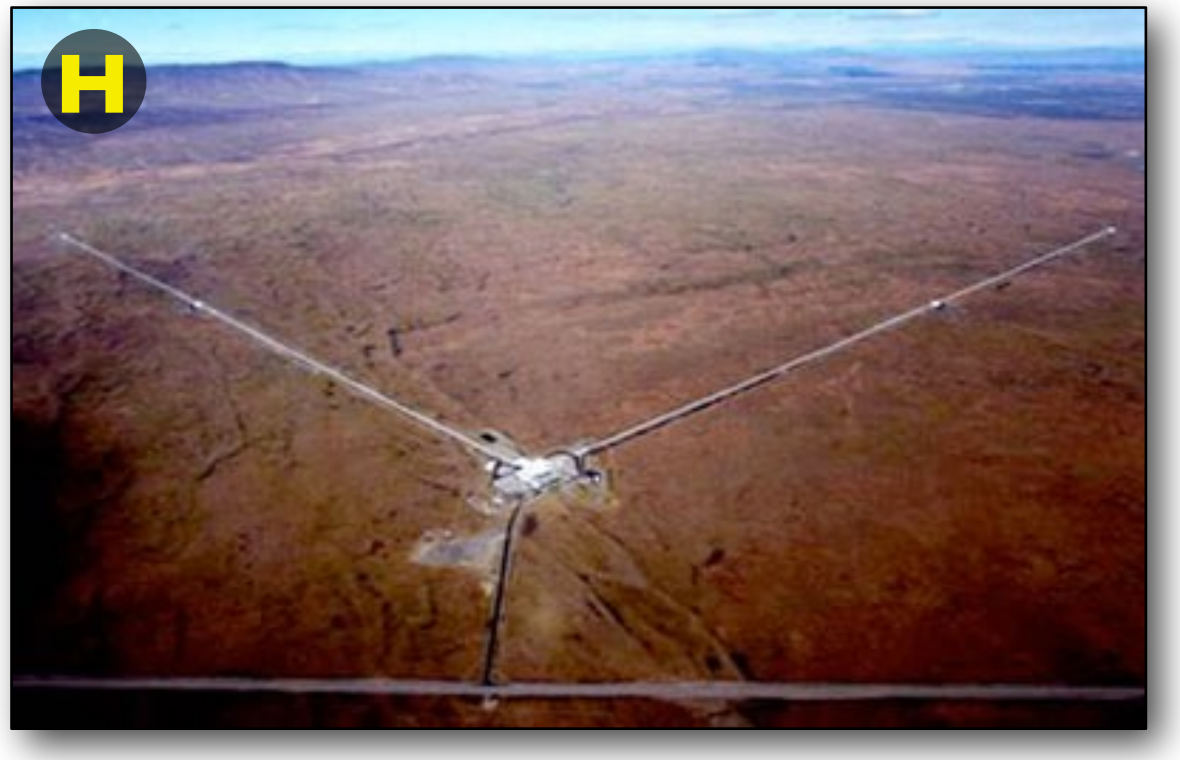
$|R^{\times}(f,\hat{n})|$

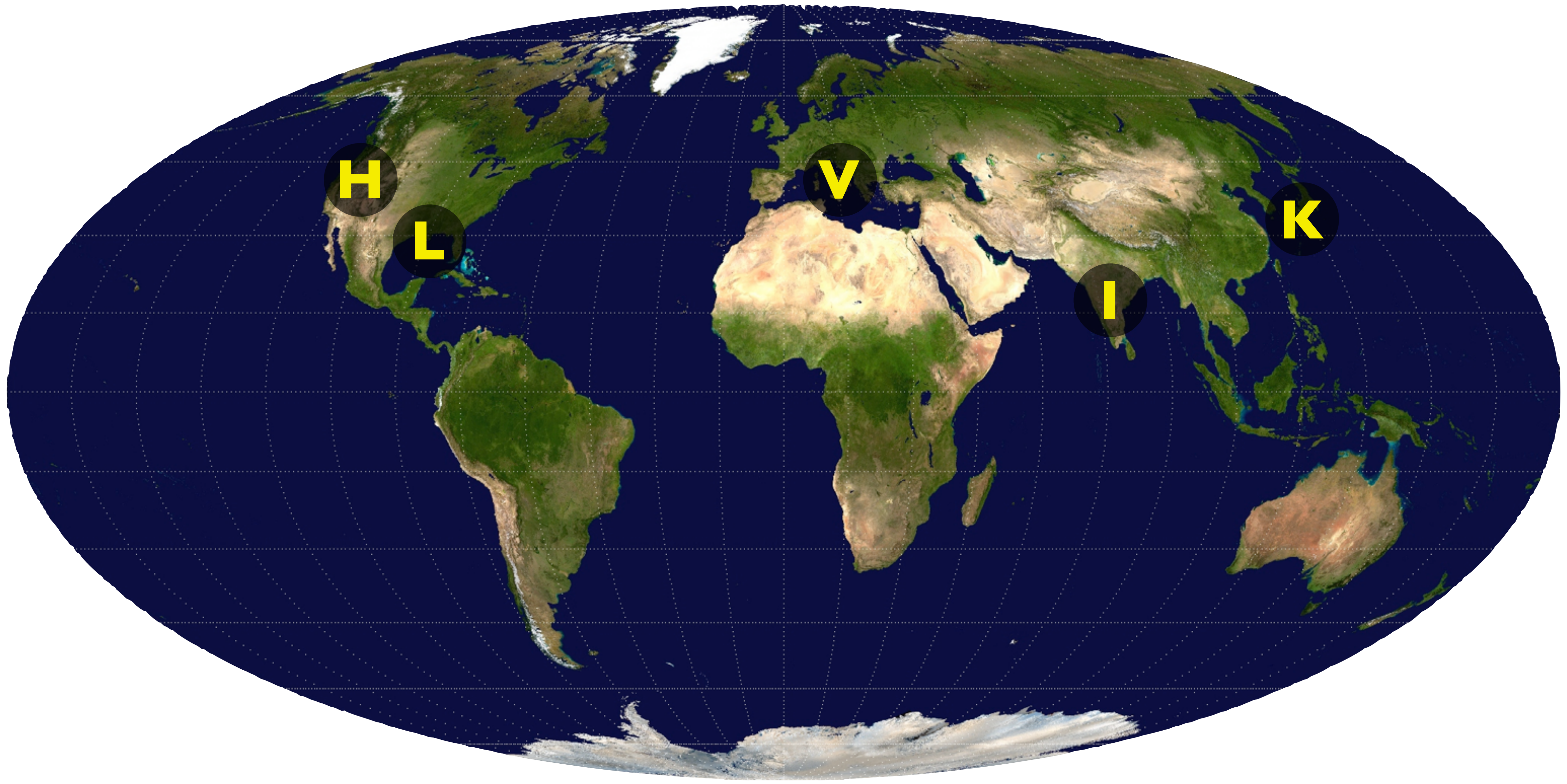


$\left(|R^{+}(f,\hat{n})|^2+|R^{\times}(f,\hat{n})|^2\right)^{1/2}$

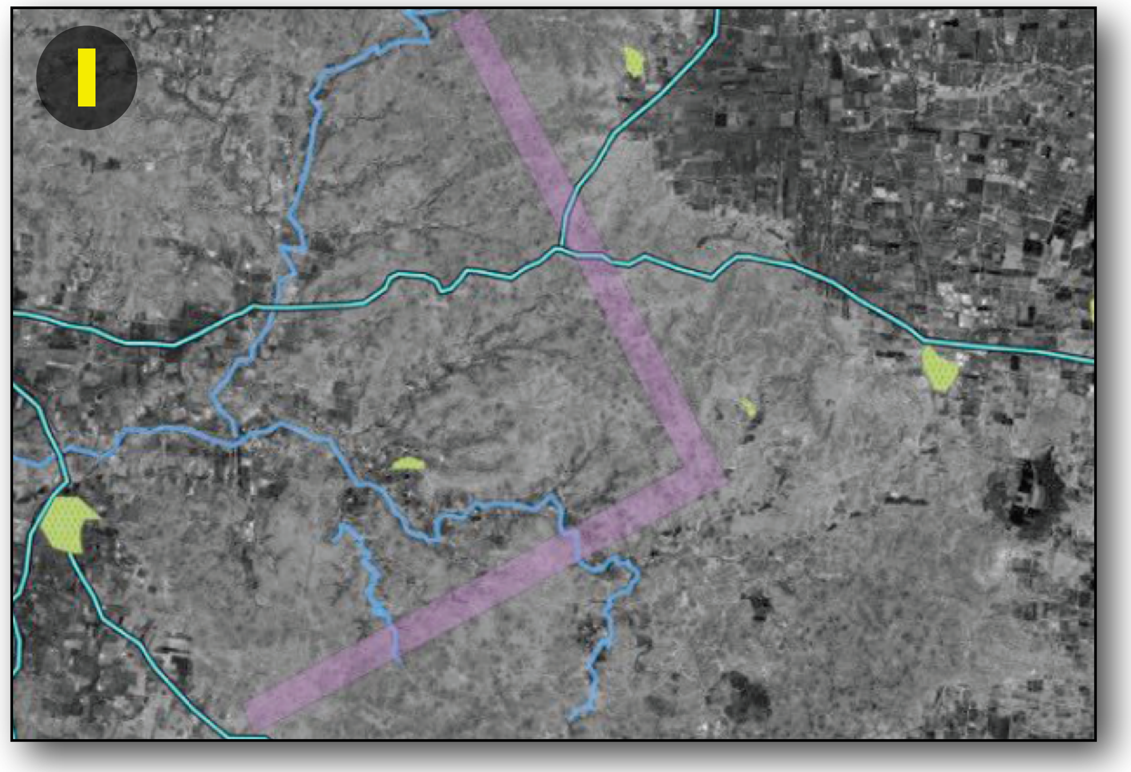
(f < a few kHz)

Terrestrial Network

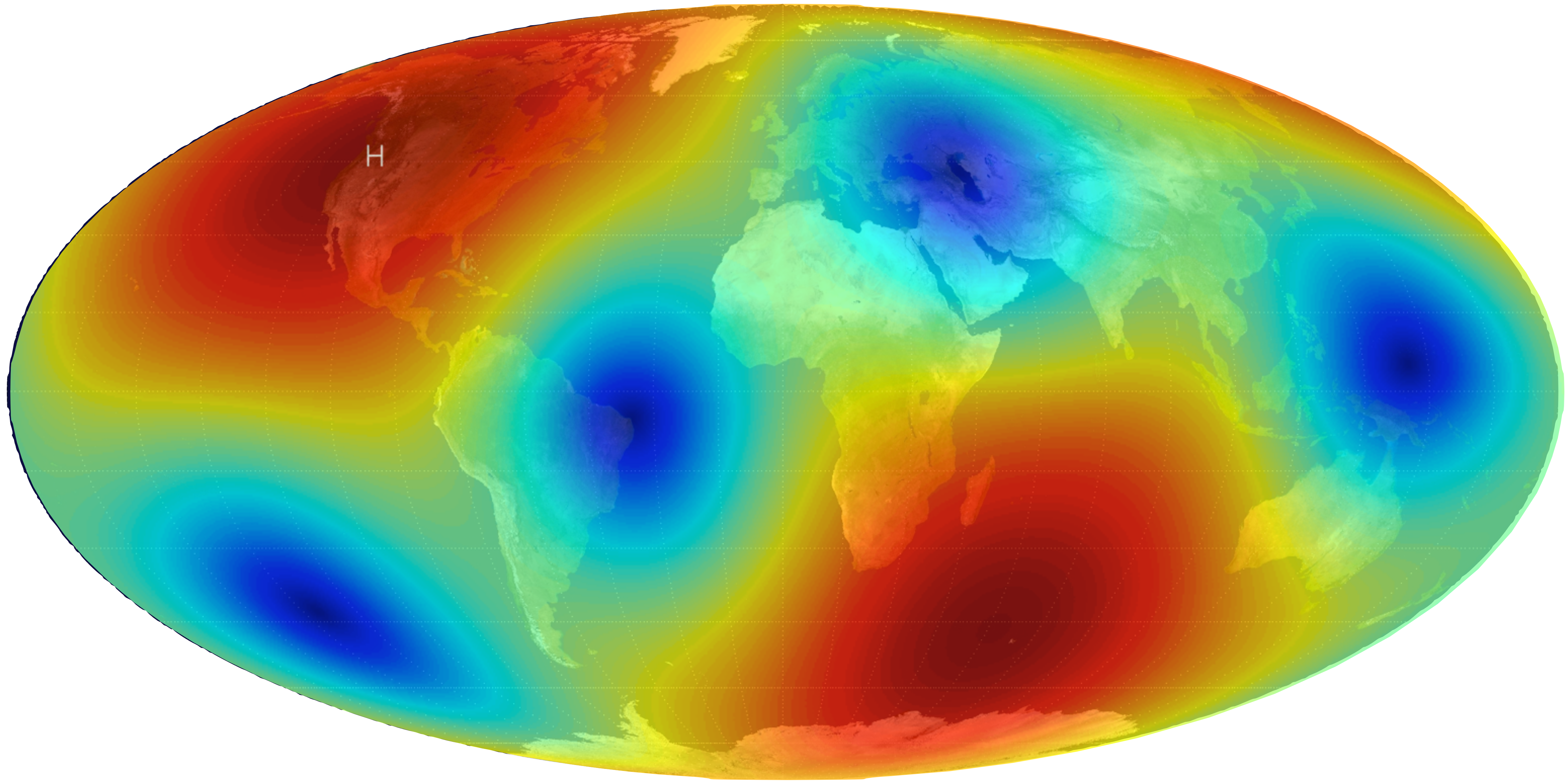




(credit: N. Cornish)



Terrestrial Network



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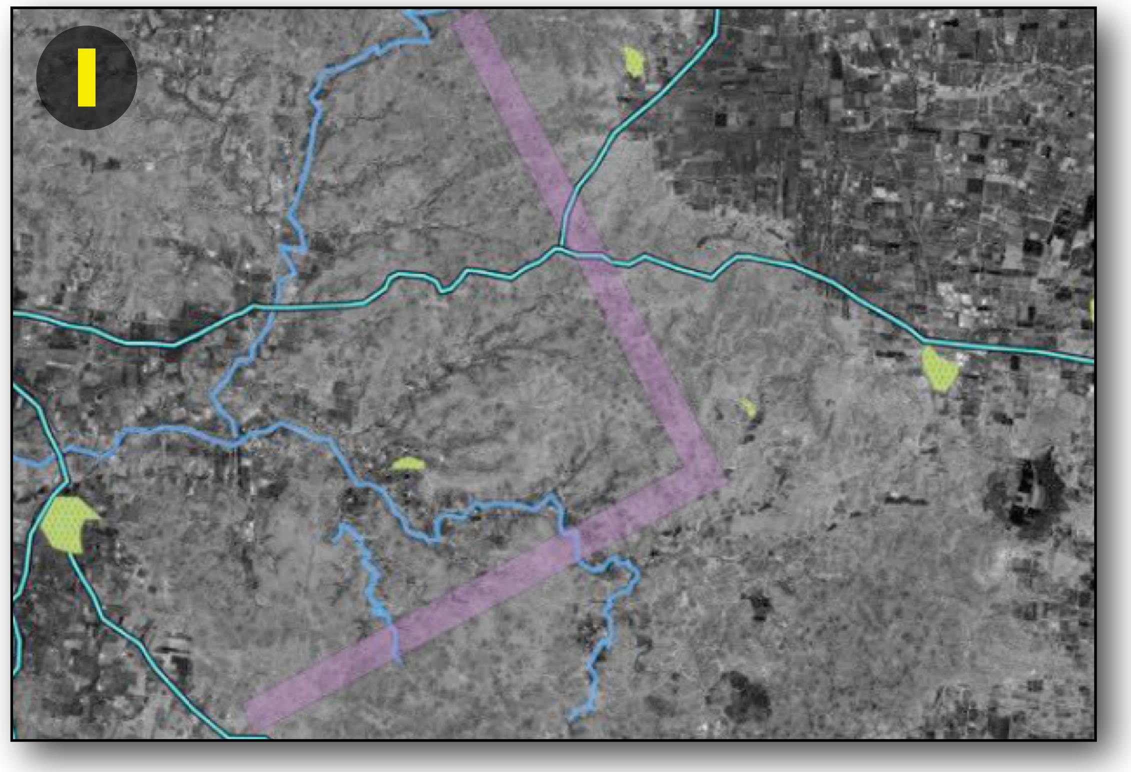

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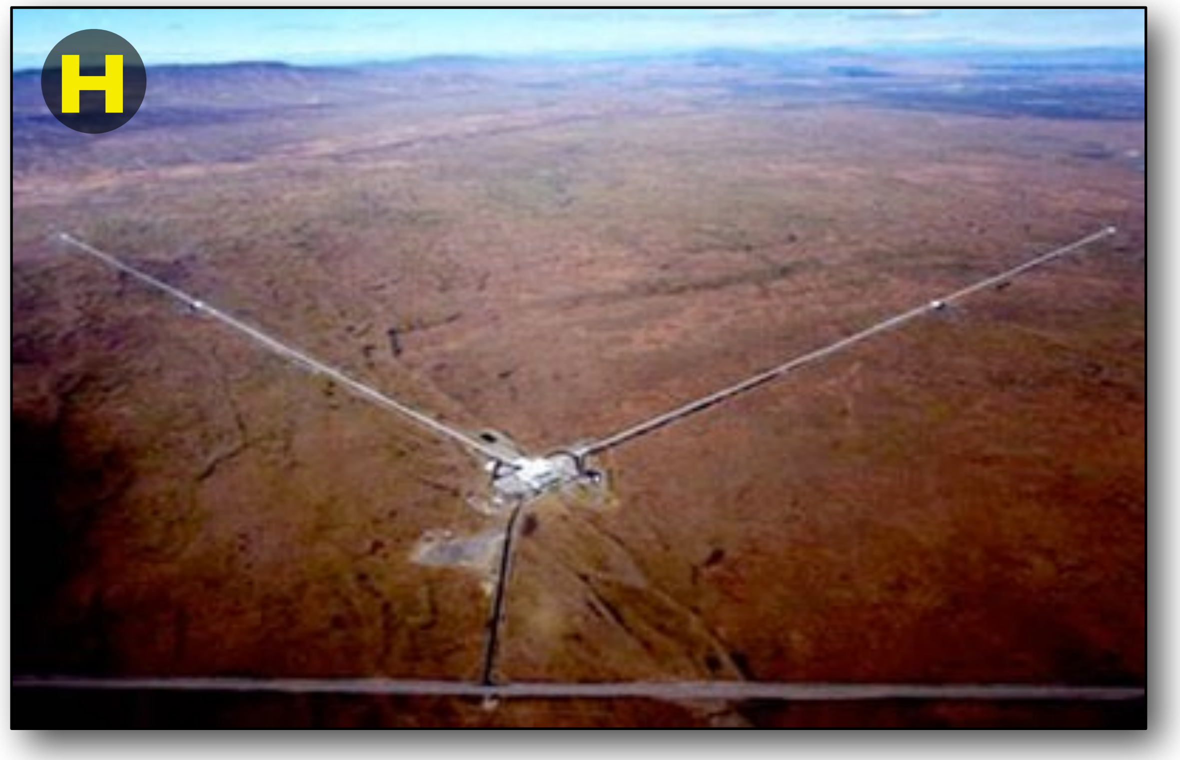

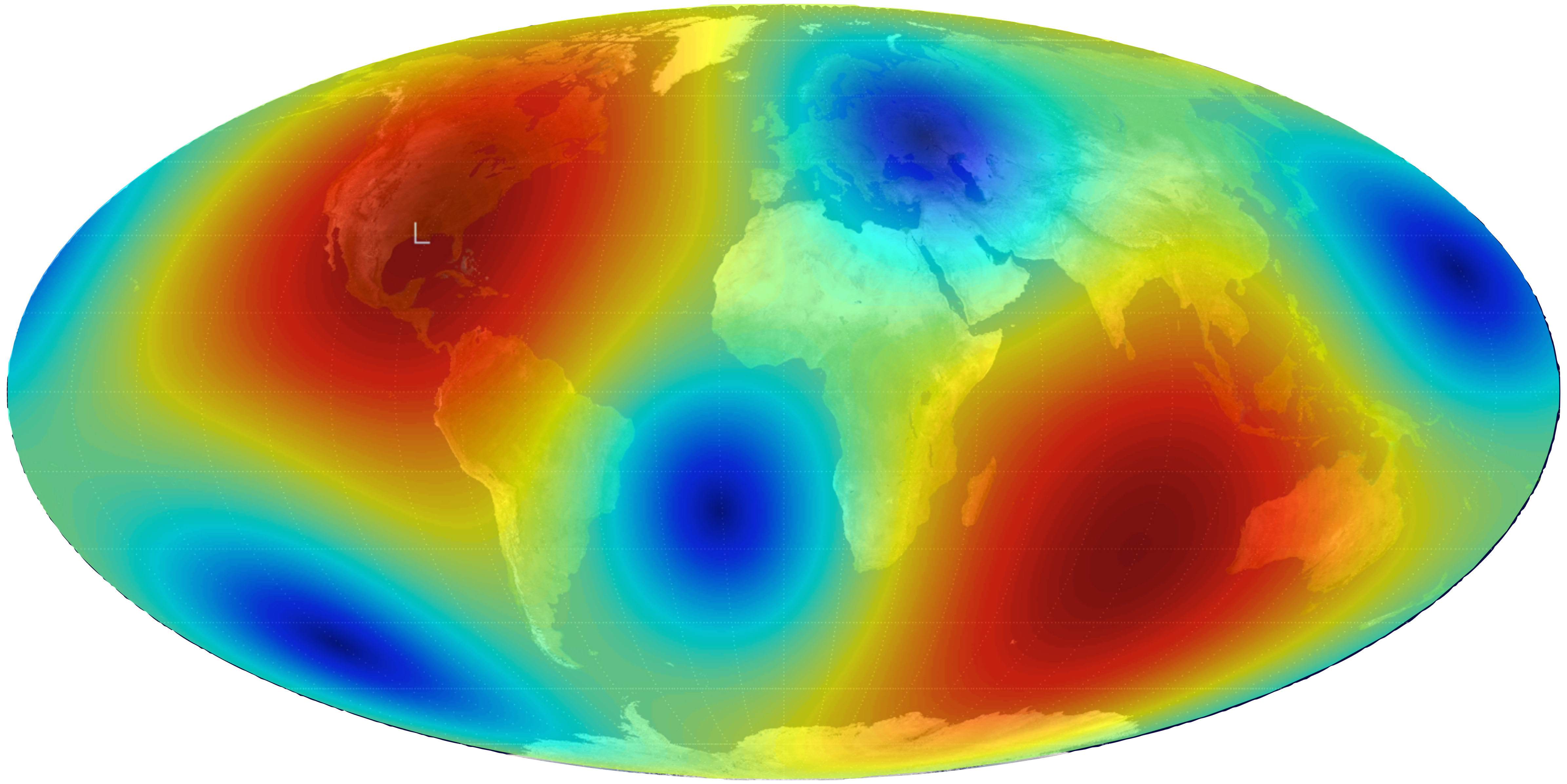

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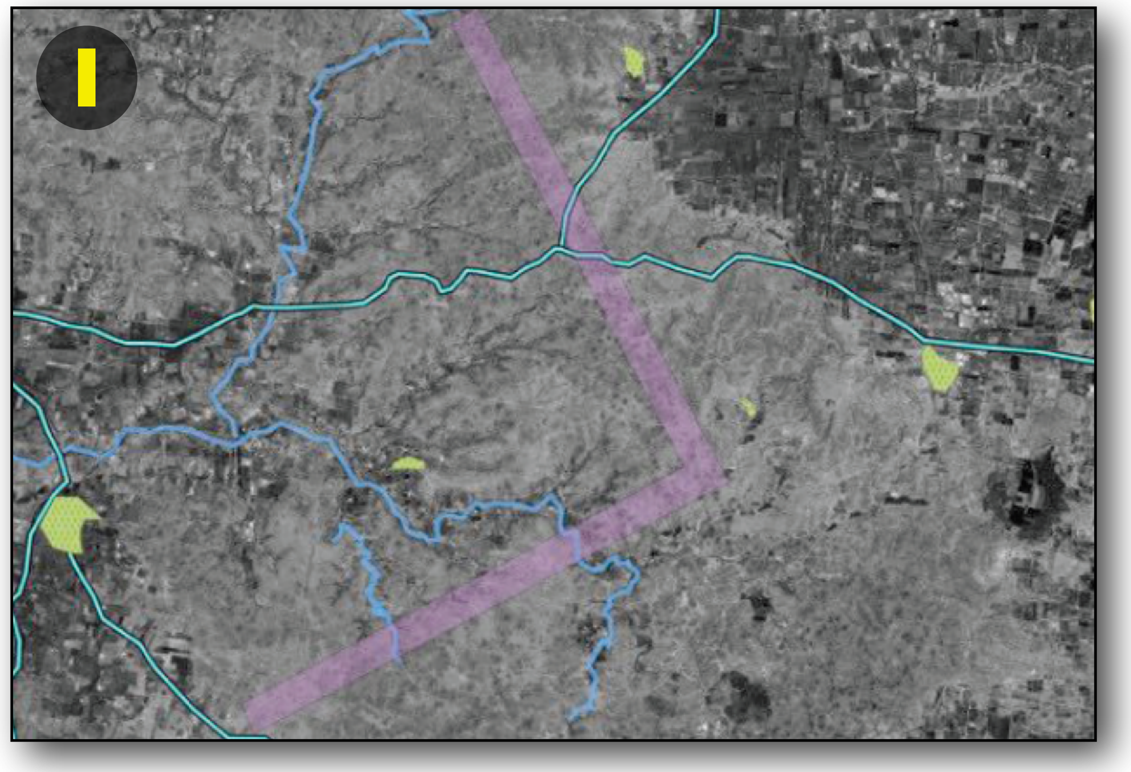

