Physics for Scientists and Engineers II
Lecture summaries

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Abstract

Disclaimer: These notes *summarize* the key concepts and formulae discussed during lecture. They should be supplemented by any standard calculus-based physics textbook on these topics (e.g., by Serway and Jewett) or internet resource to fill in any gaps. Please send any comments, criticisms, suggestions to: `joseph.d.romano@gmail.com`.

Contents

1 Electric charge, electric force

1.1 Electric charge

- Properties of electric charge:

  (i) Two types: *positive* and *negative* (+ and −), whose effects tend to cancel. Opposite charges attract, like charges repel.

  (ii) Total electric charge is *conserved* (cannot be created or destroyed, only transferred from one object to another).

  (iii) Electric charge is *quantized* (integer multiples of the charge of an proton $e = 1.602 \times 10^{-19}$ C; an electron has charge $-e$).

- Materials:

  (i) *Conductors*: one or more electrons per atom are able to move freely through the material. Examples: aluminum, copper, gold, ... (metals).

  (ii) *Insulators*: electrons are bound to atomic nuclei. Examples: rubber, wood, paper, ...

- Some demos:

  1) A plastic rod rubbed with fur is negatively charged; a glass rod rubbed with silk is positively charged. Show that opposite charges attract, like charges repel.

  2) An uncharged aluminum can will be attracted to both negative and positive rods. Electrons in the aluminum can (a conductor) move away from the negative rod or toward the positive rod leading to an attractive force in bothe cases.

  3) Bits of uncharged paper will be attracted to both negative and positive rods due to charge separation within the atoms / molecules in the paper (an insulator).

  4) A charged rubber ballon sticks to the white board for the same reason as (3).

1.2 Electric force

- *Coulomb's law*: The electric force on point charge q_1 due to point charge q_2, at rest with respect to one another and separated by a distance r:

$$\vec{F}_{12} = \frac{k_e q_1 q_2}{r^2}\,\hat{r}\,, \quad \text{where} \quad k_e = 8.99 \times 10^9\ \frac{\mathrm{N \cdot m^2}}{\mathrm{C^2}} \tag{1}$$

  and $\hat{r}$ is a unit vector pointing from q_2 to q_1.

- The force is attractive (in the $-\hat{r}$ direction) if the two charges have opposite signs; the force is repulsive (in the $+\hat{r}$ direction) if the two charges have the same signs.

- One sometimes writes

$$k_e = \frac{1}{4\pi\epsilon_0}\,, \quad \text{where} \quad \epsilon_0 = 8.85 \times 10^{-12}\ \frac{\mathrm{C^2}}{\mathrm{N \cdot m^2}} \tag{2}$$

  where ϵ_0 is the so-called *permitivity* of free space (vacuum).

- The electric force on point charge Q due to a set of point charges q_1, q_2, ..., q_N is given by the vector sum of the forces

$$\vec{F} = \sum_{i=1}^{N} \frac{k_e Q q_i}{r_i^2}\,\hat{r}_i \tag{3}$$

  where r_i and $\hat{r}_i$ are the separation and unit vector pointing from q_i to Q. This is called the *superposition principle*.

2 Electric field, electric field lines

2.1 Electric field

- The electric force between charges is a *non-contact* force. A set of charges creates a *field* to which another charge responds.
- The electric field at a point P in space is a vector $\vec{E}$, whose direction is given by the electric force $\vec{F}_e$ on a small, positive (test) charge q_0 placed at P, and whose magnitude is given by F_e/q_0:

$$\vec{E} = \vec{F}_e/q_0 \tag{4}$$

- For a single point charge q:

$$\vec{E} = \frac{k_e q}{r^2}\,\hat{r} \tag{5}$$

- For a set of discrete point charges q_i, $i = 1, 2, \cdots, N$ or a continuous charge distribution:

$$\vec{E} = \sum_{i=1}^{N} \frac{k_e q_i}{r_i^2}\,\hat{r}_i \quad \text{(discrete)}, \qquad \vec{E} = \int \frac{k_e dq}{r^2}\,\hat{r} \quad \text{(continuous)} \tag{6}$$

  where

$$dq = \rho\, dV\,, \quad dq = \sigma\, dA\,, \quad dq = \lambda\, dl\,, \tag{7}$$

  for volume, surface, and line charge densities ρ, σ, and λ, respectively.

2.2 Worked examples

- Electric dipole: Two charges, $+q$ and $-q$ separated by distance a along the x-axis (located at $x = -a/2$, $x = a/2$, respectively). Find $\vec{E}$ at point $P = (0, y)$.

$$E_y(P) = 0\,, \qquad E_x(P) = \frac{k_e q a}{[y^2 + (a/2)^2]^{3/2}} \tag{8}$$

  For $y \gg a$, $E_x(P) \simeq k_e q a/y^3$ (note $1/y^3$ dependence).