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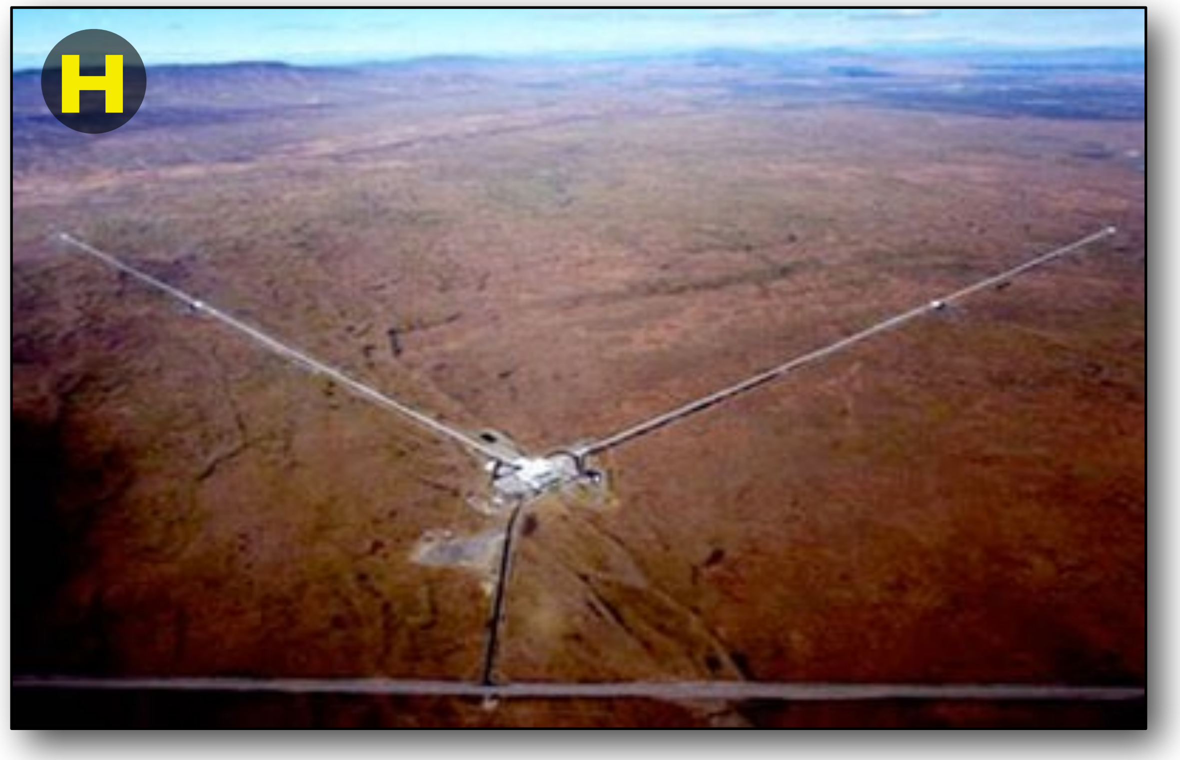

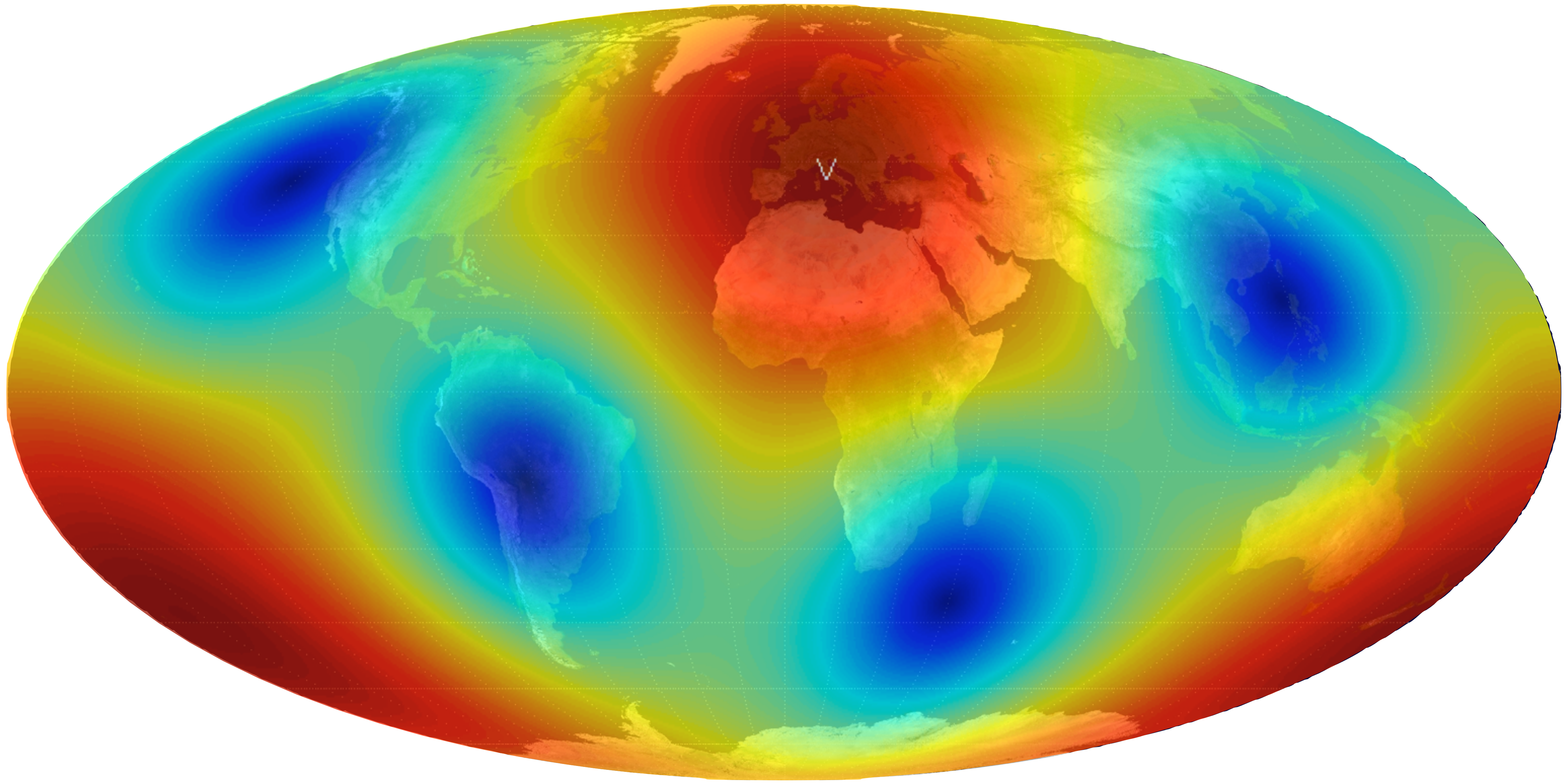

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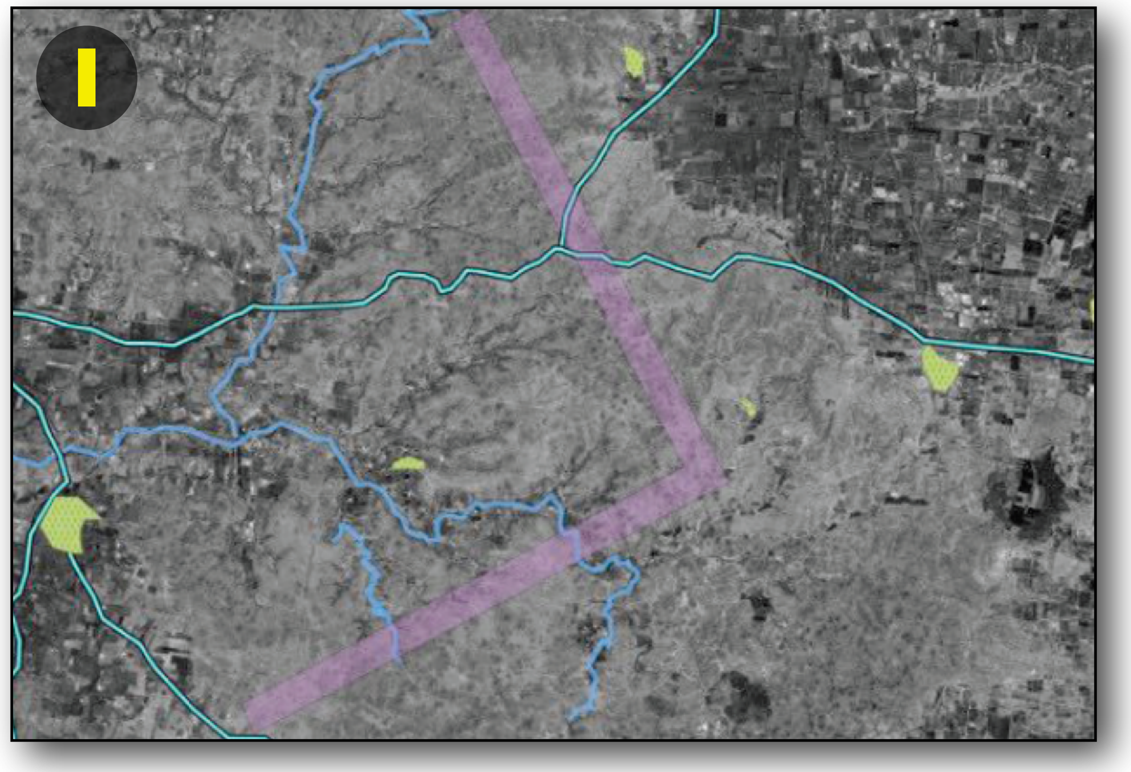

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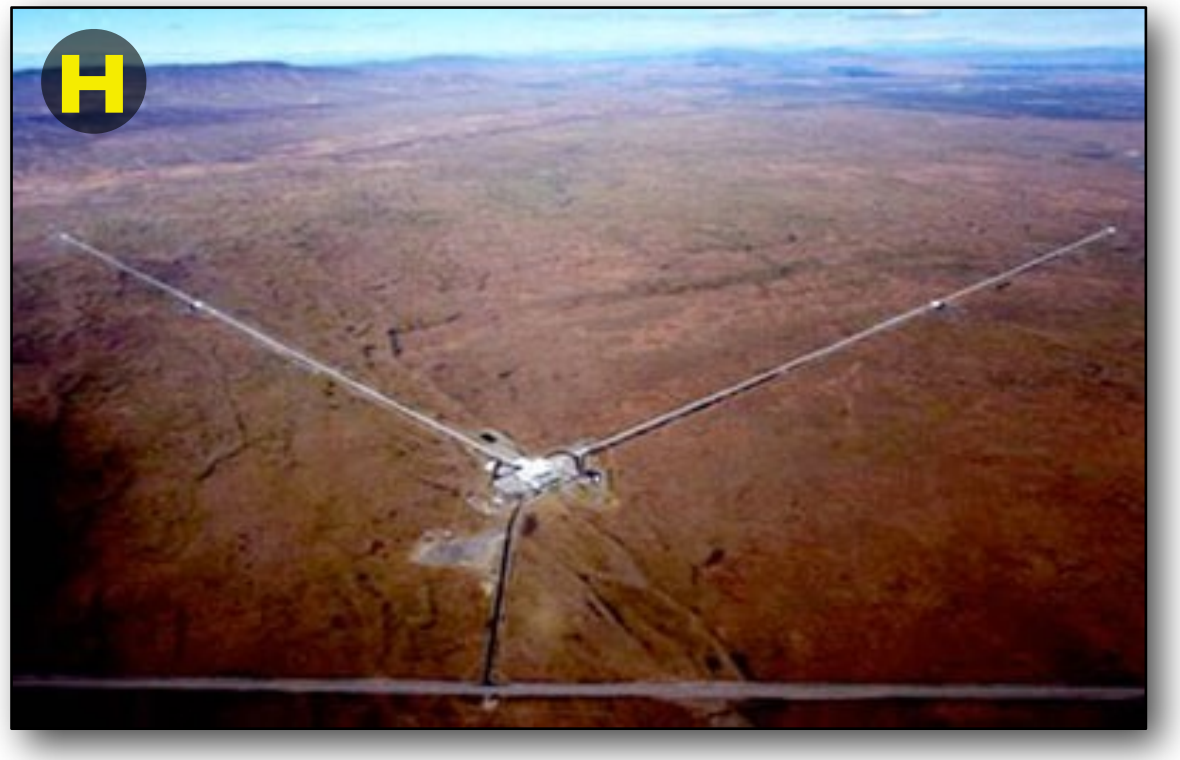

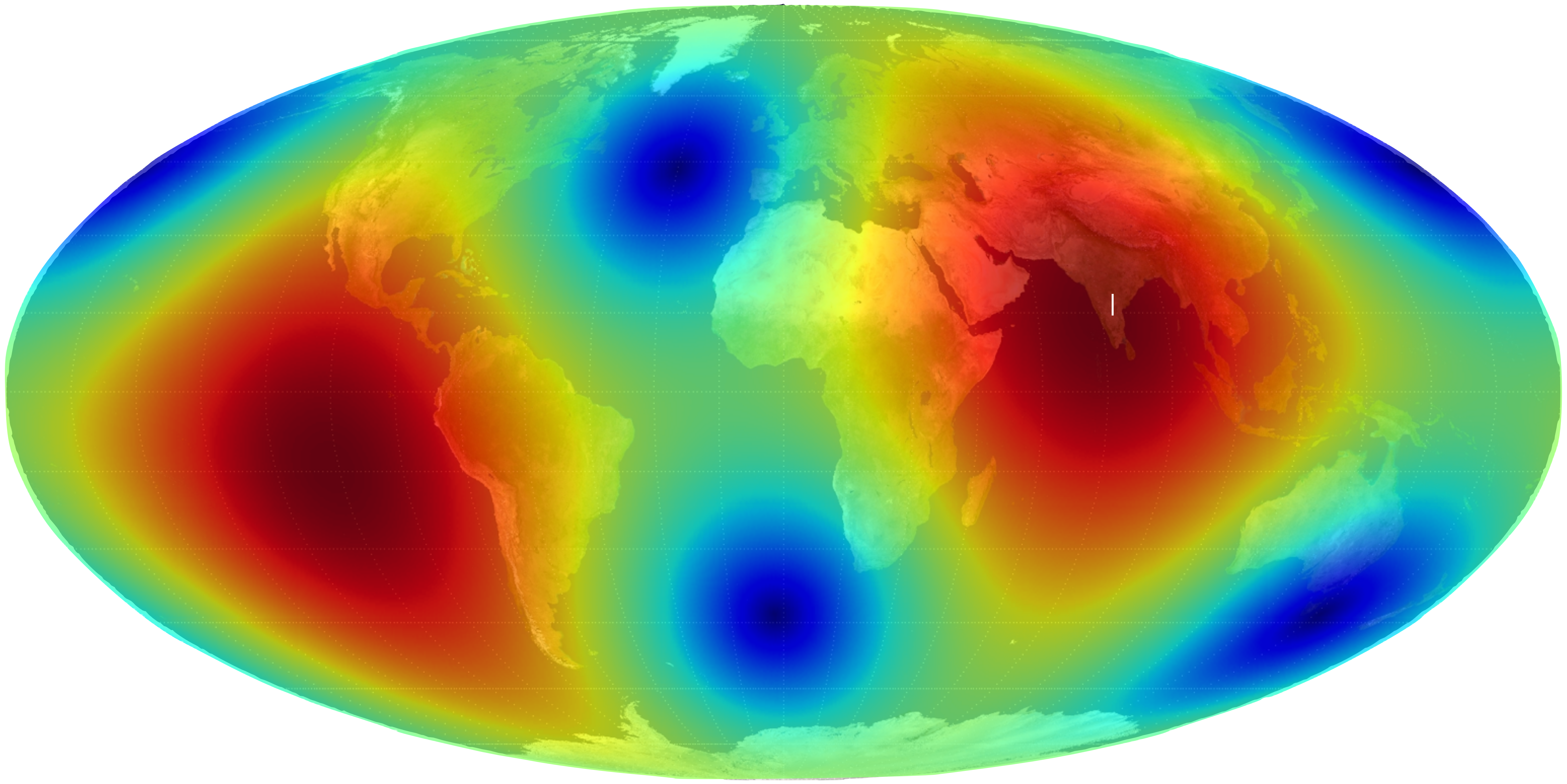

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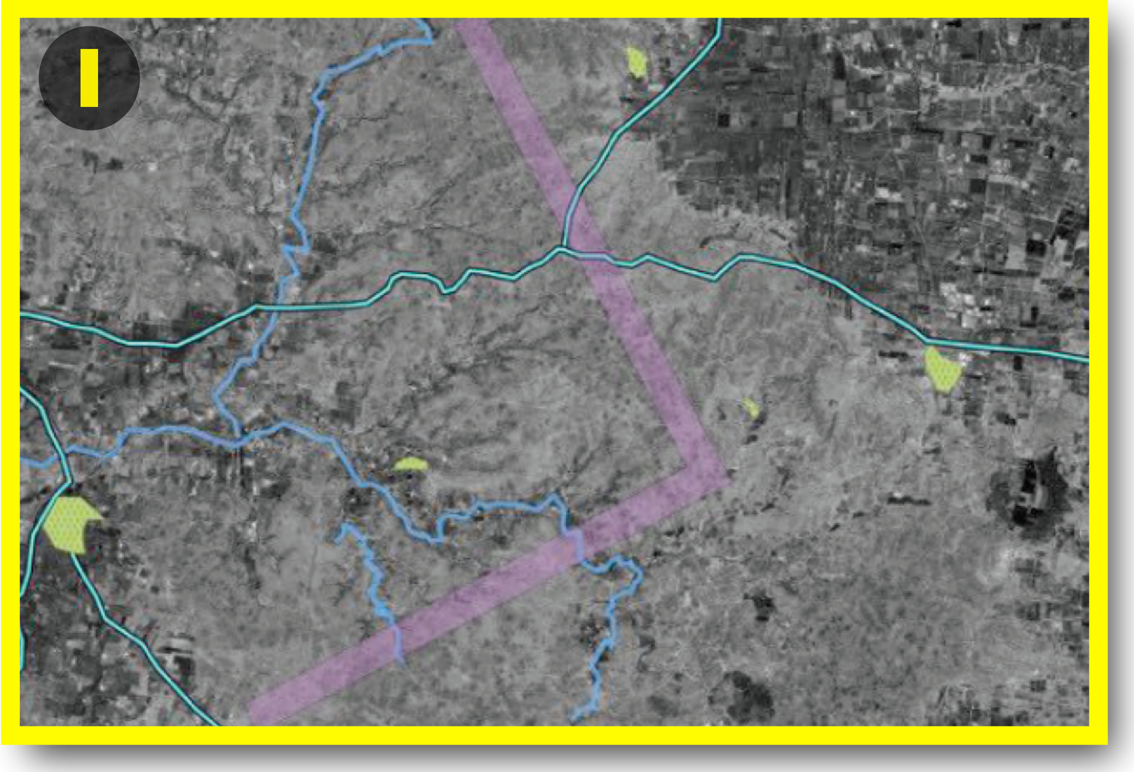

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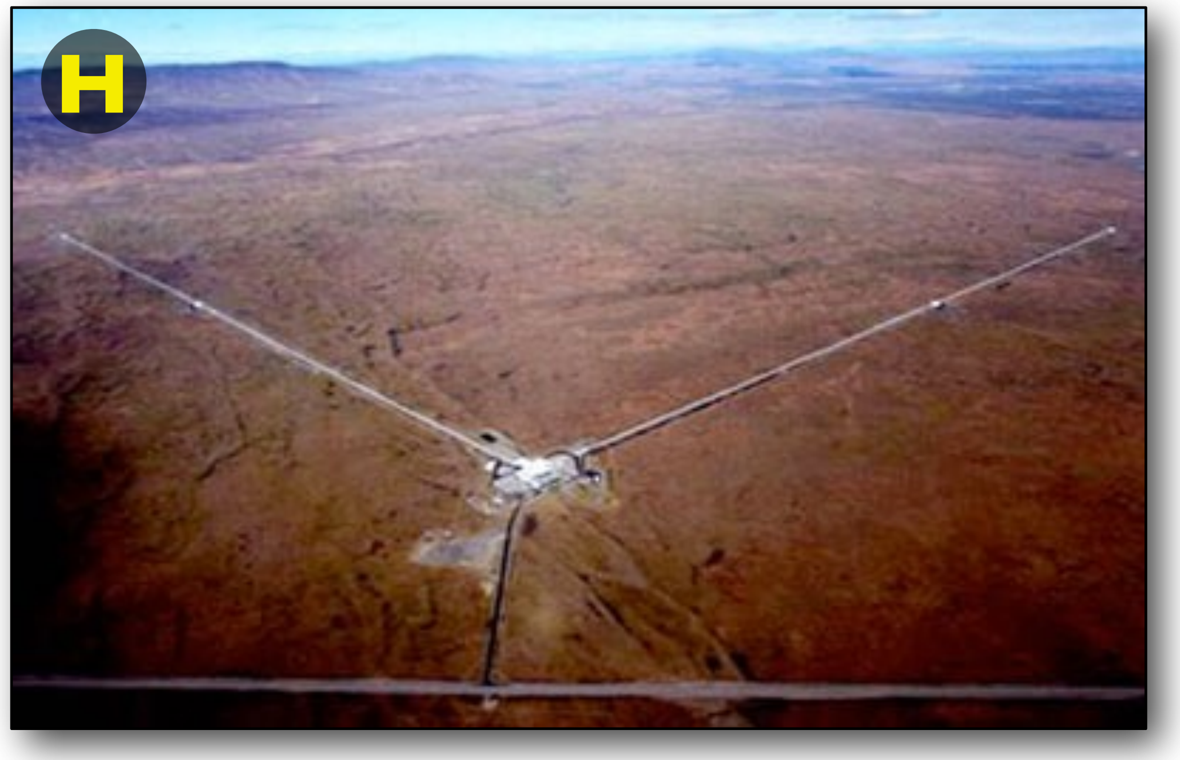

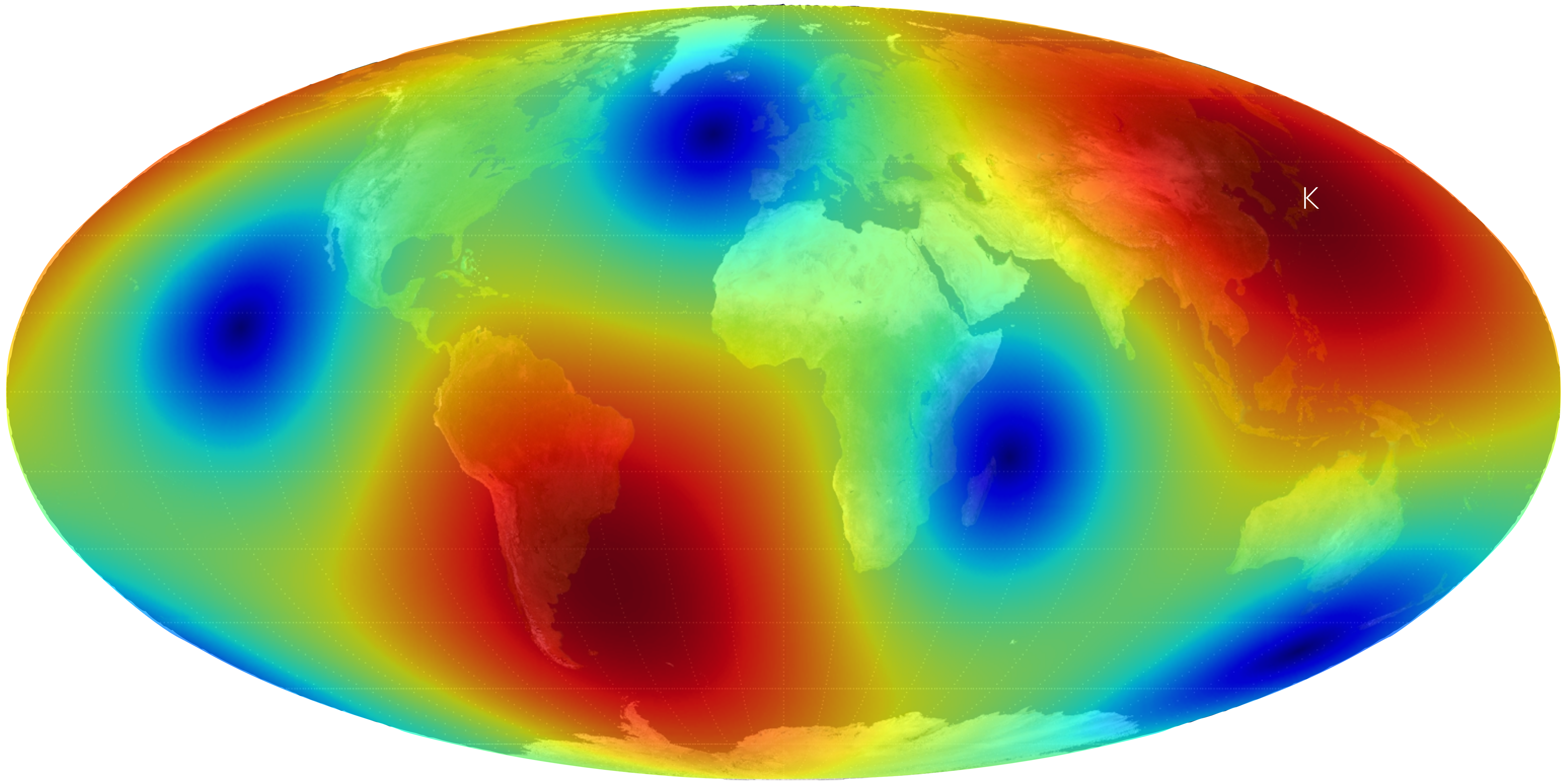

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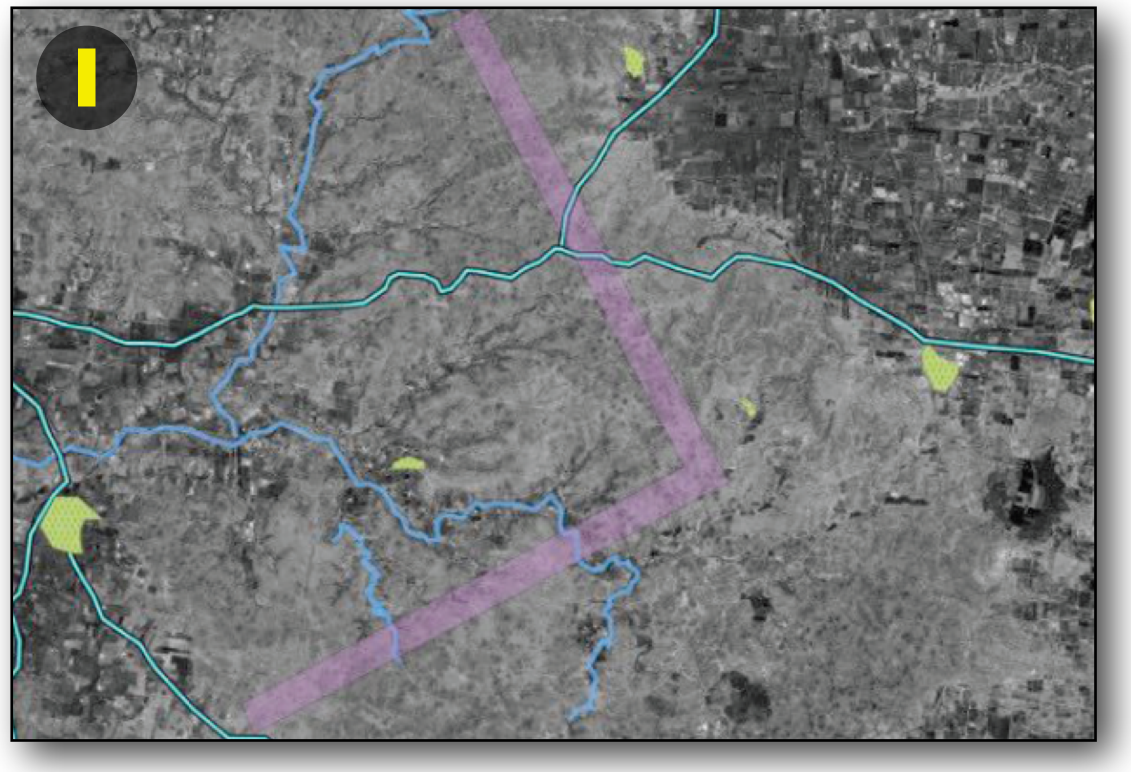
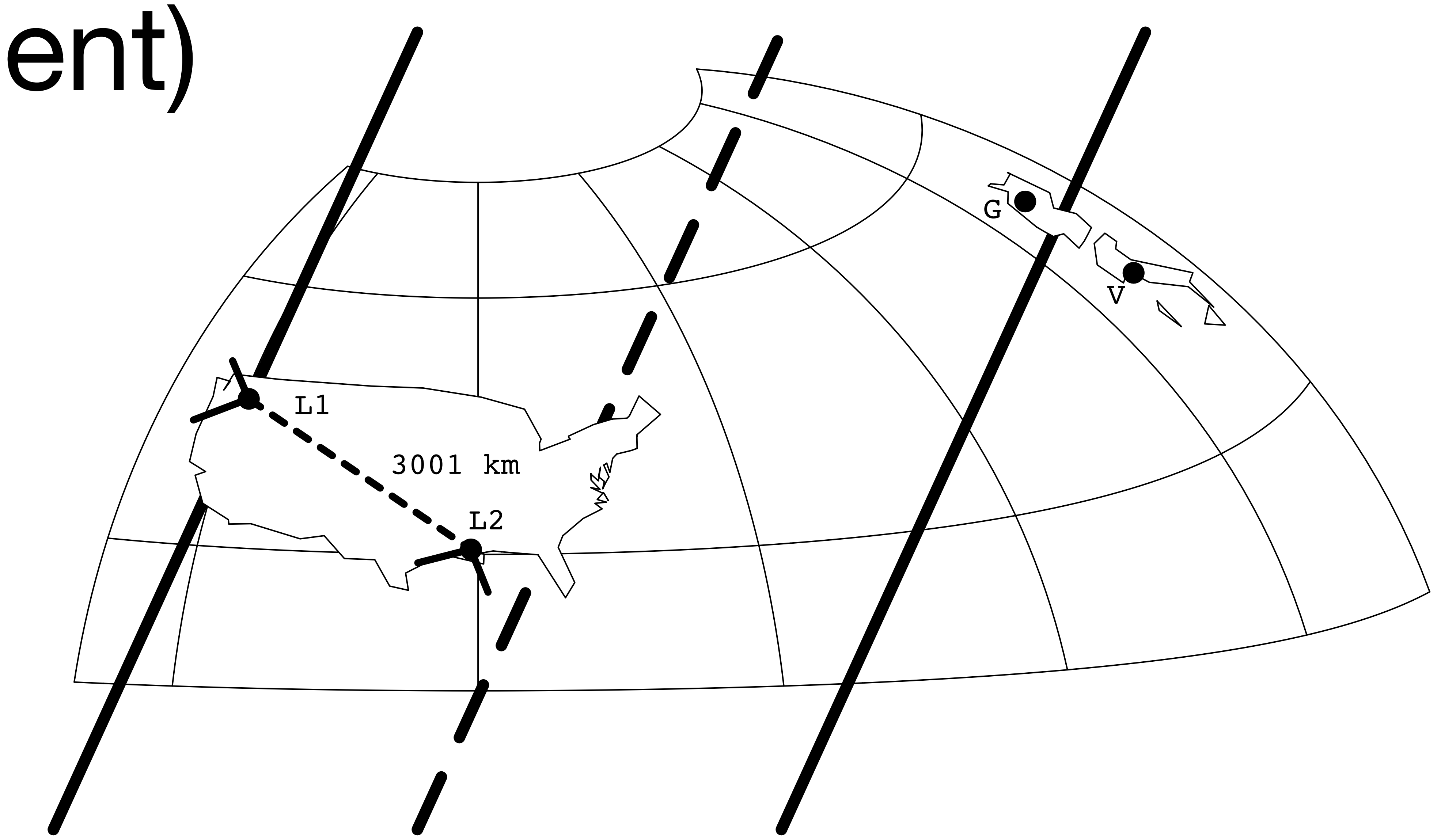


2. Non-trivial overlap functions

Overlap function (correlation coefficient)

- Detectors in different locations and with different orientations respond differently to a passing GW.
- Overlap function encodes reduction in sensitivity of a cross-correlation analysis due to separation and misalignment of the detectors



(B. Allen, Les Houches 1995)

Expected correlation:

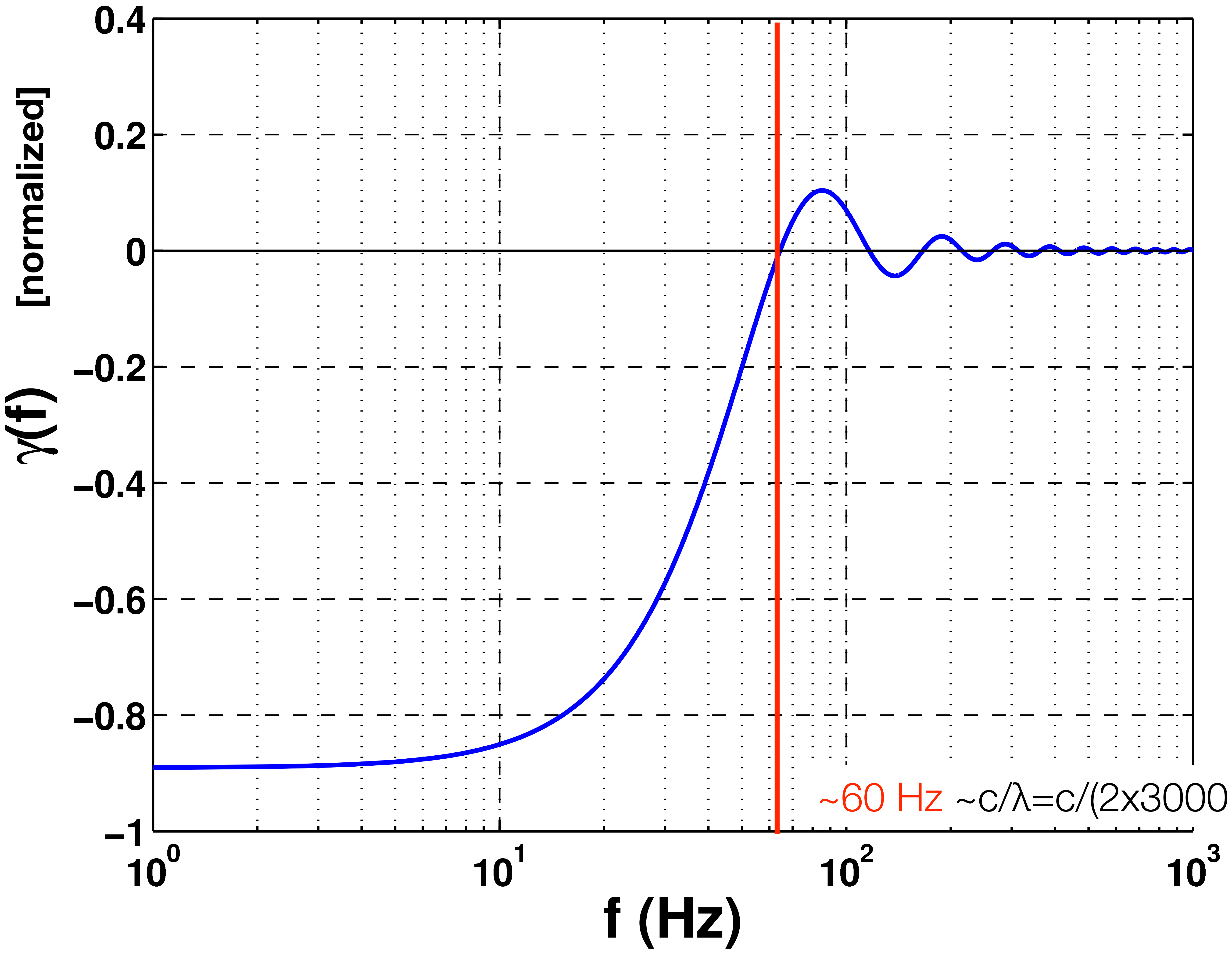
$$\langle h_I(t) h_J(t') \rangle = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}f \, e^{i2\pi f(t-t')} \Gamma_{IJ}(f) S_h(f) \quad \Longleftrightarrow \quad \langle \tilde{h}_I(f) \tilde{h}_J^*(f') \rangle = \frac{1}{2} \delta(f - f') \Gamma_{IJ}(f) S_h(f)$$

$$\Gamma_{IJ}(f) = \frac{1}{8\pi} \int \mathrm{d}^2\Omega_{\hat{n}} \sum_A R_I^A(f, \hat{n}) R_J^{A*}(f, \hat{n})$$

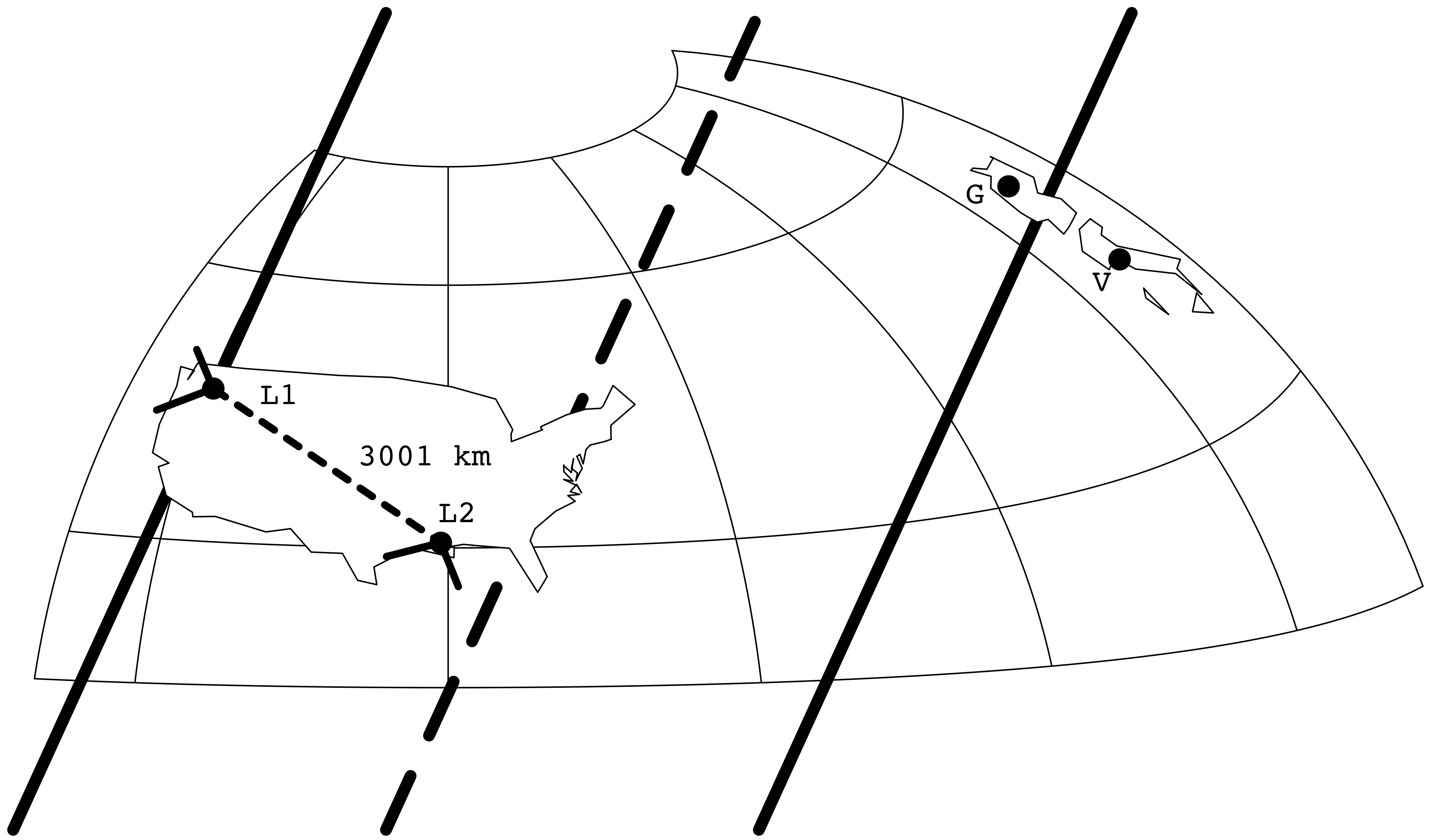
(unpolarized, stationary, isotropic background)

$\Gamma_{IJ}(f)$ is the transfer function between GW power and detector cross-power; integrand of $\Gamma_{IJ}(f)$ is important for anisotropic stochastic backgrounds

LIGO Hanford-LIGO Livingston overlap function (small-antenna approximation)

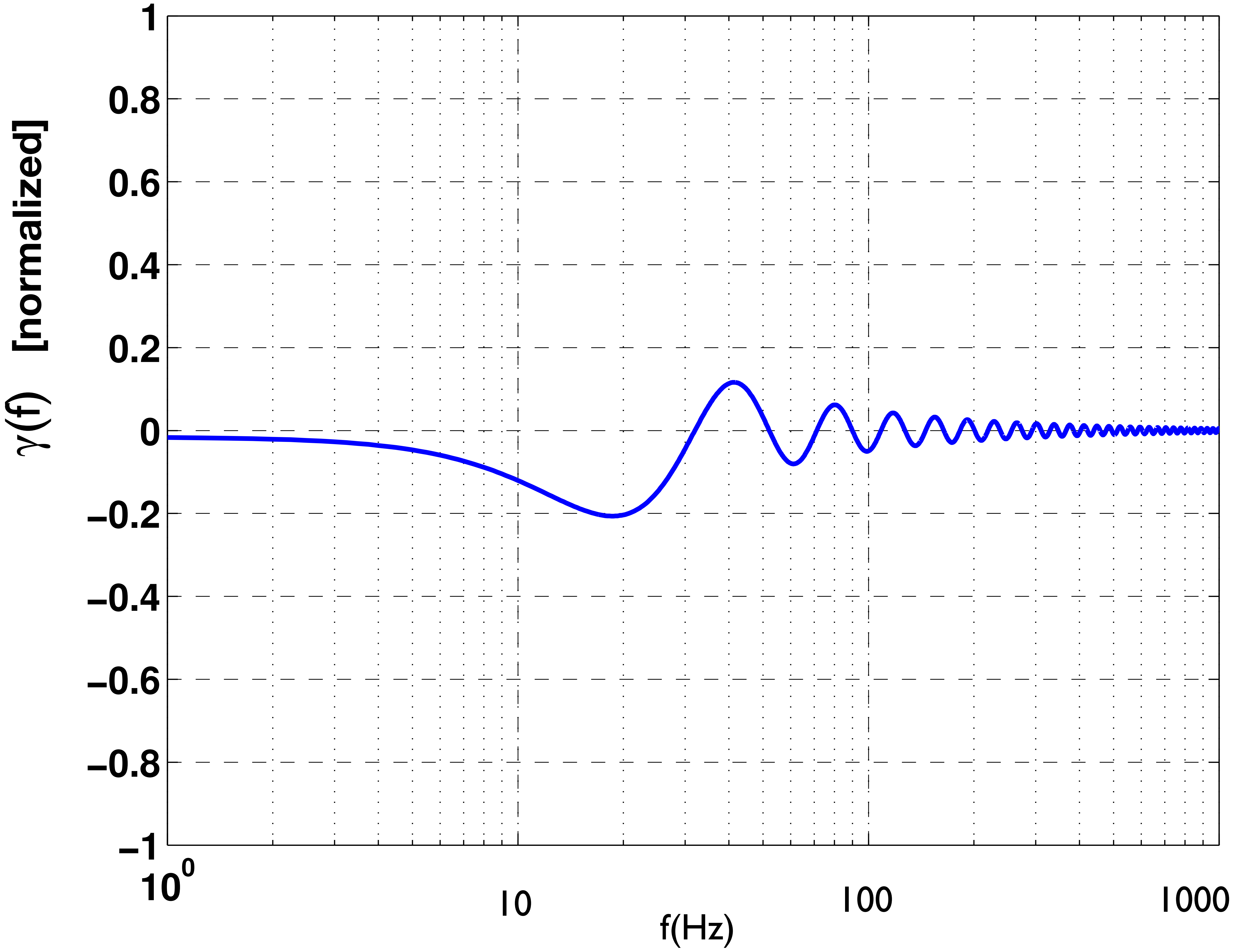


~60 Hz ~c/λ=c/(2x3000 km)