- Uniform line charge λ: Length l, charge Q, located on the x-axis with ends at $x = -l/2$, $x = l/2$. Find $\vec{E}$ at point $P = (0, y)$.

$$E_x(P) = 0\,, \qquad E_y(P) = \frac{k_e Q}{y} \frac{1}{\sqrt{y^2 + (l/2)^2}} \tag{9}$$

  For $y \gg l$, $E_y(P) \simeq k_e Q/y^2$ (point charge).

  For $y << l$, $E_y(P) \simeq \lambda/2\pi\epsilon_0 y$ (infinite line charge).

2.3 Electric field lines

- Pictorial representation of the electric field $\vec{E}$. Direction of field lines point in the direction of $\vec{E}$; density of field lines (number per area perpendicular to $\vec{E}$) is proportional to the magnitude of $\vec{E}$.
- Rules for drawing electric field lines:

  (i) lines begin on + charges, end on − charges (or may extend to infinity);

  (ii) the number of lines leaving or terminating on a charge should be proportional to the magnitude of the charge;

  (iii) electric field lines cannot cross.

- For a point charge, the density of lines at a distance r from the charge is given by $N/4\pi r^2$, which is proportional to the magnitude of the field $E = k_e q/r^2$ as it should.

3 Electric flux, Gauss's law

3.1 Electric flux

- Electric flux Φ_E is the "flow" of the electric field through a surface. It is proportional to the (net) number of electric field lines passing through the surface.
- For a uniform electric field $\vec{E}$ and flat surface S with area A and unit normal $\hat{n}$,

$$\Phi_E = EA\cos\theta = E_\perp A = EA_\perp \tag{10}$$

  where θ is the angle between $\vec{E}$ and $\hat{n}$.
- $E_\perp$ is the component of $\vec{E}$ perpendicular to the surface (so in the same direction as $\hat{n}$); $A_\perp$ is the component of the area perpendicular to the electric field lines.
- If the surface S is curved or the electric field $\vec{E}$ is non-uniform, we must sum up the contributions associated with infinitesimal area elements $d\vec{A} = \hat{n}\,dA$:

$$\Phi_E = \int_S \vec{E}\cdot\hat{n}\,dA \tag{11}$$

- For a closed surface S, $\hat{n}$ is taken to be the *outward-pointing* normal.

3.2 Gauss's law

- *Gauss's law*:

$$\oint_S \vec{E}\cdot\hat{n}\,dA = \frac{q_{\rm enc}}{\epsilon_0} \tag{12}$$

  where $q_{\rm enc}$ is the (net) electric charge enclosed by the surface S.
- If $q_{\rm enc} = 0$, the flux of the electric field through S is zero.
- It is easy to "prove" Gauss's law for a single point charge q enclosed by a spherical surface centered on q.

  (i) The result is independent of the radius of the sphere, since the surface area of the sphere is proportional to r^2, while the magnitude of the electric field falls-off like $1/r^2$.

  (ii) The result is also independent of the shape of the surface enclosing q, since

$$\hat{r}\cdot\hat{n}\,dA = r^2\sin\theta\,d\theta\,d\phi \tag{13}$$

  for any area element $\hat{n}\,dA$ (i.e., the surface need not be a sphere centered on the charge).
- For something more complicated than a single point charge, use the superposition principle to calculate $\vec{E}$; the RHS is the (net) sum of the charges making up the distribution that are enclosed by the surface S.

4 Gauss's law examples, conductors

4.1 Gauss's law examples

- Gauss's law can be used to calculate the electric field created by highly-symmetric charge distributions. These include charge distributions that are spherically symmetric, cylindrically symmetric, or planar symmetric; or some combinations of these.
- Spherical symmetry:

  Example: Find the electric field $\vec{E}$ both inside and outside a uniformly-charged sphere of radius a and charge Q.

  Answer: : The electric field is directed radially outward with magnitude

  $$E = \begin{cases} k_e Q/r^2, & r > a \\ k_e Q r/a^3, & 0 < r < a \end{cases} \tag{14}$$

  so the field is that of a point charge Q at the center of the sphere if $r > a$.

  Example: Find the electric field $\vec{E}$ both inside and outside a uniformly-charged spherical shell of radius a and charge Q.

  Answer: : The electric field is directly radially outward with magnitude

  $$E = \begin{cases} k_e Q/r^2\,, & r > a \\ 0\,, & 0 < r < a \end{cases} \tag{15}$$

- Cylindrical symmetry:

  Example: Find the electric field $\vec{E}$ a perpendicular distance r away from an infinite line charge with uniform linear charge density λ.

  Answer: The electric field is directed radially outward from the line charge with magnitude $E = \lambda/2\pi\epsilon_0 r$.