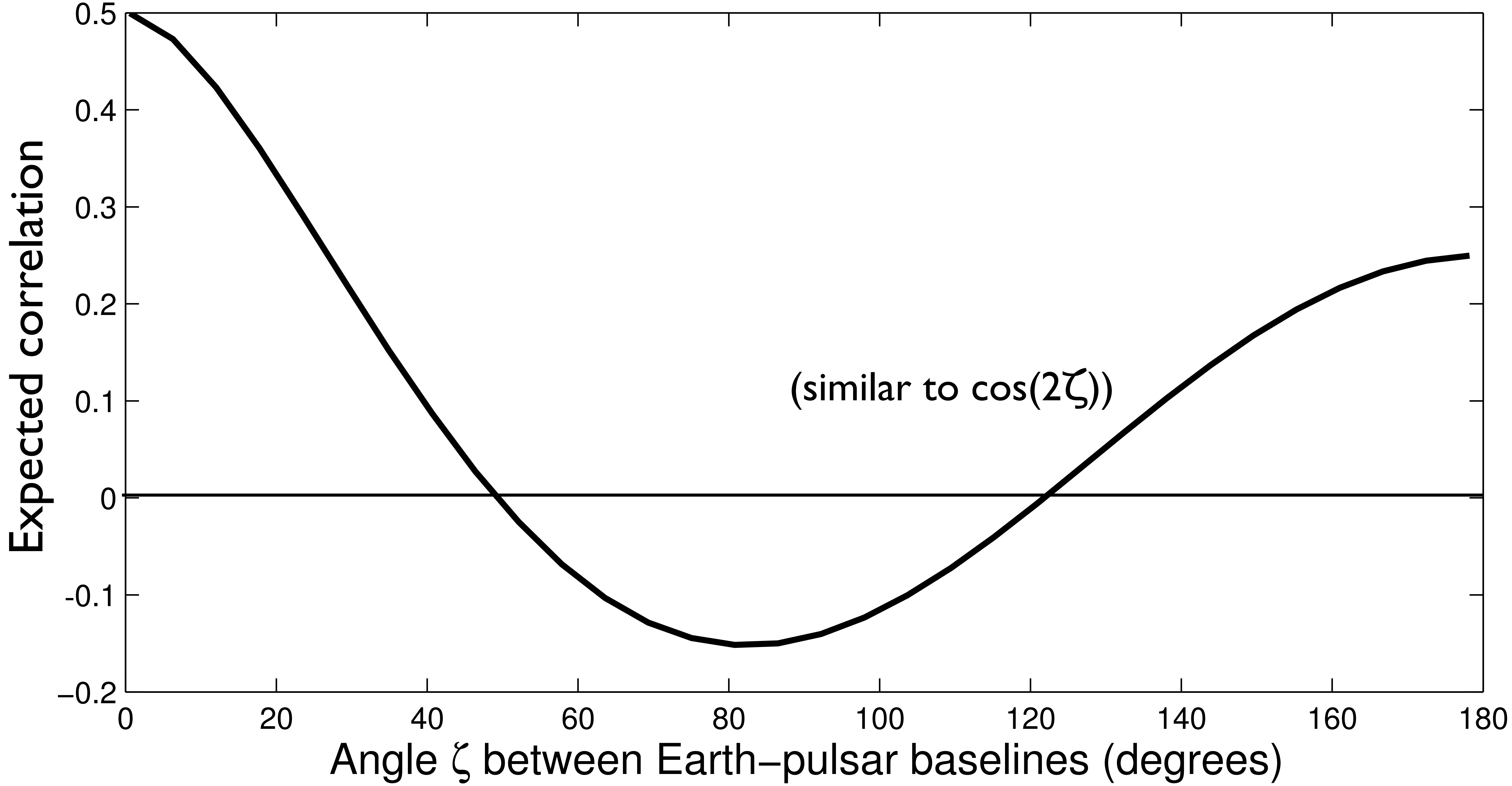


(B. Allen, Les Houches 1995)

LIGO Hanford-Virgo overlap function (small-antenna approximation)



Pulsar timing correlations (Hellings & Downs curve)
(correlation for an isotropic, unpolarized GW background in GR)

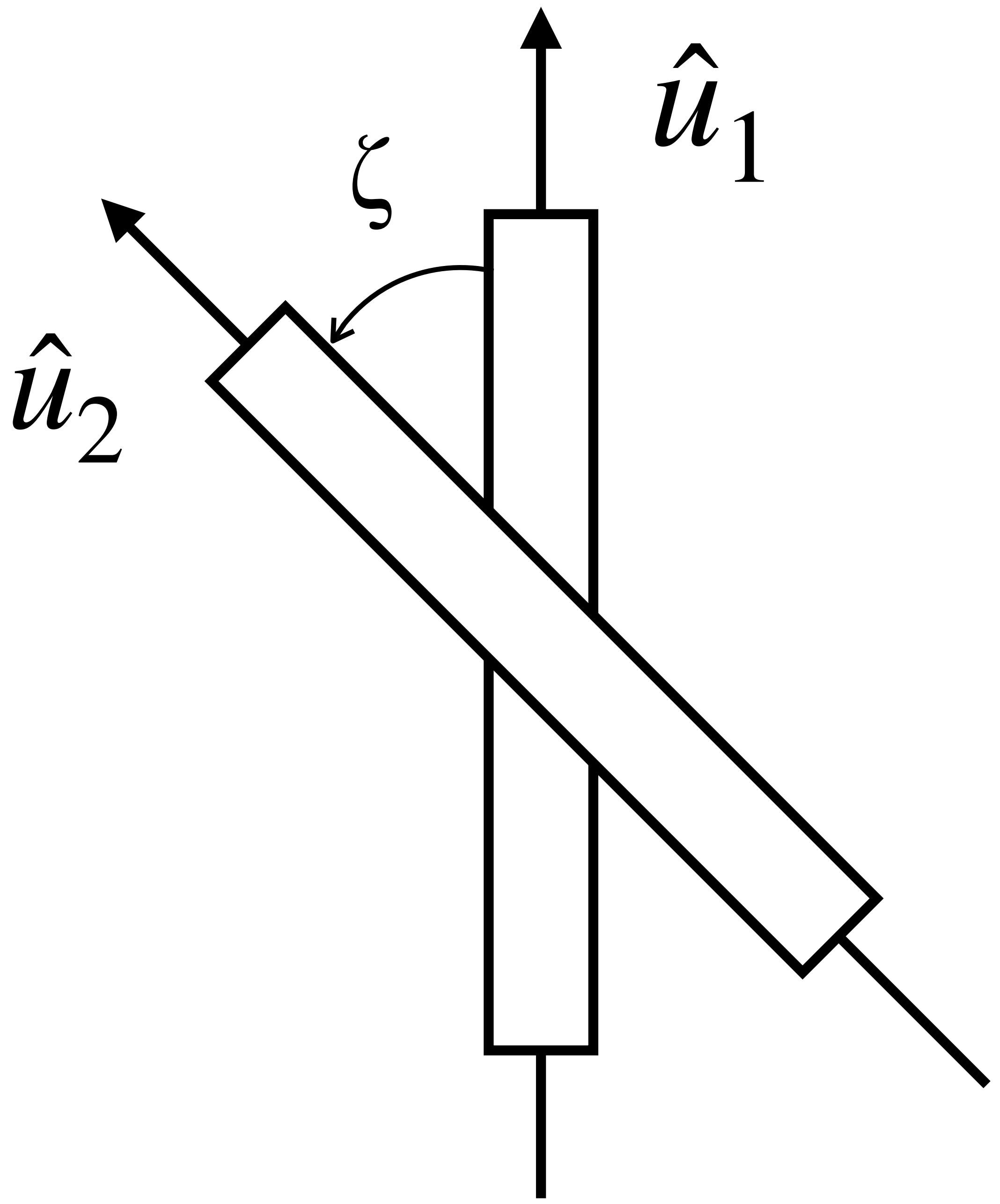


Exercise 7: Show that the overlap function for a pair of short, colocated electric dipole antennae pointing in direction u_1 and u_2 is given by

$$\Gamma_{12} \propto \hat{u}_1 \cdot \hat{u}_2 \equiv \cos\zeta$$

for an unpolarized, isotropic electromagnetic field.

Jenet and Romano, AJP 83 (7), 2015



Hint: $r_I(t) = \hat{u}_I \cdot \vec{E}(t, \vec{x}_0)$

$$\vec{E}(t, \vec{x}) = \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{n}} \sum_{\alpha=1}^{2} \tilde{E}_\alpha(f, \hat{n})\hat{\epsilon}_\alpha(\hat{n}) e^{i2\pi f(t+\hat{n}\cdot\vec{x}/c)}$$

$$\hat{\epsilon}_1(\hat{n}) = \hat{\theta}, \quad \hat{\epsilon}_2(\hat{n}) = \hat{\phi}$$

etc. …

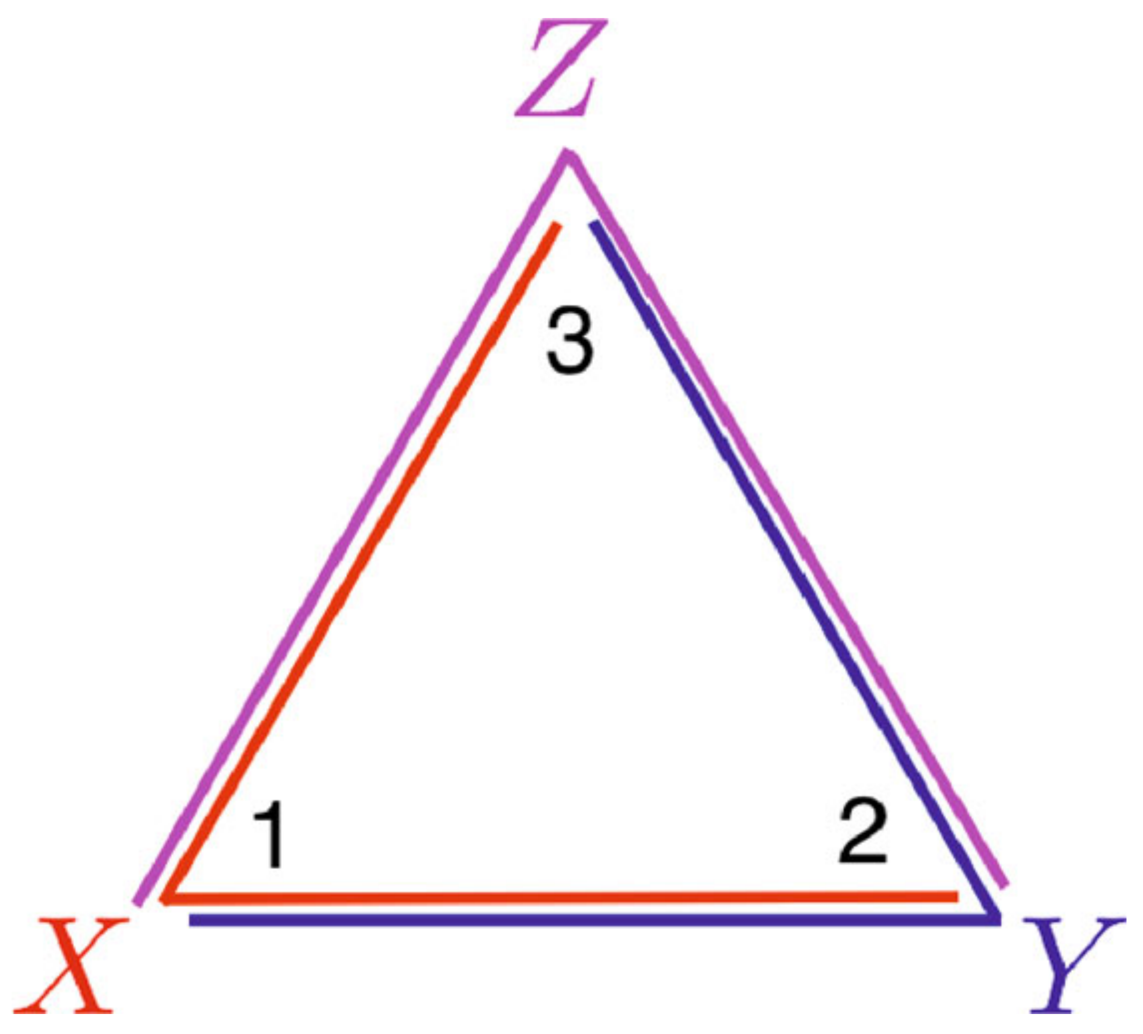
3. What to do in the absence of correlations (e.g., for LISA)?

LISA (Laser Interferometer Space Antenna)

Cross-correlation is not an option for LISA

(at least for low frequencies)

- Although there are 3 Michelson combinations (X,Y,Z), they have common noise (since they share arms)
- Can diagonalize the noise covariance matrix to obtain noise-orthogonal combinations (A,E,T), which also turn out to be signal orthogonal
  - A, E: two Michelsons rotated by 45 degrees
  - T: relatively insensitive to GW (null channel)
- Nonetheless, proper modeling of instrumental noise, astrophysical foregrounds (galactic WD binaries), and GWB allows you to discriminate all three components (Adams & Cornish, 2010, 2014)

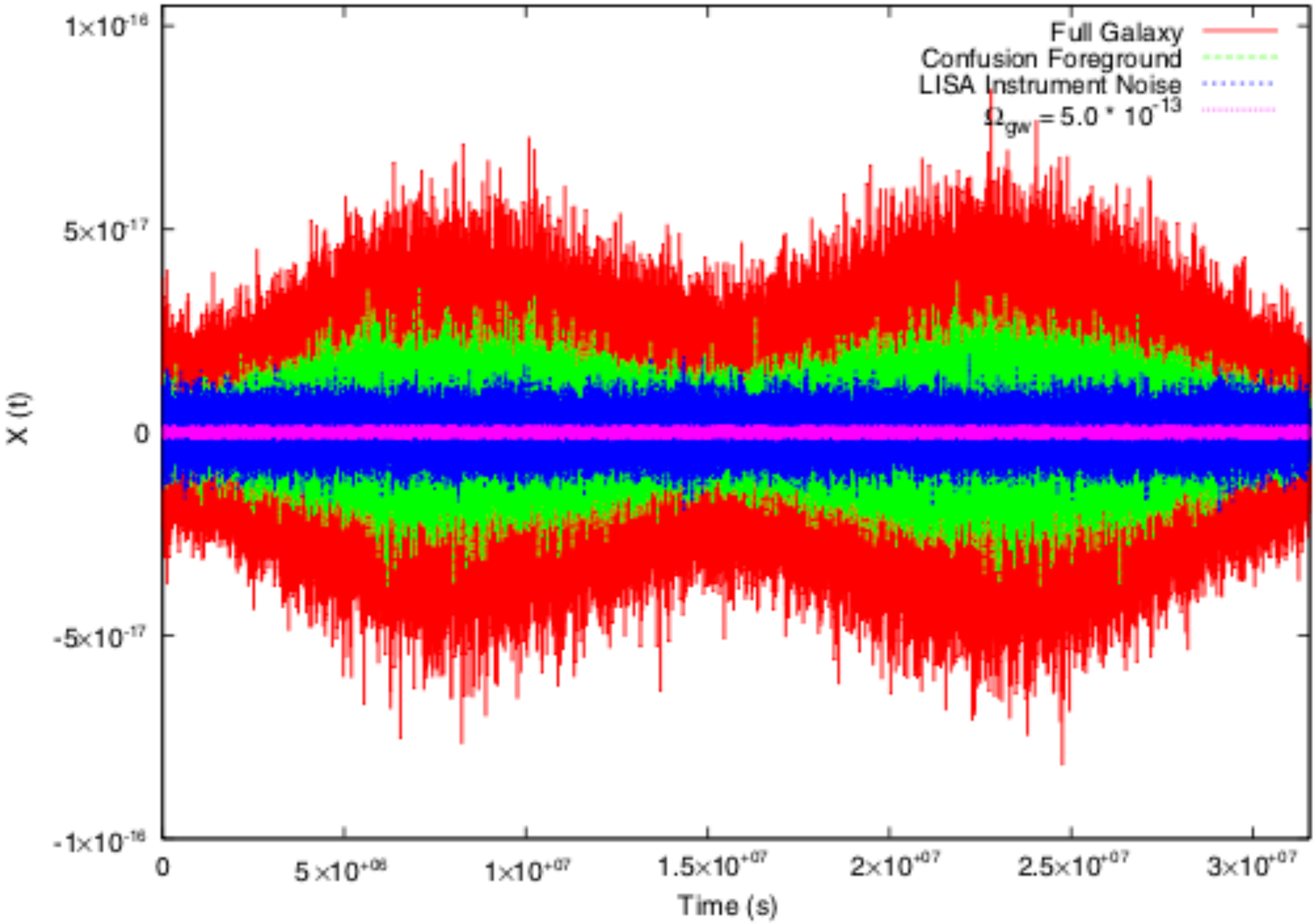


$$A \equiv \frac{1}{3}(2X - Y - Z),$$

$$E \equiv \frac{1}{\sqrt{3}}(Z - Y),$$

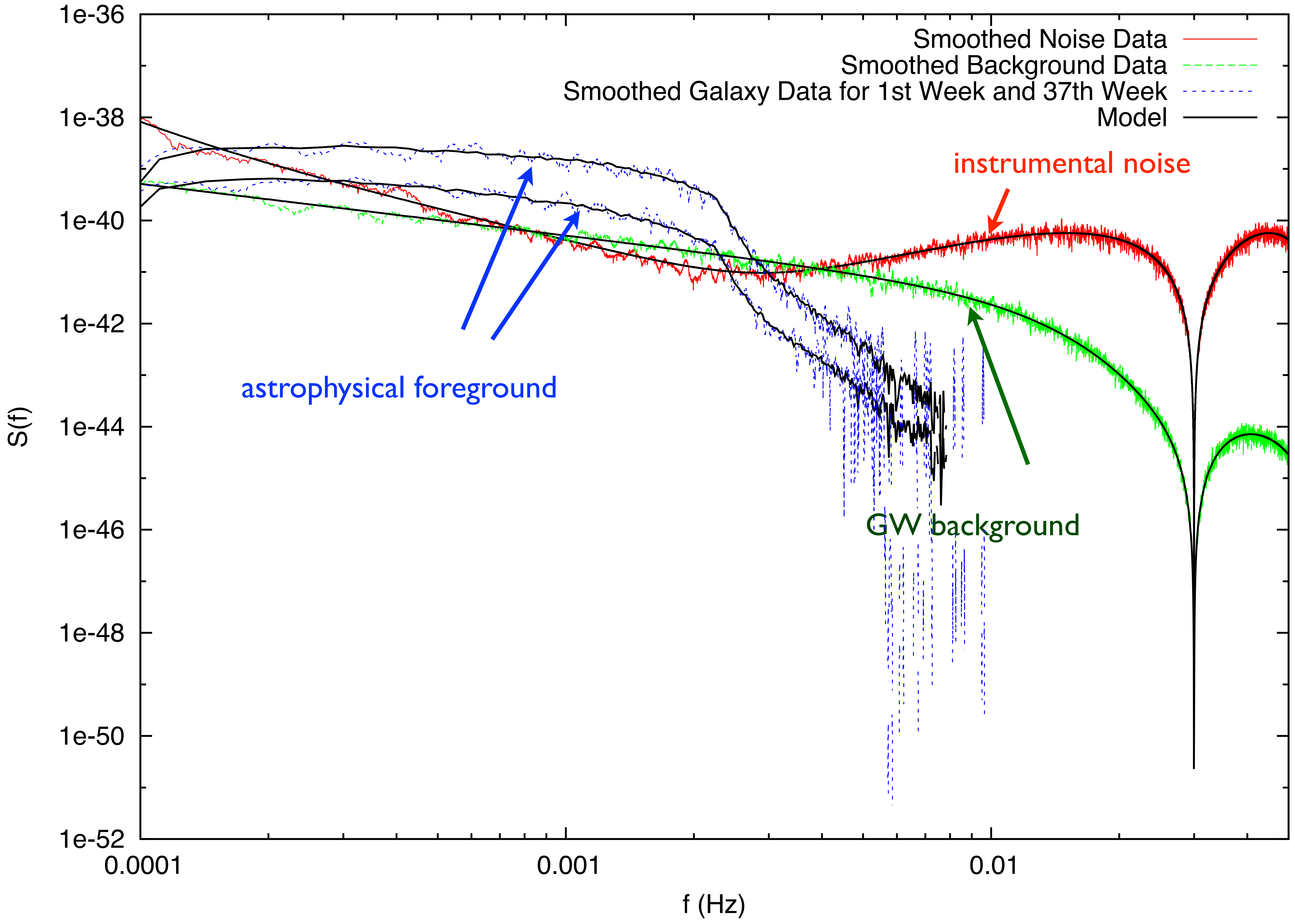
$$T \equiv \frac{1}{3}(X + Y + Z).$$

Detailed questions? Ask Neil when he arrives!



Full Galaxy
Confusion Foreground
LISA Instrument Noise
$\Omega_{gw} = 5.0 * 10^{-13}$
X (t)
Time (s)
1×10^{-16}
5×10^{-17}
0
-5×10^{-17}
-1×10^{-16}
0
$5\times10^{+06}$
$1\times10^{+07}$
$1.5\times10^{+07}$
$2\times10^{+07}$
$2.5\times10^{+07}$
$3\times10^{+07}$

Different spectra => differentiate different noise components



4. Frequentist and Bayesian methods

Frequentist statistics	Bayesian infererence
Probabilities are long-run relative occurrences of outcomes of repeatable expts —> can't be assigned to hypotheses	Probabilities are degree of belief —> can be assigned to hypotheses
Usually start with a likelihood function p(d|H)	Same as frequentist
Construct statistics for parameter estimation / hypothesis testing	Specify priors for parameters and hypotheses
Calculate probability distribution of the statistics (e.g., using time slide)	Use Bayes' theorem to update degree of belief
Calculates confidence intervals and p-values	Construct posteriors and odds ratios (Bayes factors)

Likelihood function

Starting point for both frequentist & Bayesian analyses:

$$\text{likelihood} = p(\text{data}\,|\,\text{parameters}, \text{model})$$

Gaussian detector noise and GWB:

$$p(d\,|\,C_n, \mathcal{M}_0) = \frac{1}{\sqrt{\det(2\pi C_n)}}\,\exp\left[-\frac{1}{2}d^T C_n^{-1} d\right]$$

(noise-only model)

$$p(d\,|\,C_n, S_h, \mathcal{M}_1) = \frac{1}{\sqrt{\det(2\pi C)}}\,\exp\left[-\frac{1}{2}d^T C^{-1} d\right]$$

(signal+noise model)

N samples of white noise, white GWB, in two colocated and coaligned detectors:

$$C_n = \begin{bmatrix} S_{n_1}\,1_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & S_{n_2}\,1_{N\times N} \end{bmatrix} \qquad \& \qquad C = \begin{bmatrix} (S_{n_1} + S_h)\,1_{N\times N} & S_h\,1_{N\times N} \\ S_h\,1_{N\times N} & (S_{n_2} + S_h)\,1_{N\times N} \end{bmatrix}$$

Frequentist analysis

Use maximum-likelihood (ML) ratio for detection, and maximum-likelihood parameter values as estimators

Maximum-likelihood detection statistic:

$$\Lambda_{\mathrm{ML}}(d) \equiv \frac{\max_{S_{n_1},S_{n_2},S_h} p(d\,|\,S_{n_1}S_{n_2}, S_h, \mathscr{M}_1)}{\max_{S_{n_1},S_{n_2}} p(d\,|\,S_{n_1}, S_{n_2}, \mathscr{M}_0)}$$

$$\Lambda(d) \equiv 2\ln(\Lambda_{\mathrm{ML}}(d)) \simeq \frac{\hat{S}_h^2}{\hat{S}_{n_1}\hat{S}_{n_2}/N} \longleftarrow \mathrm{SNR}^2$$

Maximum-likelihood estimators:

$$\hat{S}_h \equiv \frac{1}{N}\sum_{i=1}^{N} d_{1i}d_{2i} \longleftarrow \text{cross-correlation statistic}$$

$$\hat{S}_{n_1} \equiv \frac{1}{N}\sum_{i=1}^{N} d_{1i}^2 - \hat{S}_h$$

$$\hat{S}_{n_2} \equiv \frac{1}{N}\sum_{i=1}^{N} d_{2i}^2 - \hat{S}_h$$

Exercise 8: Verify the expressions for the ML estimators.

Exercise 9: Verify the expression for the detection statistic $2\ln(\Lambda_{\mathrm{ML}}(d))$

Bayesian analysis

Use Bayes' theorem to calculate posterior distributions for parameter estimation and odds ratios (Bayes factors) for model selection

Bayes' theorem:

likelihood

prior

$$p(H|d) = \frac{p(d|H)p(H)}{p(d)}$$

normalization factor

Posteriors:

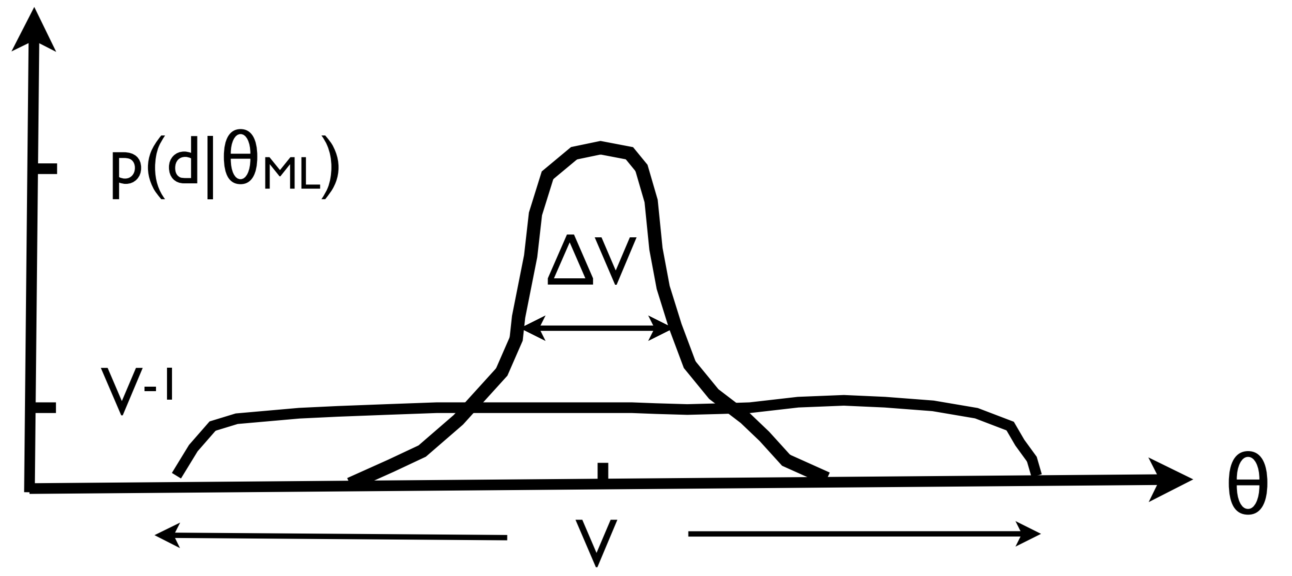
$$p(S_{n_1}, S_{n_2}, S_h | d, \mathscr{M}_1) = \frac{p(d|S_{n_1}, S_{n_2}, S_h, \mathscr{M}_1)p(S_{n_1}, S_{n_2}, S_h|\mathscr{M}_1)}{p(d|\mathscr{M}_1)}$$

$$p(S_h|d, \mathscr{M}_1) = \int dS_{n_1} \int dS_{n_2}\, p(S_{n_1}, S_{n_2}, S_h|d, \mathscr{M}_1)$$

Model selection:

Bayes factor $\mathscr{B}_{10}(d)$

$$\frac{p(\mathscr{M}_1|d)}{p(\mathscr{M}_0|d)} = \boxed{\frac{p(d|\mathscr{M}_1)}{p(d|\mathscr{M}_0)}}\frac{p(\mathscr{M}_1)}{p(\mathscr{M}_0)}$$



Relationship to frequentist approach:

$$\mathscr{B}_{10}(d) \equiv \frac{p(d|\mathscr{M}_1)}{p(d|\mathscr{M}_0)} = \frac{\int dS_{n_1}\int dS_{n_2}\int dS_h\, p(d|S_{n_1}, S_{n_2}, S_h, \mathscr{M}_1)p(S_{n_1}, S_{n_2}, S_h|\mathscr{M}_1)}{\int dS_{n_1}\int dS_{n_2}\, p(d|S_{n_1}, S_{n_2}, \mathscr{M}_0)p(S_{n_1}, S_{n_2}|\mathscr{M}_0)} \simeq \Lambda_{\mathrm{ML}}(d)\,\frac{\Delta V_1/V_1}{\Delta V_0/V_0}$$

Example: Derivation of standard stochastic likelihood by marginalizing over a stochastic signal prior

Generic likelihood:

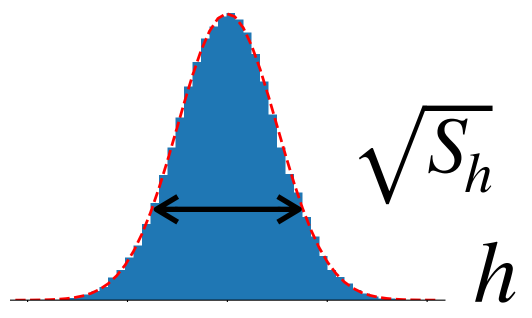
$$p(d\,|\,C_n, h) \equiv p_n(d-h\,|\,C_n) = \frac{1}{\sqrt{\det(2\pi C_n)}}\,\exp\left[-\frac{1}{2}(d-h)^T C_n^{-1}(d-h)\right]$$

signal model

covariance matrix for noise, e.g., $C_n = \begin{bmatrix} S_{n_1} & 0 \\ 0 & S_{n_2} \end{bmatrix}$

stochastic signal model:

$$p(h\,|\,S_h) = \frac{1}{\sqrt{2\pi S_h}}\,\exp\left[-\frac{1}{2}\frac{h^2}{S_h}\right]$$



Marginalized likelihood:

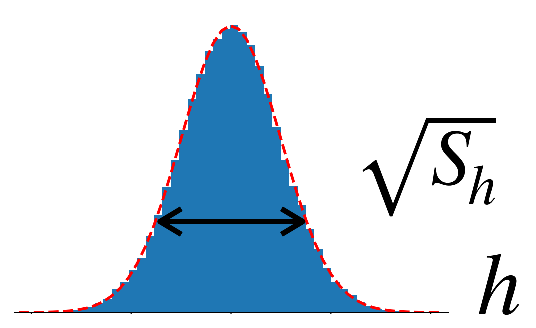
$$p(d\,|\,C_n, S_h) = \int \mathrm{d}h\, p_n(d-h\,|\,C_n)p(h\,|\,S_h) \quad = \frac{1}{\sqrt{\det(2\pi C)}}\,\exp\left[-\frac{1}{2}d^T C^{-1} d\right]$$

covariance matrix for signal + noise $\longrightarrow$ $C = \begin{bmatrix} S_{n_1} + S_h & S_h \\ S_h & S_{n_2} + S_h \end{bmatrix}$

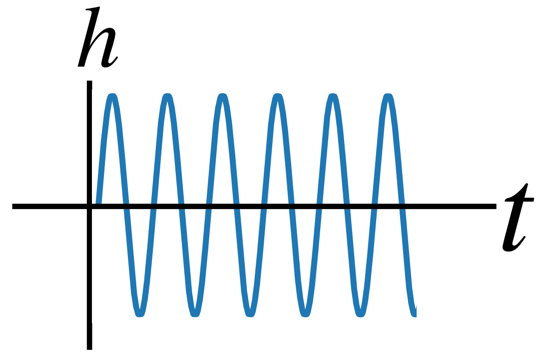
Exercise 10: Do the marginalization over h to obtain this final result.

Signal priors define the signal model…

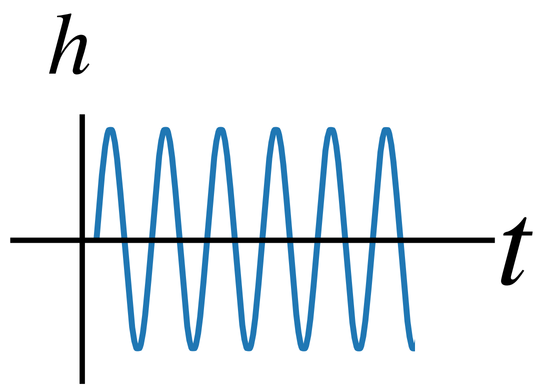
stochastic: $$p(h\,|\,S_h) = \frac{1}{\sqrt{2\pi S_h}}\,\exp\left[-\frac{1}{2}\frac{h^2}{S_h}\right]$$



deterministic: $$p(h\,|\,A, t_0, f_0) = \delta\left(h - A\sin[2\pi f_0(t - t_0)]\right)$$



hybrid: $$p(h\,|\,\xi, A, t_0, f_0) = \xi\,\delta\left(h - A\sin[2\pi f_0(t - t_0)]\right) + (1-\xi)\,\delta(h)$$



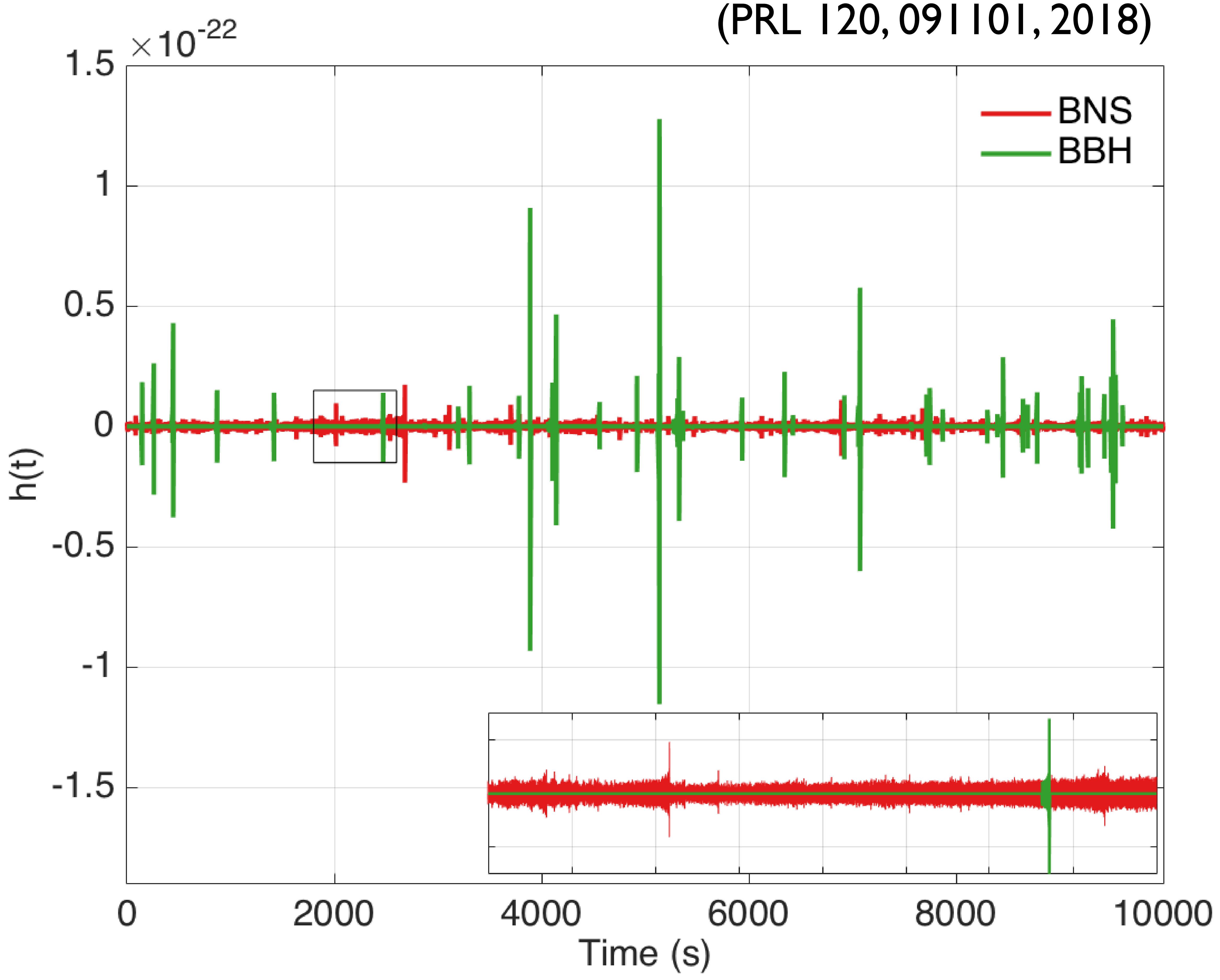
ξ percent or (1-ξ) percent

5. Example: searching for the background from BBH mergers

Recall: Non-stationary background from BBH mergers is a potential signal for advanced LIGO, Virgo

- Recent detections of BBH and BNS mergers by advanced LIGO, VIrgo imply the existence of a stochastic background of weaker events
- Smith & Thrane (PRX 8, 021019,2018) have proposed an alternative method to search for the BBH component, optimally suited for the non-stationarity
- Describe BBH background with a hybrid signal model
- Average over chirp parameters to infer only rate of mergers
- Use two detectors to discriminate against glitches

(PRL 120, 091101, 2018)



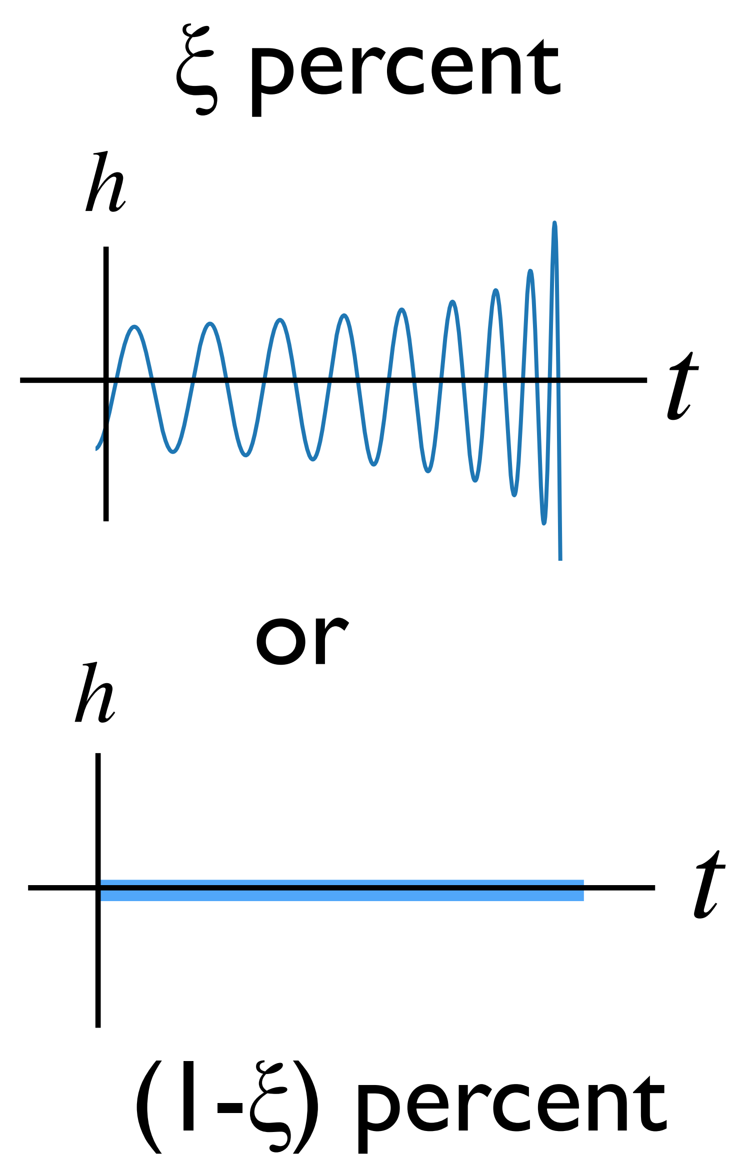
Mathematical details

Split data in short (e.g., 4 sec) segments, which should contain at most 1 BBH merger.
For each segment we have:

Likelihood: $$p(d\,|\,C_n, h) \equiv p_n(d - h\,|\,C_n)$$

Hybrid signal model:

$$p(h\,|\,\xi, \vec{\lambda}) = \xi\,\delta\left(h - \mathrm{chirp}(\vec{\lambda})\right) + (1 - \xi)\,\delta(h)$$



Marginalized likelihoods:

$$p(d\,|\,\xi, \vec{\lambda}) = \int \mathrm{d}h\, p(d\,|\,C_n, h) p(h\,|\,\xi, \vec{\lambda}) \;\; = \xi\, p_n(d - \mathrm{chirp}(\vec{\lambda})\,|\,C_n) + (1 - \xi)\, p_n(d\,|\,C_n)$$

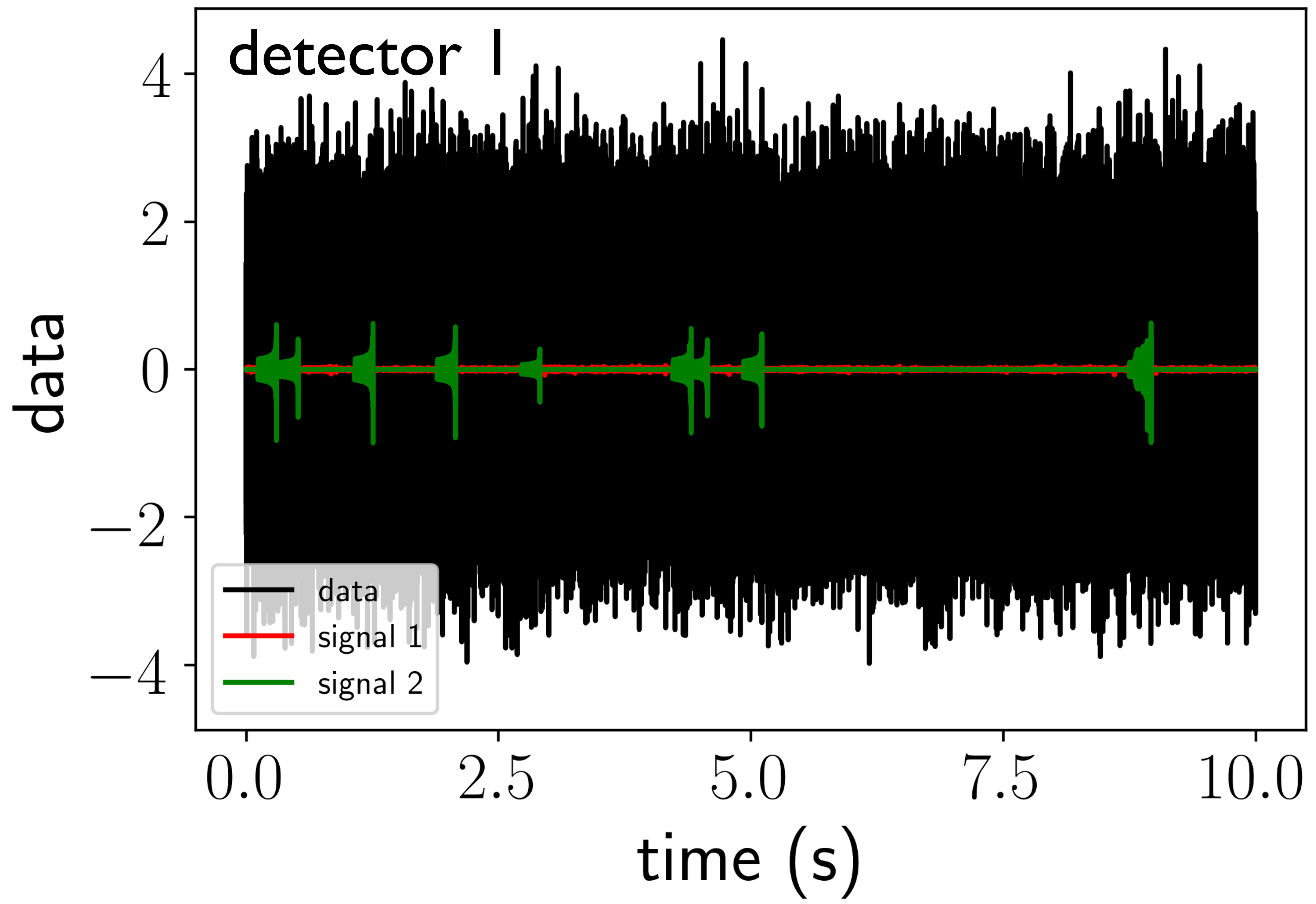
"mixture" gaussian

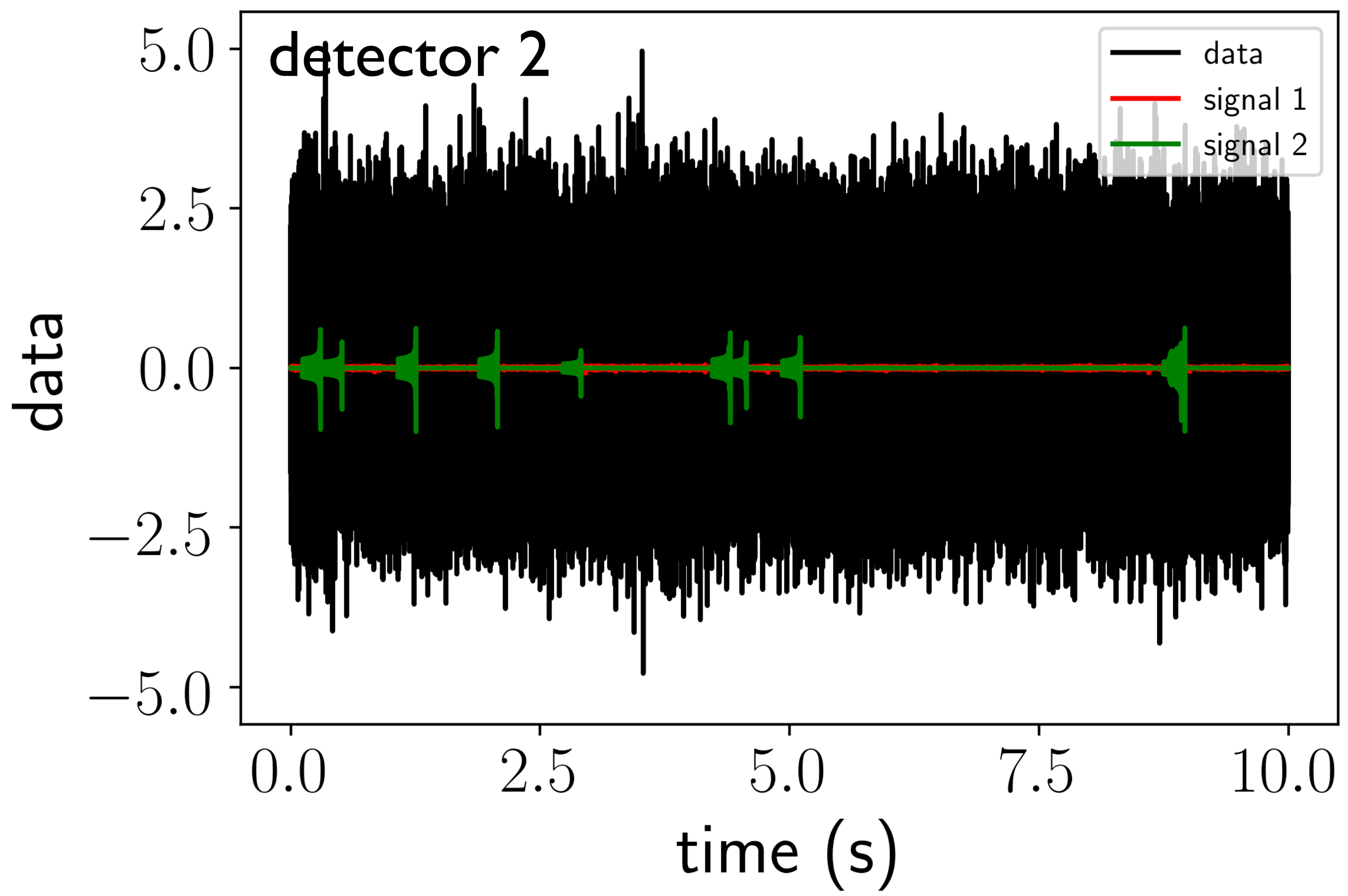
$$p(d\,|\,\xi) = \int \mathrm{d}\vec{\lambda}\; p(d\,|\,\xi, \vec{\lambda})\, p(\vec{\lambda}) \;\; = (S - N)\xi + N$$