- Planar symmetry:

  Example: Find the electric field $\vec{E}$ a perpendicular distance r away from an infinite planar sheet of charge with uniform surface charge density σ.

  Answer: The electric fields is directed perpendicular to the surface with magnitude $E = \sigma/2\epsilon_0$.

4.2 Conductors in electrostatic fields

- Properties:

  1) $\vec{E} = \vec{0}$ everywhere inside a conductor, for both solid conductors and conductors with *empty* cavities.

  2) If an isolated conductor carries a charge, then that excess charge resides on the outer surface of the conductor.

  3) The electric field just outside the surface of a charged conductor is perpendicular to the surface with magnitude $E = \sigma/\epsilon_0$, where σ is the surface charge density.

  4) For irregularly shaped conductors, the surface charge density σ is greatest where the radius of curvature of the surface is smallest (i.e., where the surface is sharpest).

- Property (1) arises since charges in the conductor rearrange themselves in response to an applied electric field such that the *induced* field associated with these charges exactly cancels the applied field in the conductor, $\vec{E} + \vec{E}_{\text{ind}} = \vec{0}$.
- Property (1) implies that any external electric field is shielded from a cavity inside a conductor. That's why you're safe inside a car during a lightning storm.
- Fields within a cavity, produced e.g., by a charge, do make it outside of the conductor. So the outside of a conductor is not shielded from the fields inside a cavity of a conductor.

5 Electrostatic potential

5.1 Work and potential energy

- Recall: The work done by a force in moving a particle from point A to point B along some path is given by
$$W_{A\to B} = \int_A^B \vec{F}\cdot d\vec{s} \tag{16}$$
- For a general force, the value of the integral depends on the path connecting A and B. If the force is *conservative*, then the integral is path independent and depends only on the endpoints.
- Examples of conservative forces are gravity, the spring force, and the electrostatic force. An example of a non-conservative force is friction.
- For conservative forces, one can a associate a *potential energy* U with the work done by the force in moving the particle from A to B:
$$\Delta U \equiv U_B - U_A = -W_{A\to B} = -\int_A^B \vec{F}\cdot d\vec{s} \tag{17}$$
- Note the minus sign in the above expression. The potential energy of the system decreases if the force does positive work. For example, as a mass m falls in a gravitational field $\vec{g}$, it loses potential energy, while gaining kinetic energy.

5.2 Electrostatic potential

- The above results apply to the electrostatic force acting on a charge q. We define
$$\Delta U = -\int_A^B \vec{F}_e\cdot d\vec{s} = -q\int_A^B \vec{E}\cdot d\vec{s} \tag{18}$$
- The electrostatic potential V is then defined as the electrostatic potential energy per unit charge:
$$\Delta V \equiv \frac{\Delta U}{q} = -\int_A^B \vec{E}\cdot d\vec{s} \tag{19}$$
which has units of Joules per Coulomb, or volts (1 volt $\equiv$ 1 J/C).
- Note that changing the electrostatic potential V by an additive constant ($V' = V + \text{const}$) does not change the potential difference, since $\Delta V = \Delta V'$. So the zero (or reference point) of the electrostatic potential can be set in whatever way is most convenient for the problem at hand.
- If $\vec{E}$ is everywhere orthogonal to $d\vec{s}$, then $\Delta V = 0$. This means that the electric field $\vec{E}$ is *perpendicular* to an *equipotential* surface.

5.3 Obtaining $\vec{E}$ from V

- The electrostatic potential V is given by integrating the electric field $\vec{E}$. Conversely, by considering two nearby points A and B, one can show that the electric field $\vec{E}$ is given by differentiating V:
$$E_x = -\frac{\partial V}{\partial x}\,, \quad E_y = -\frac{\partial V}{\partial y}\,, \quad E_z = -\frac{\partial V}{\partial z} \quad \Leftrightarrow \quad \vec{E} = -\vec{\nabla} V \tag{20}$$
- This relationship is very useful since it is often much easier to first calculate the electrostatic potential V for a distribution of charges, and then differentiate V to obtain the electric field $\vec{E}$.
- Recall that it is typically hard to calculate $\vec{E}$ for a distribution of charges since one has to sum up *vectors* for each charge in the distribution. Summing up scalars (for the potential V) is much easier.
- Note that the units of $\vec{E}$ can also be expressed in terms of volt per meter: 1 N/C = 1 volt/m.