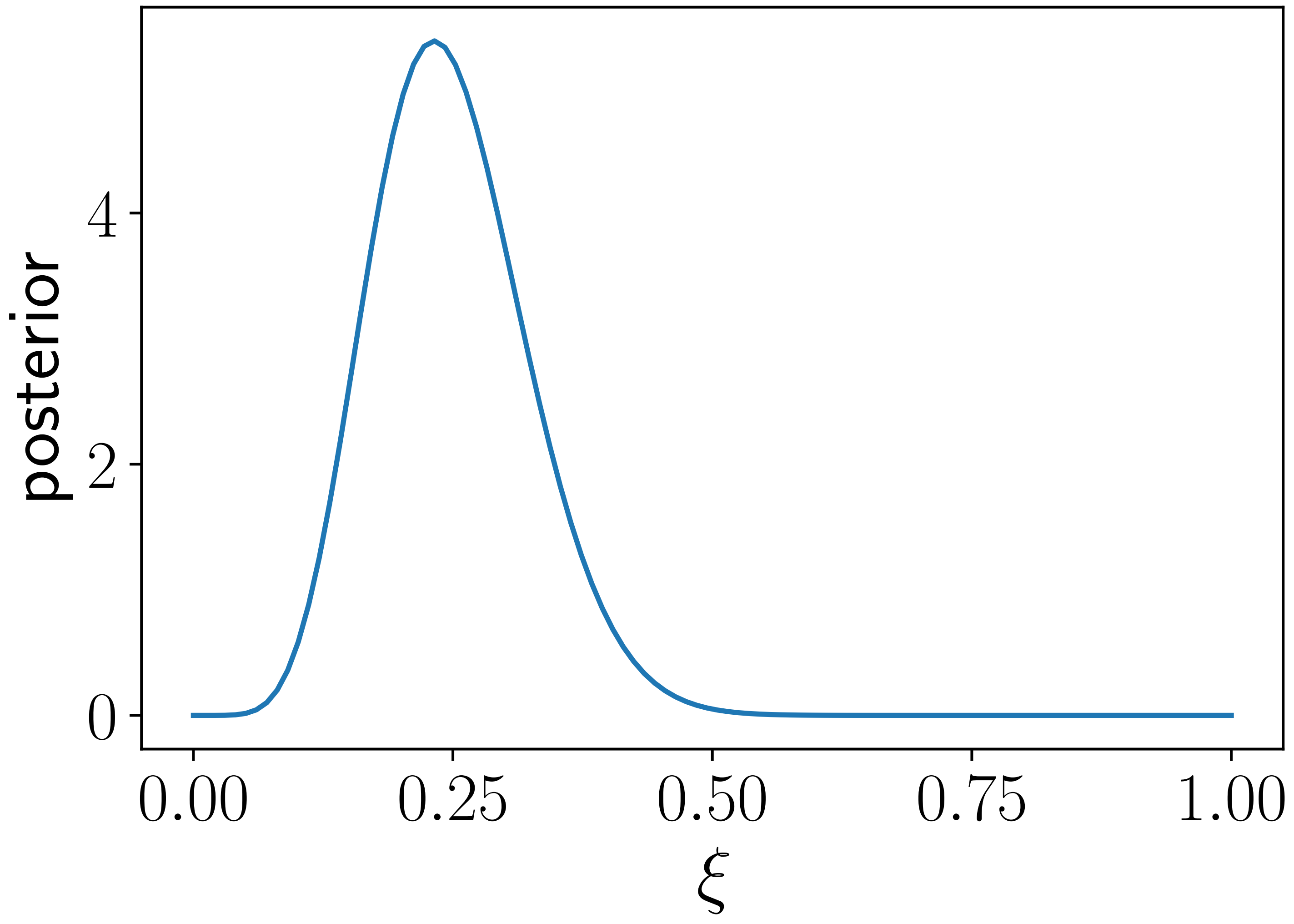
Posterior: $$p(\xi\,|\,d) = \frac{p(d\,|\,\xi) p(\xi)}{p(d)}$$

Combine segments by multiplying likelihoods, ...

Example: Simulated BBH background in white detector noise and confusion-limited BNS background





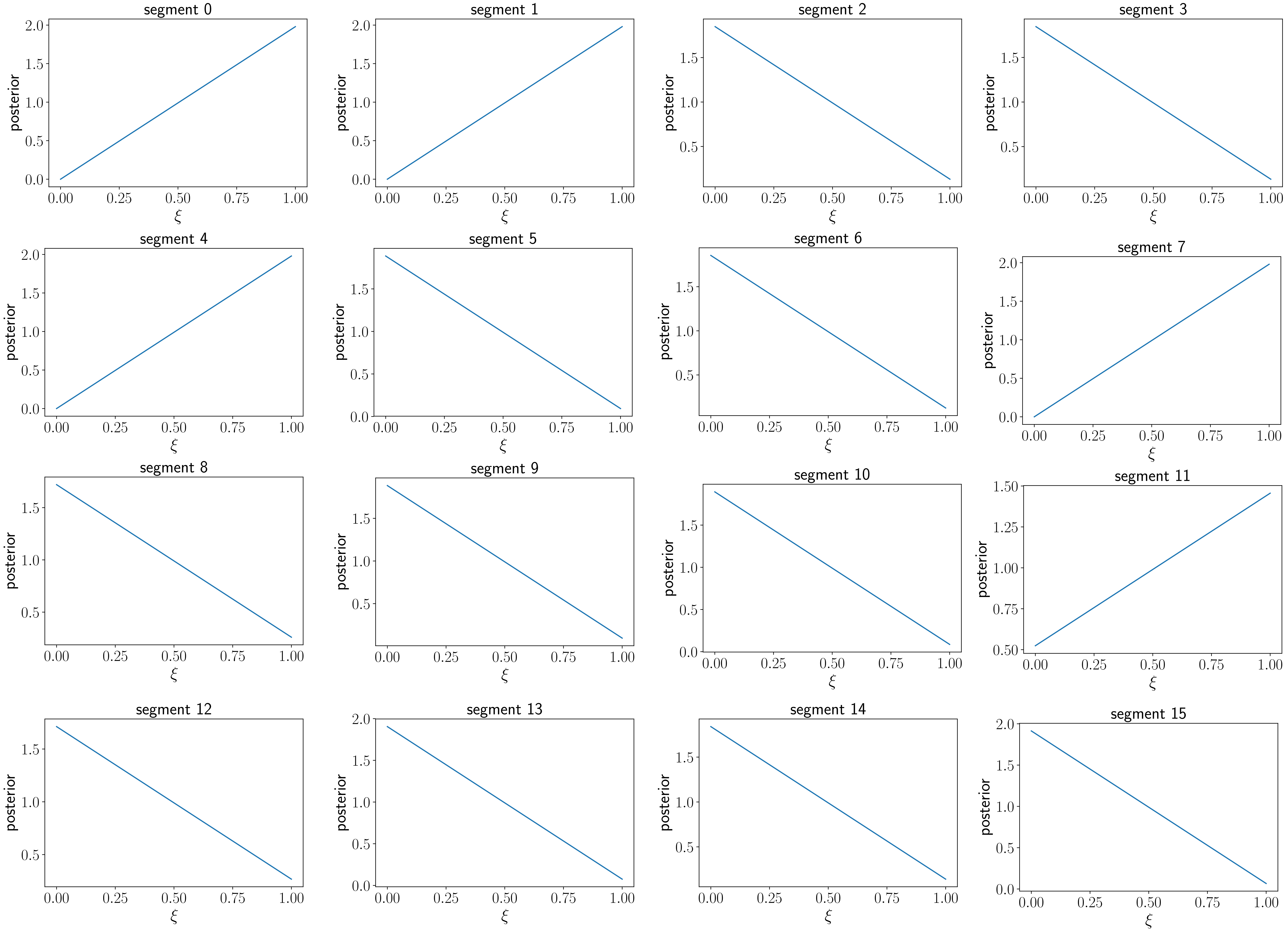


$\mathrm{SNR}_{\text{non-stationary}} = 15.3$

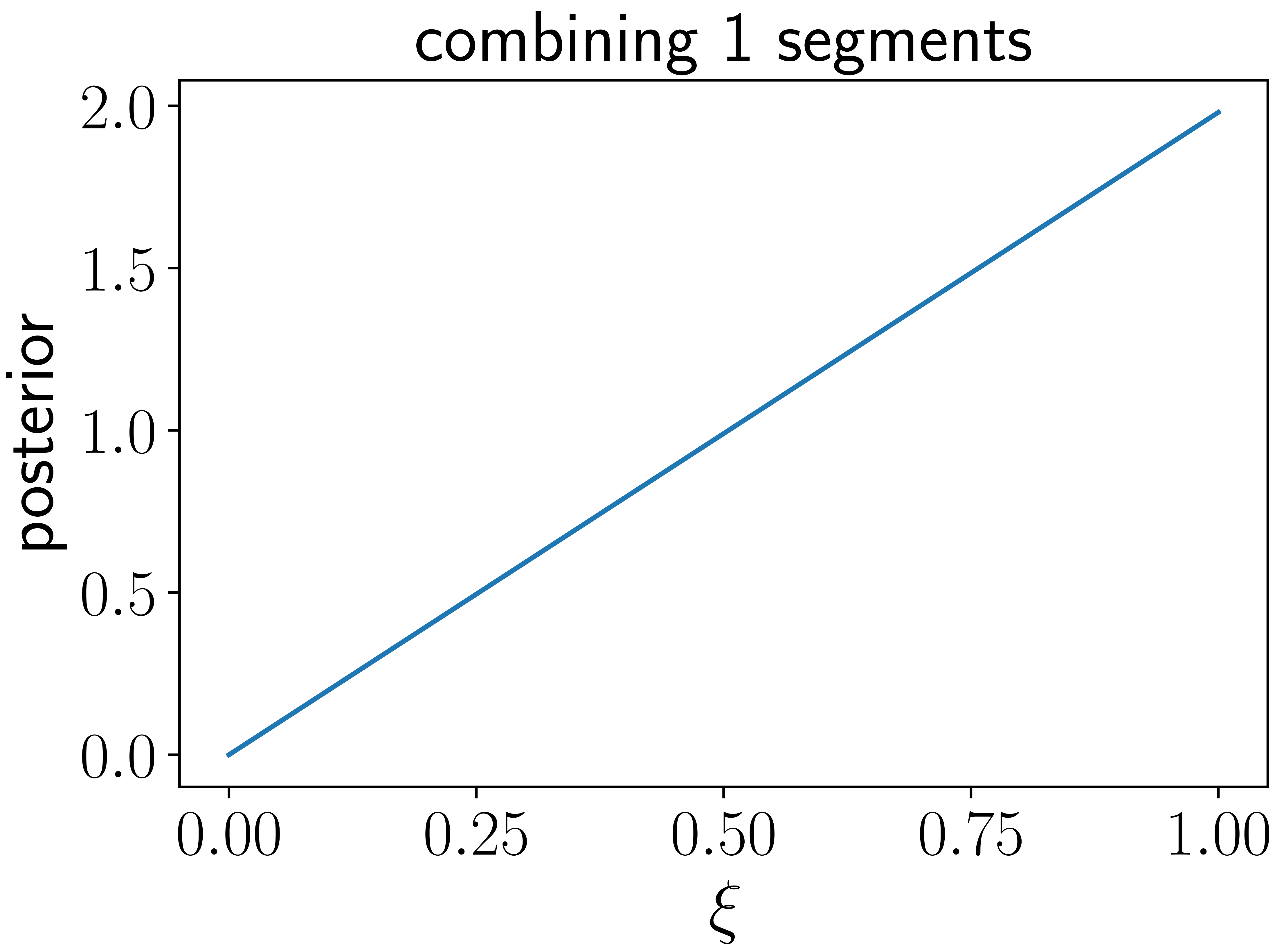
vs

$\mathrm{SNR}_{\text{stationary}} = 8.9$

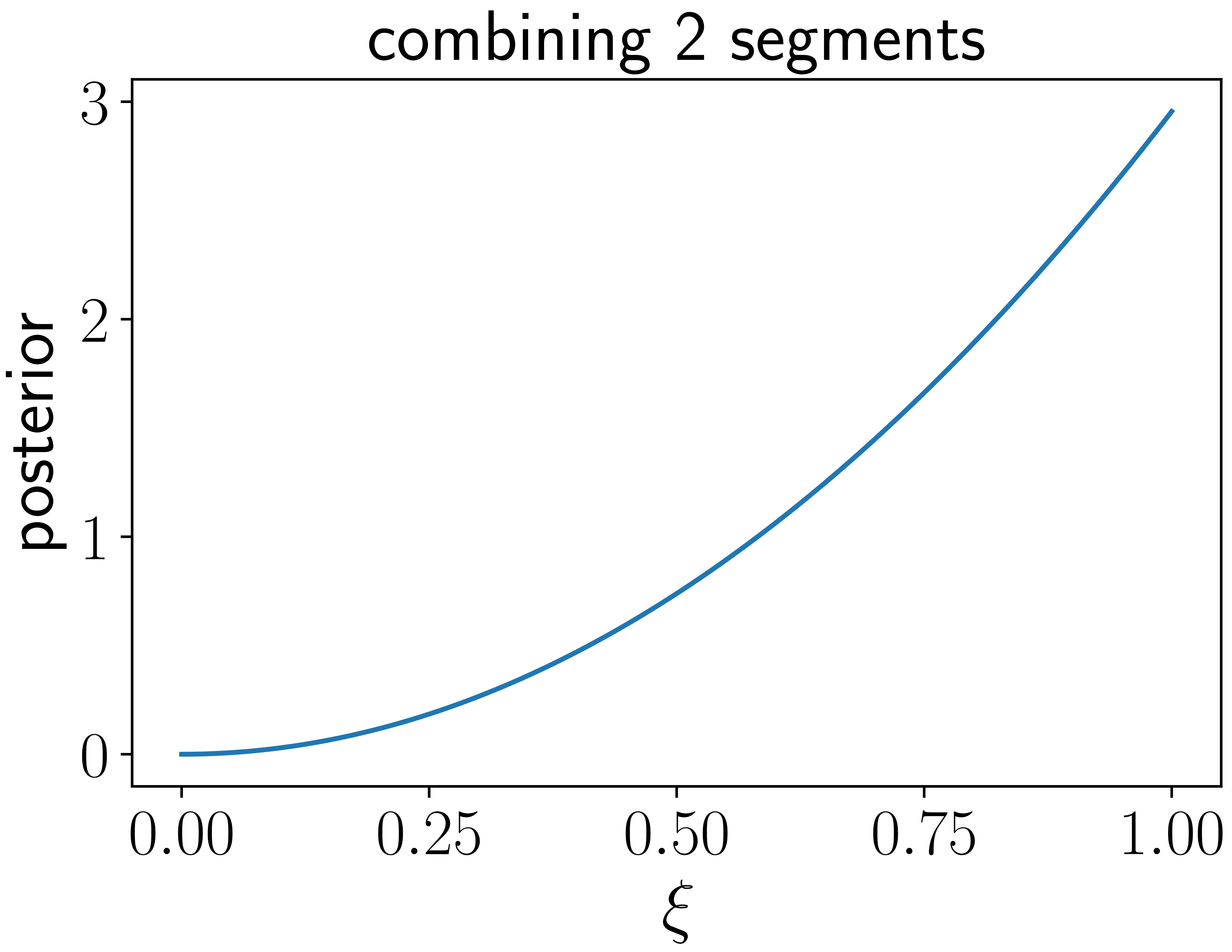
Posteriors $p(\xi|d)$ for individual segments



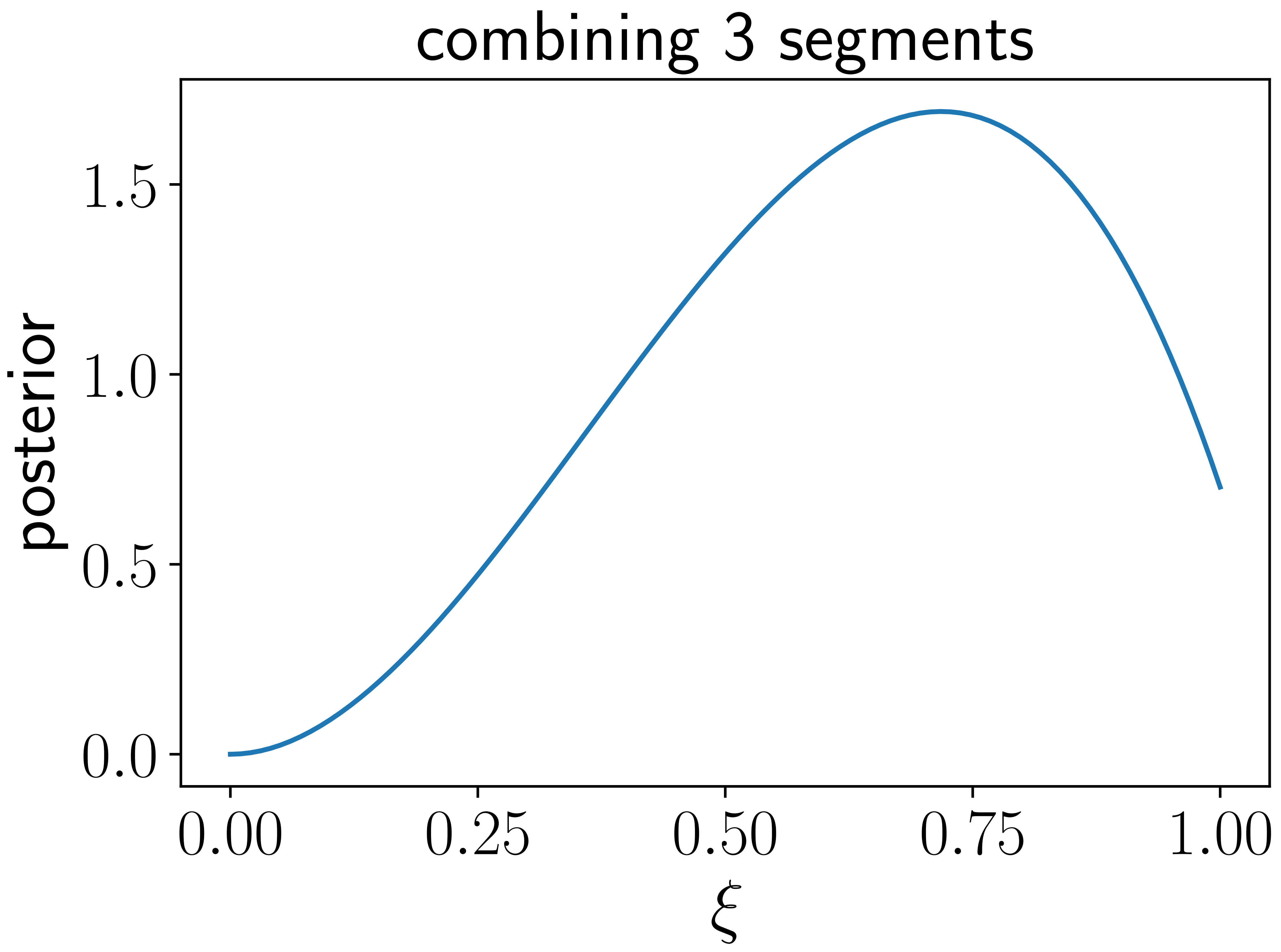
Combining segment posteriors



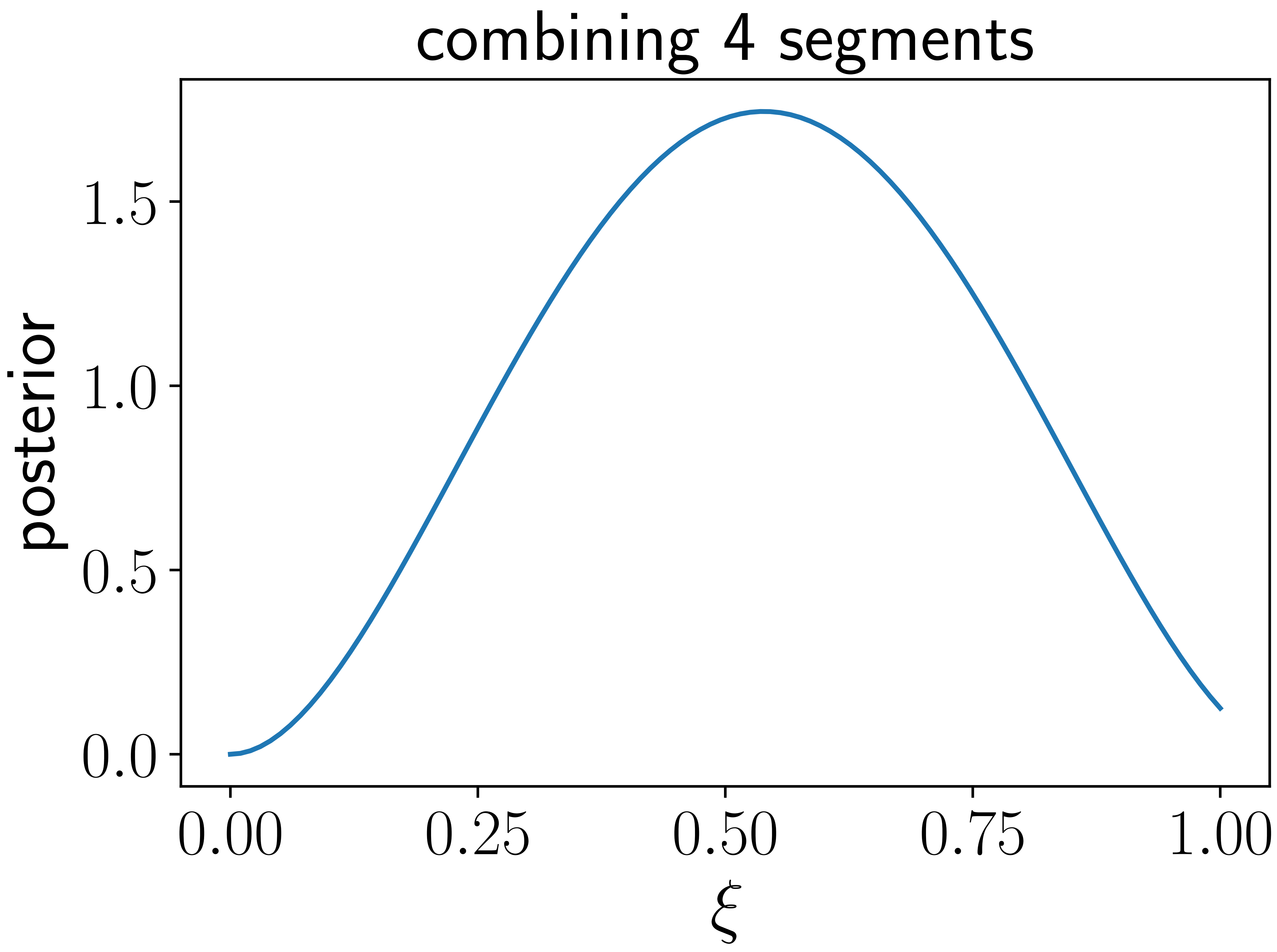
Combining segment posteriors



Combining segment posteriors



Combining segment posteriors



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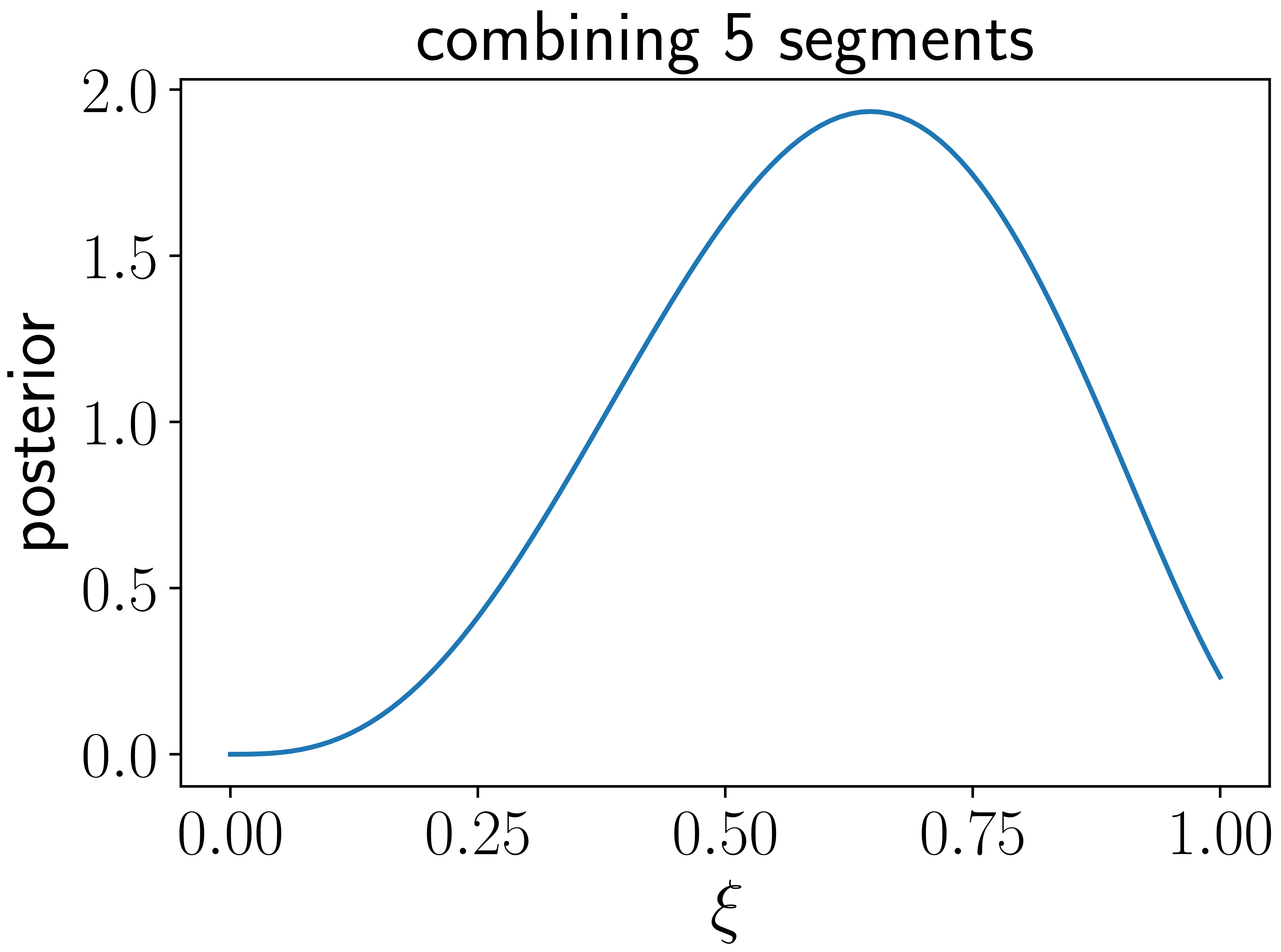

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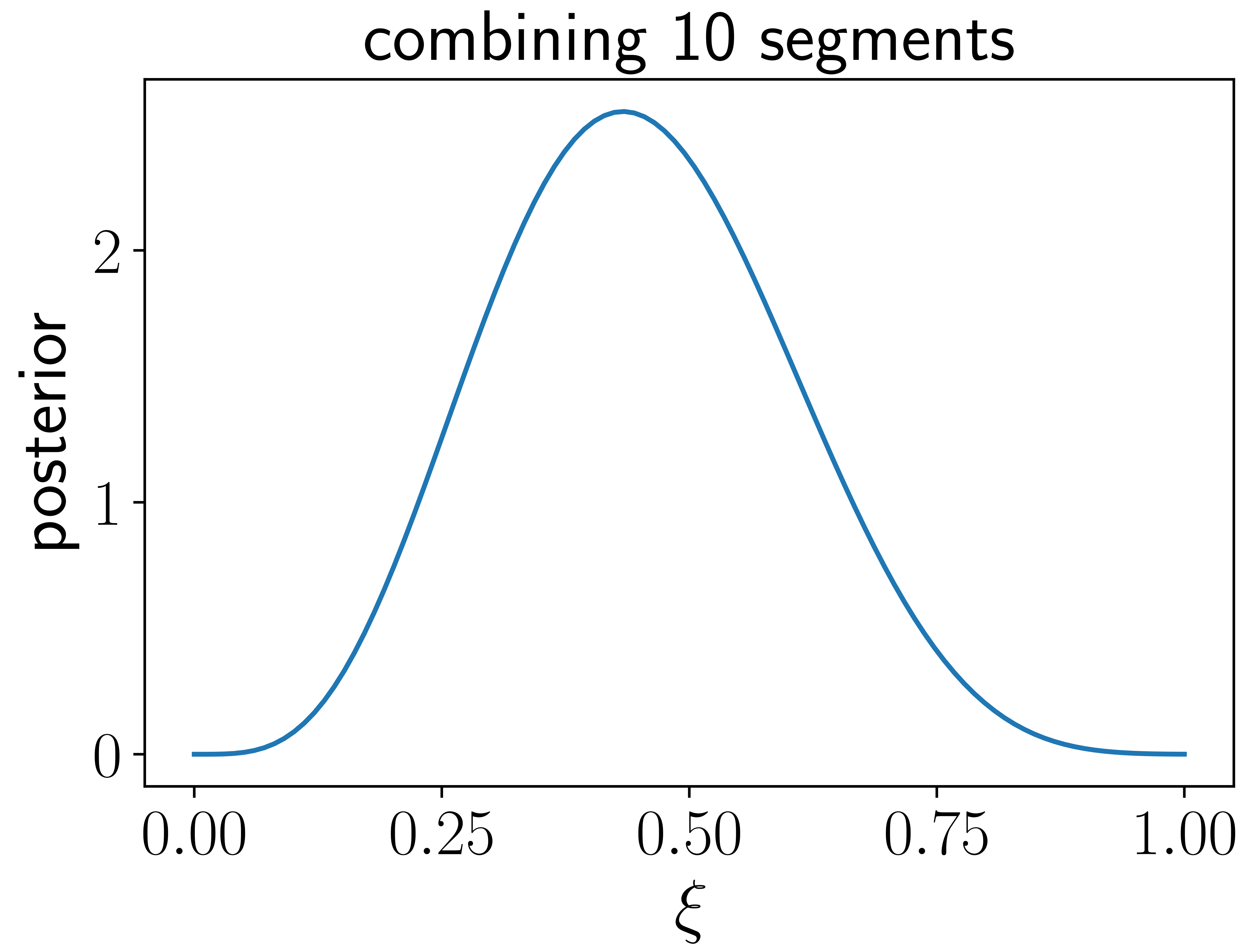
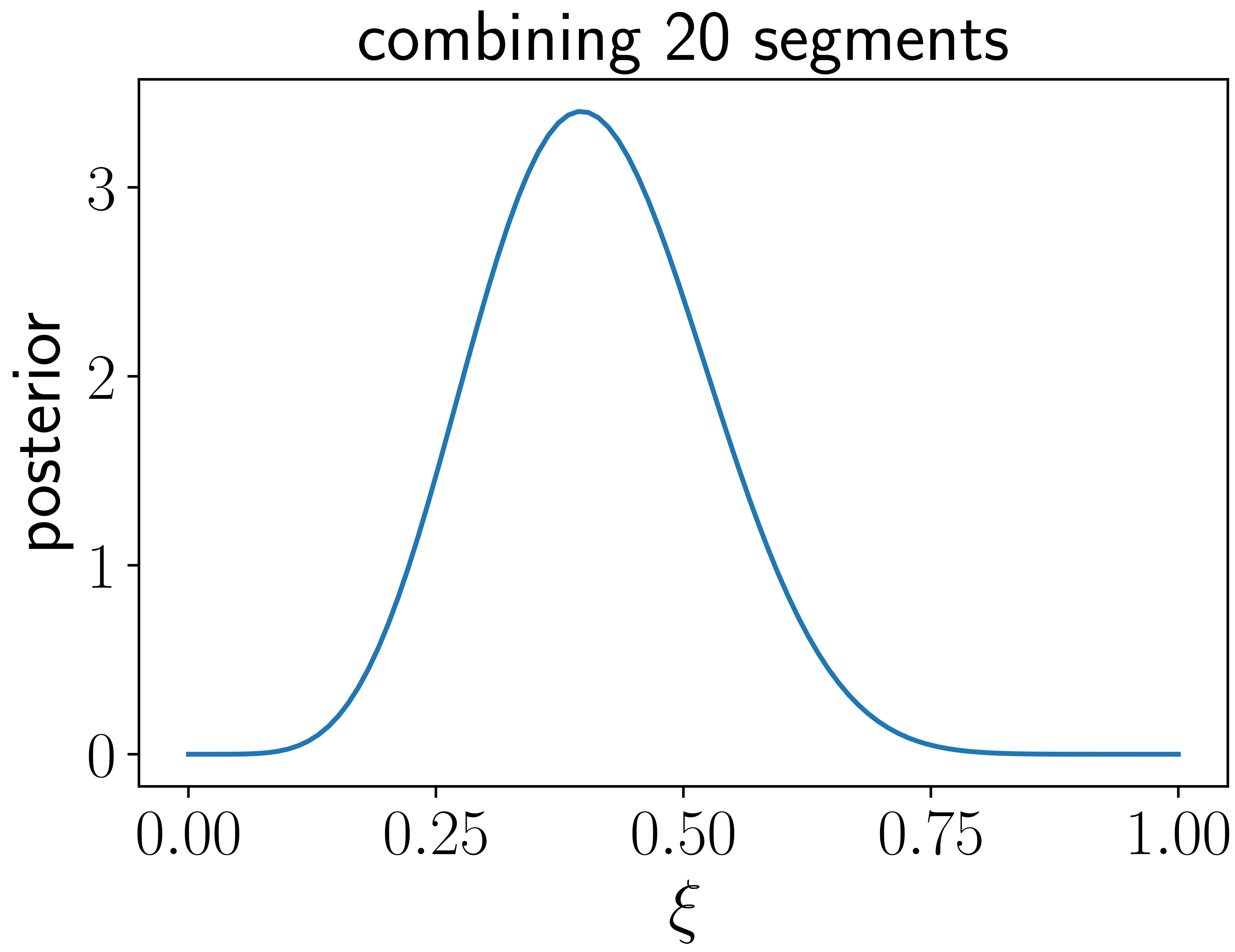
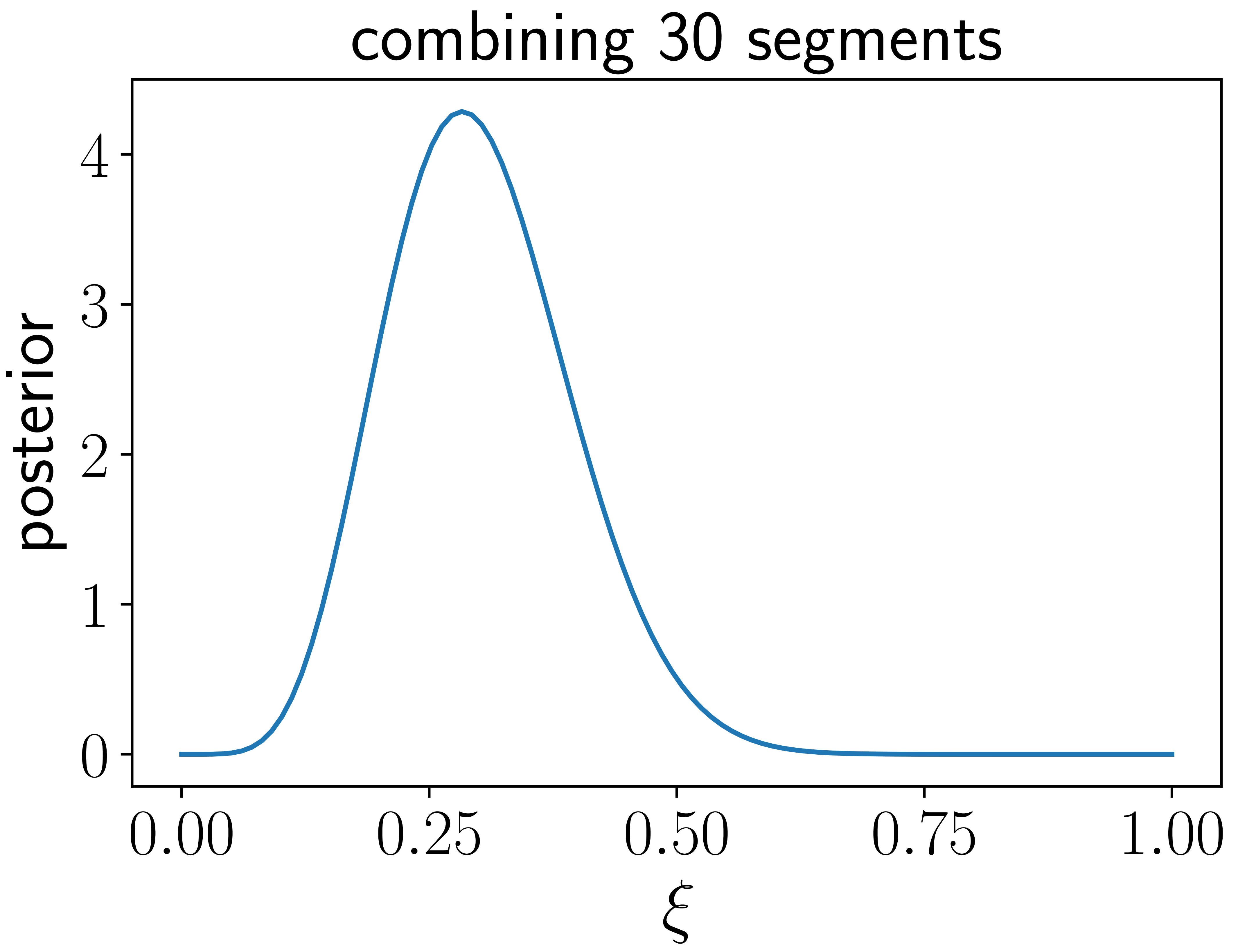


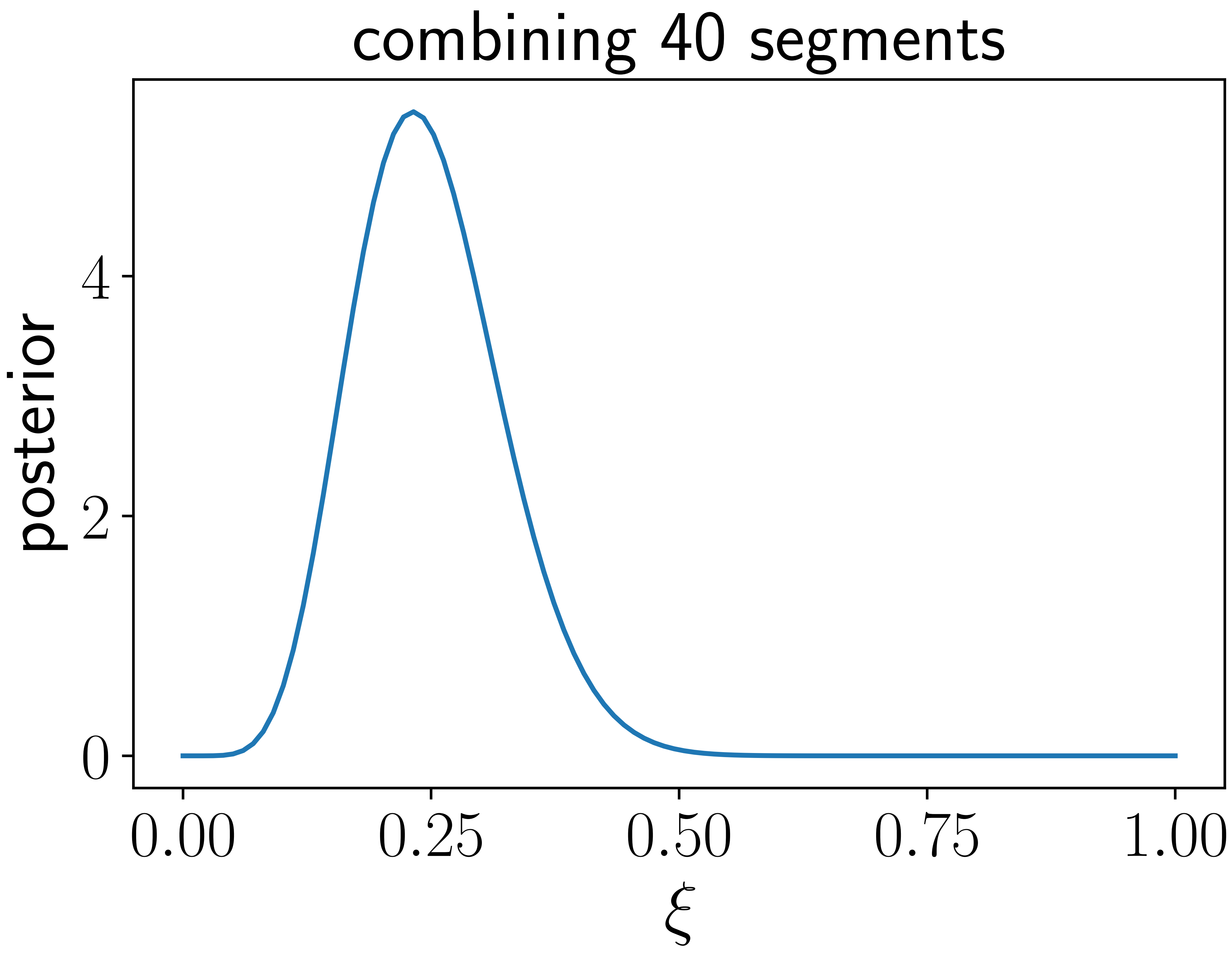
Combining segment posteriors



Combining segment posteriors



Combining segment posteriors



The optimal analysis reduces time to detection because…

- All segments contribute to estimating probability parameter ξ
- BBH chirp signal is deterministic and not stochastic

$$\frac{\mathrm{SNR}_{\mathrm{non-stationary}}}{\mathrm{SNR}_{\mathrm{stationary}}} \sim \sqrt{\frac{N_{\mathrm{cycles}}}{\xi}}$$

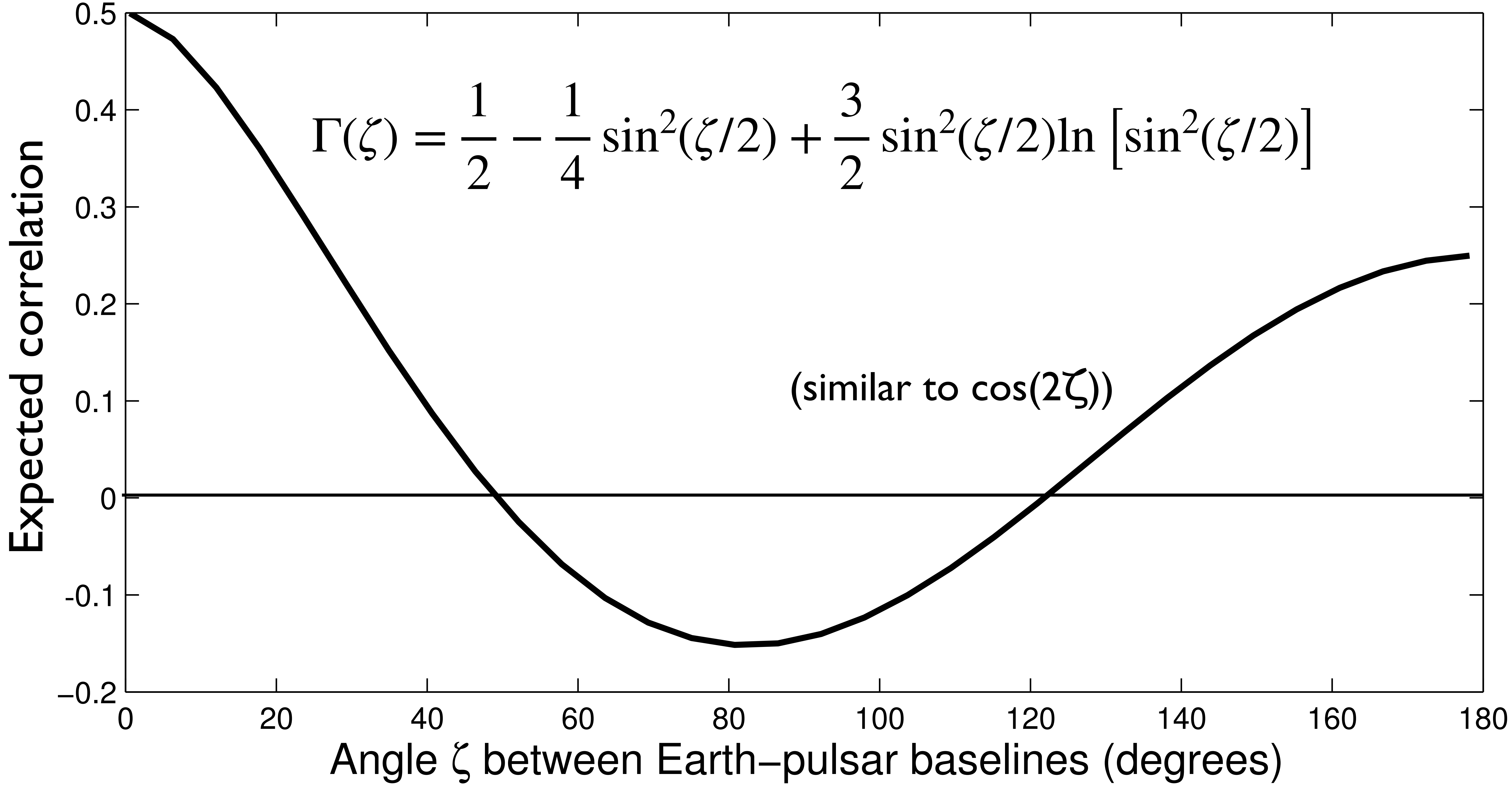
haven't been able to rigorously prove the N_{cylces} part!!

~40 months of observation reduces to ~1 day!!

So stay tuned!!

extra slides

Pulsar timing correlations (Hellings & Downs curve)
(correlation for an isotropic, unpolarized GW background in GR)



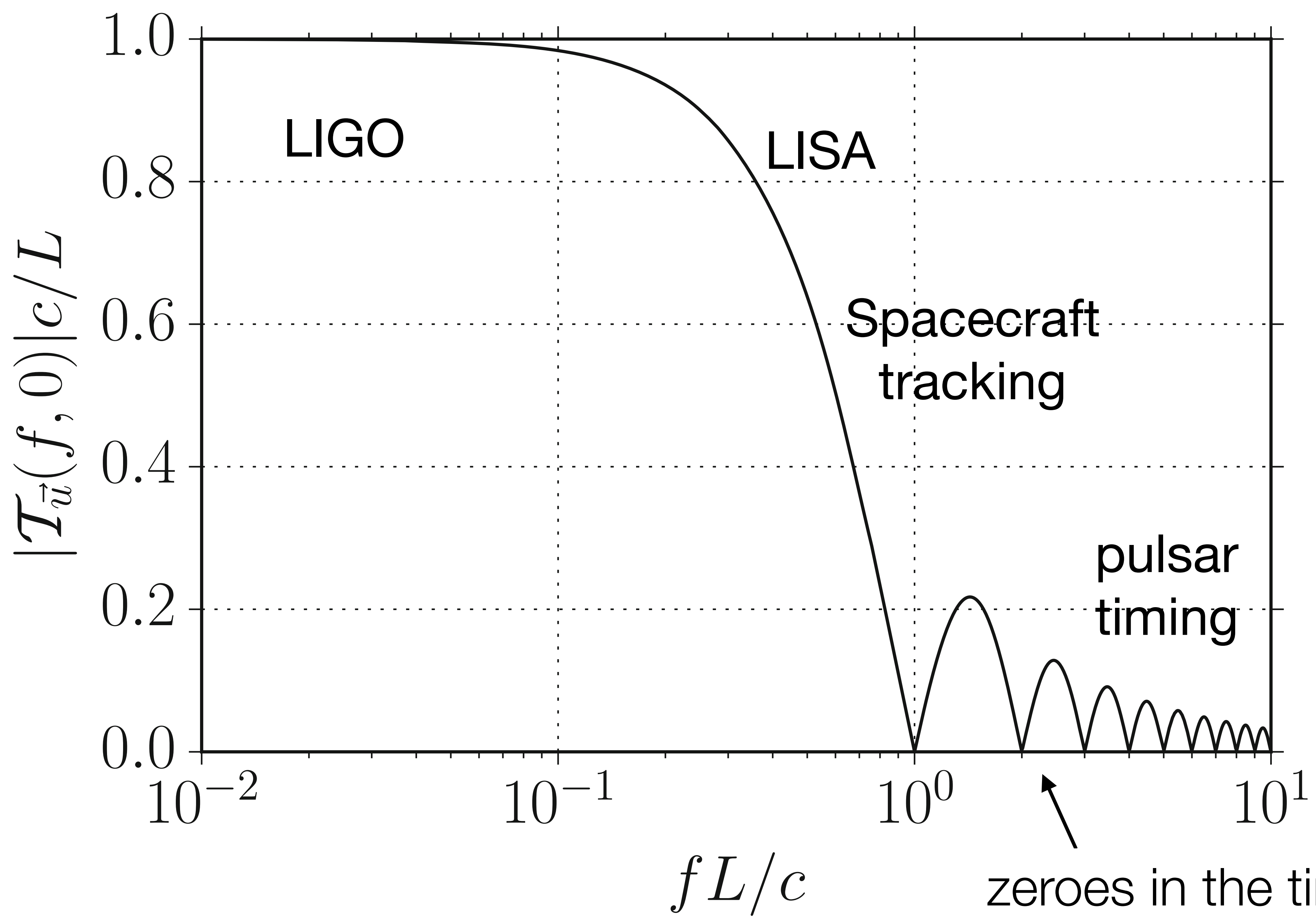
Beyond the short-antenna limit

LISA, spacecraft Doppler tracking and pulsar timing all operate outside of the short-antenna limit

Recall response function:
(one-arm, one-way)

$$R^A(f,\hat{n}) = \frac{1}{2}u^a u^b e^A_{ab}(\hat{n}) \boxed{\frac{1}{i2\pi f}\frac{1}{1+\hat{n}\cdot\hat{u}}\left[1-e^{-\frac{i2\pi fL}{c}(1+\hat{n}\cdot\hat{u})}\right]} e^{i2\pi f\hat{n}\cdot\vec{r}_2/c}$$

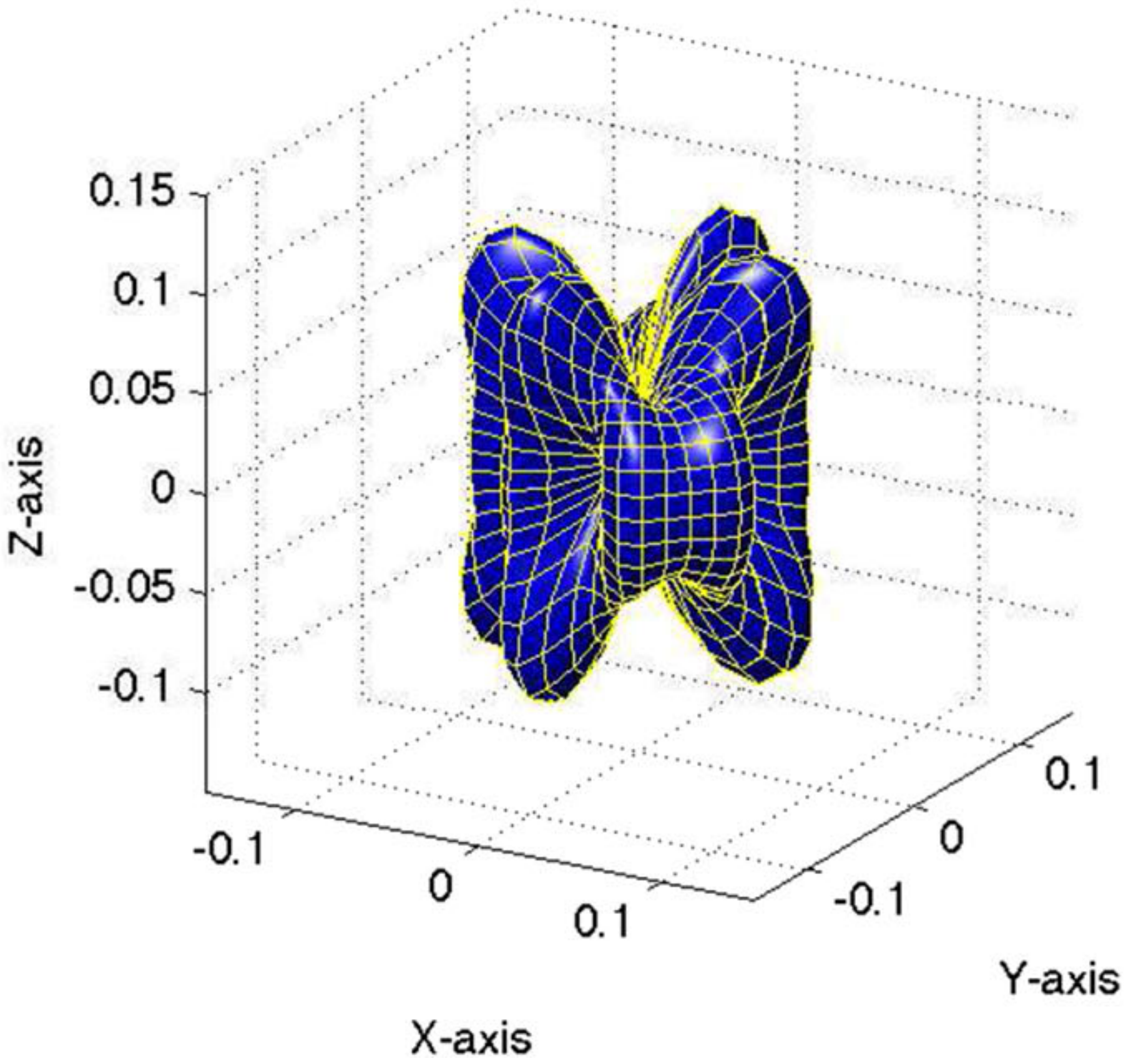
$$\mathcal{T}_{\vec{u}}(f,\hat{n}\cdot\hat{u}) \equiv \frac{1}{i2\pi f}\frac{1}{1+\hat{n}\cdot\hat{u}}\left[1-e^{-\frac{i2\pi fL}{c}(1+\hat{n}\cdot\hat{u})}\right] = \frac{L}{c}e^{-\frac{i\pi fL}{c}(1+\hat{n}\cdot\hat{u})}\,\mathrm{sinc}\left(\frac{\pi fL}{c}[1+\hat{n}\cdot\hat{u}]\right)$$



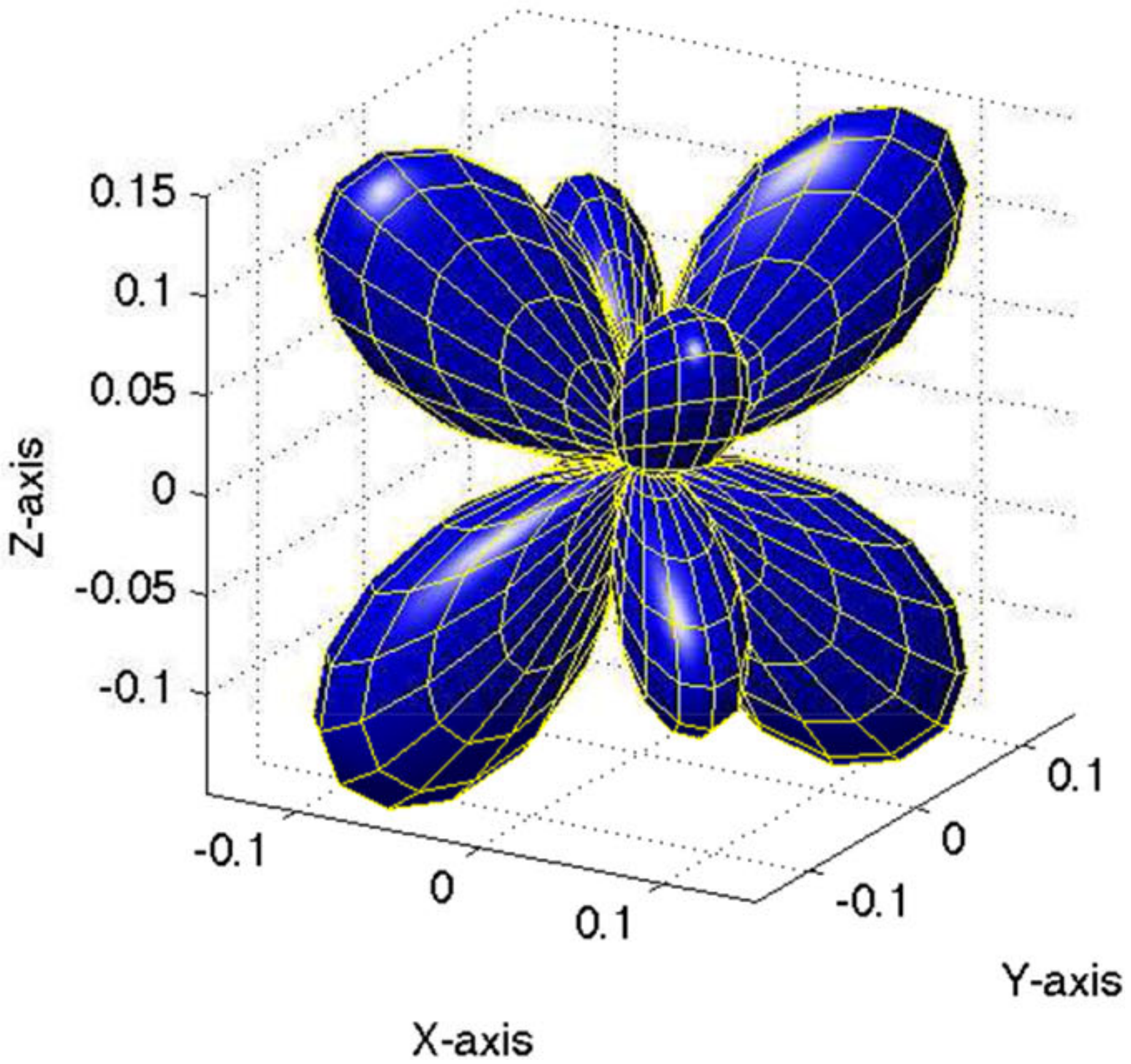
Beam detector	L (km)	f_* (Hz)	f (Hz)	f/f_*	Relation
Ground-based interferometer	~ 1	$\sim 10^5$	$10 - 10^4$	$10^{-4} - 10^{-1}$	$f \ll f_*$
Space-based interferometer	$\sim 10^6$	$\sim 10^{-1}$	$10^{-4} - 10^{-1}$	$10^{-3} - 1$	$f \lesssim f_*$
Spacecraft Doppler tracking	$\sim 10^9$	$\sim 10^{-4}$	$10^{-6} - 10^{-3}$	$10^{-2} - 10$	$f \sim f_*$
Pulsar timing	$\sim 10^{17}$	$\sim 10^{-12}$	$10^{-9} - 10^{-7}$	$10^3 - 10^5$	$f \gg f_*$

zeroes in the timing transfer function at multiples of $f_* \equiv c/L$

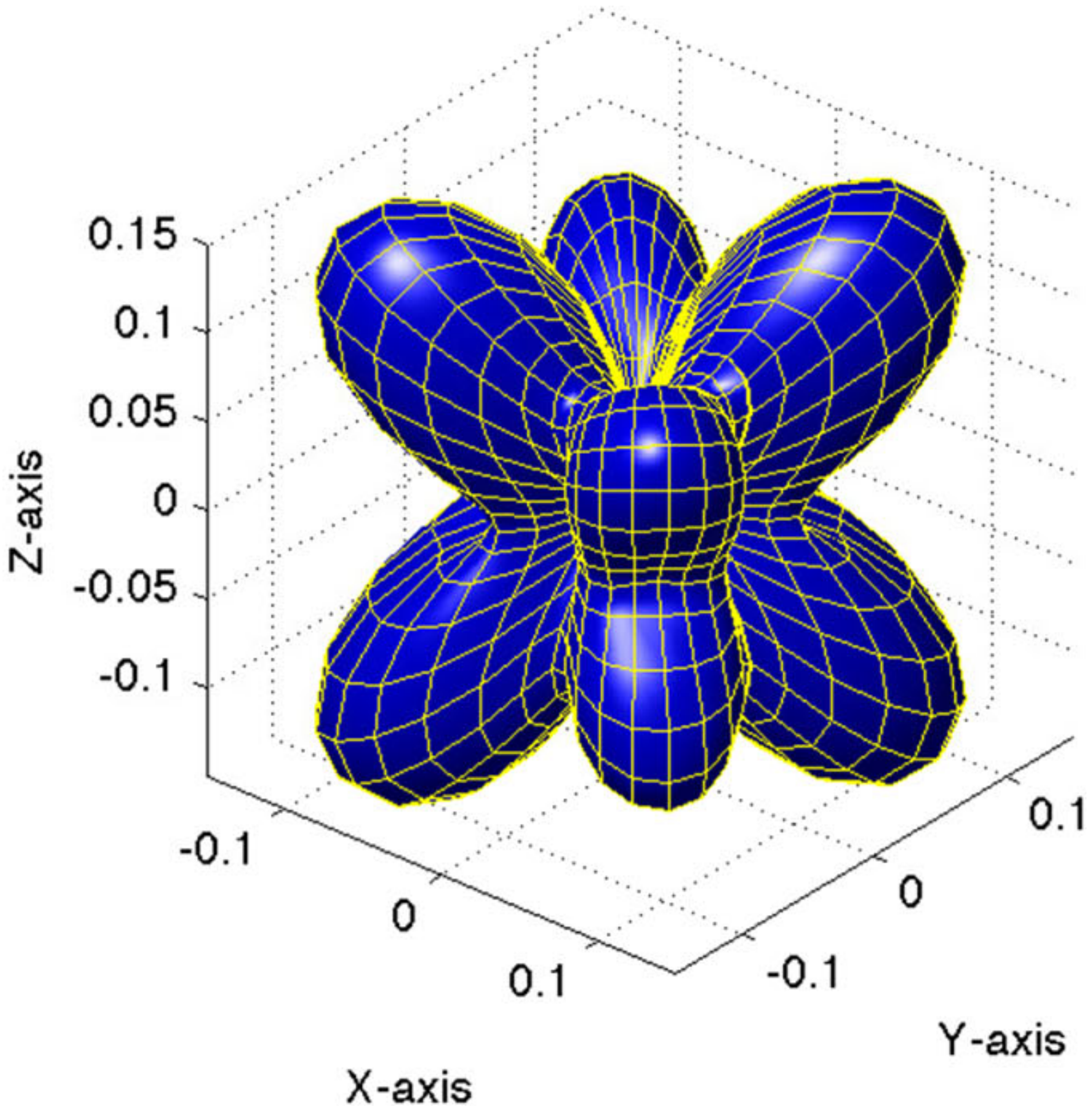
Beam pattern functions



$|R^{+}(f, \hat{n})|$

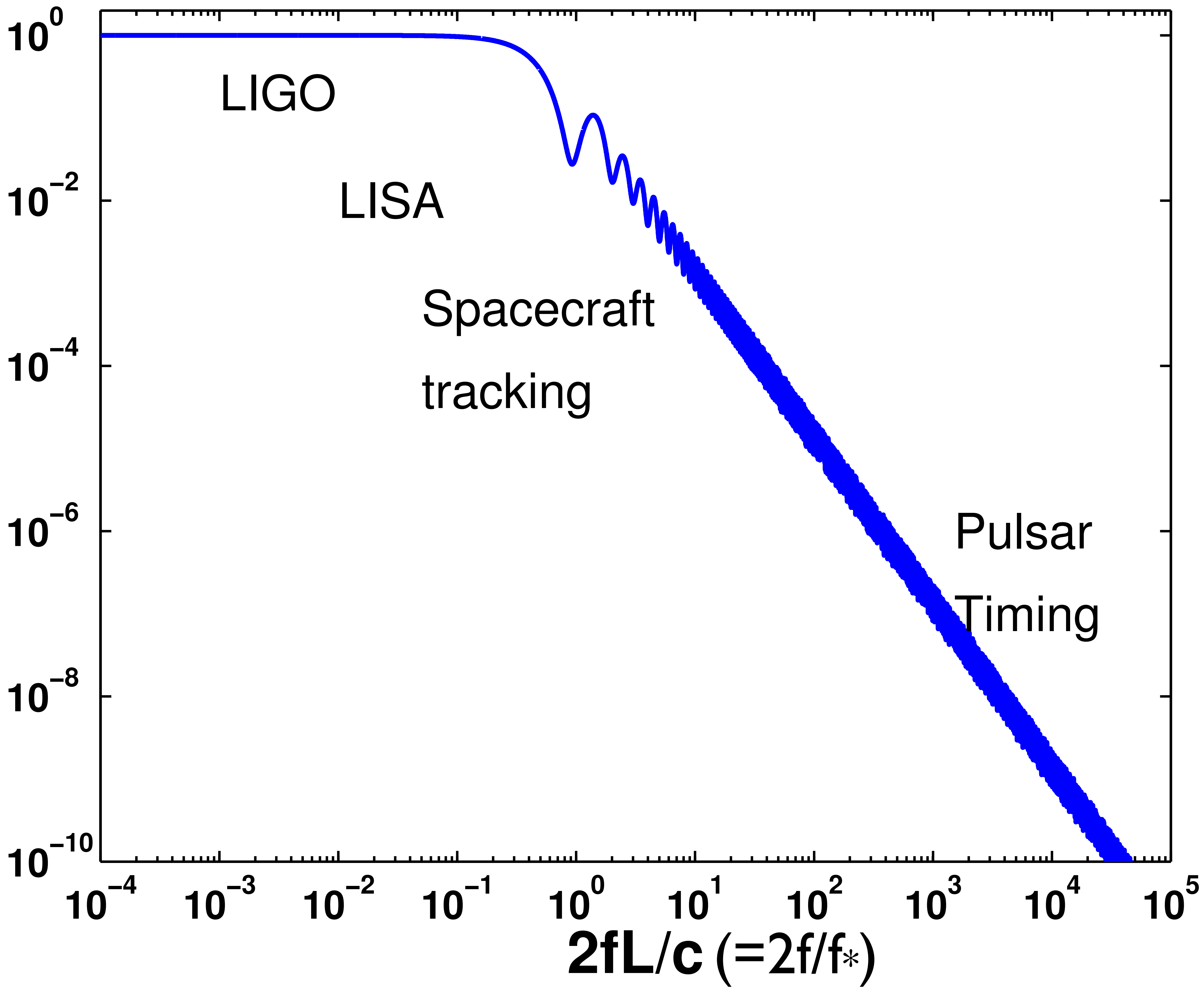


$|R^{\times}(f, \hat{n})|$



$\left(|R^{+}(f, \hat{n})|^2 + |R^{\times}(f, \hat{n})|^2\right)^{1/2}$

($f = c/(2L) = 37.5$ kHz)



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Pulsar timing	$\sim 10^{17}$	$\sim 10^{-12}$	$10^{-9} - 10^{-7}$	$10^3 - 10^5$	$f \gg f_*$

$\mathcal{B}_{\alpha\beta}(d)$	$2 \ln \mathcal{B}_{\alpha\beta}(d)$	Evidence for model $\mathcal{M}_\alpha$ relative to $\mathcal{M}_\beta$
<1	<0	Negative (supports model $\mathcal{M}_\beta$)
1–3	0–2	Not worth more than a bare mention
3–20	2–6	Positive
20–150	6–10	Strong
>150	>10	Very strong

Adapted from Kass and Raftery (1995)

Matched-filtering determination of measured TOAs

$$C(\Delta t) = \mathcal{N} \int \mathrm{d}t \; y(t) p(t - \Delta t)$$