6 Electrostatic potential (continued)

6.1 Examples of electrostatic potentials

- For a uniform electric field $\vec{E}$, we have $\Delta V = -Ed < 0$ if we integrate in the same direction as $\vec{E}$. Thus, for $q > 0$, the electrostatic potential energy *decreases* as you move in the direction of the field.
- For a point charge q at the origin, $V(r) = k_e q/r$, where we have chosen zero potential at $r \to \infty$.
- For a set of discrete point charges or a continous charge distribution:

$$V(P) = \sum_i \frac{k_e q_i}{r_i} \quad \text{(discrete)}\,, \qquad V(P) = \int \frac{k_e dq}{r} \quad \text{(continuous)} \tag{21}$$

- There are many worked examples in the text for finding $V(P)$ for different charge distributions.

6.2 Electrostatic potential energy

- Consider three charges q_1, q_2, q_3 separated by distances r_{12}, r_{13}, r_{23}. The electrostatic potential energy associated with this system of charges can be found by calculating how much work an external agent has to do to "build" the configuration by bringing in charges one-by-one from infinity.
- You first bring q_1 to its position against no field; then q_2 to its position against the field created by q_1; and finally q_3 to its position against the field created by q_1 and q_2:

$$W = \frac{k_e q_1 q_2}{r_{12}} + \frac{k_e q_1 q_3}{r_{13}} + \frac{k_e q_2 q_3}{r_{23}} \tag{22}$$

6.3 Electrostatic potential associated with conductors

- Since $\vec{E} = \vec{0}$ everywhere inside a conductor, $\Delta V = -\int_A^B \vec{E} \cdot d\vec{s} = 0$ for any path lying within the conductor. Thus, a conductor is an *equipotential*.
- If a conductor has an empty cavity, then $\vec{E} = \vec{0}$ inside the cavity (as well as inside the "meat" of the conductor). *Proof*: If that wasn't the case, then one could integrate $\vec{E}$ along one of its field lines inside the cavity, from one side of the cavity to the other, giving $\Delta V \neq 0$ or $V_B \neq V_A$, contradicting that a conductor is an equipotential.
- The surface charge density on the surface of a conductor has its greatest magnitude where the radius of curvature is smallest. *Proof*: Model your conductor as two conducting spheres with radii r_1, r_2 (with $r_1 > r_2$) and charge q_1, q_2 connected by a metal wire. Since this configuration is an equipotential, $k_e q_1/r_1 = k_e q_2/r_2$, which implies $q_2/q_1 = r_2/r_1$. It then follows that

$$\frac{\sigma_2}{\sigma_1} = \frac{q_2}{4\pi r_2^2} \Bigg/ \frac{q_1}{4\pi r_1^2} = \frac{q_2}{q_1}\frac{r_1^2}{r_2^2} = \frac{r_2}{r_1}\frac{r_1^2}{r_2^2} = \frac{r_1}{r_2} > 1 \tag{23}$$

6.4 Demonstrations

- An uncharged (metallic-coated) pith ball is attracted to the charged surface of a van de Graaff generator. When it hits the surface of the generator, it becomes charged by conduction and is then repelled.
- We see the same behavior outside the outer surface of a charged metallic Faraday cage.
- Inside the charged Faraday cage, the electric field is zero, so an uncharged pith ball feels no force inside.
- If instead we charge the inner cylindrical surface of the Faraday cage, and place an uncharged pith ball between the inner and outer cylindrical surfaces, then we find that the pith ball is first attracted and then repelled by the inner cylinder, eventually hitting the outer cylinder and bouncing back, oscillating wildly.

7 Capacitance

7.1 Capacitors

- A capacitor is a circuit element whose primary function is to *store charge*.
- Given two conductors (of any shape and size) with charges $\pm Q$, respectively, the capacitance C is defined by

$$C \equiv Q/\Delta V \tag{24}$$

  where ΔV is the potential difference between the positively and negatively-charged conductors.
- The value of C is independent of ΔV, depending only on the geometrical arrangement of the conductors.
- Examples:

  - Parallel-plate capacitor, area A, separation d: $C = \epsilon_0 A/d$.
  - Two concentric cylinders, length L, inner and outer radii a, b: $C = 2\pi\epsilon_0 L/\ln(b/a)$.
  - Two concentric spheres, inner and outer radii a, b: $C = 4\pi\epsilon_0 ab/(b-a)$.
  - Single, isolated sphere: $C = 4\pi\epsilon_0 a$.

7.2 Series and parallel combinations

- One can connect capacitors in *parallel* or *series*, or combinations of the two. Such a combination of capacitors can be reduced to a single *equivalent* capacitance C_{eq}.
- For a parallel combination of C_1 and C_2:

$$C_{\mathrm{eq}} = C_1 + C_2 \tag{25}$$

  The proof of this relation uses the fact that the potential differences are equal, $\Delta V_1 = \Delta V_2 = \Delta V$, while the charges add $Q = Q_1 + Q_2$.
- For a series combination of C_1 and C_2:

$$\frac{1}{C_{\mathrm{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{26}$$

  The proof of this relation uses the fact that the potential differences add, $\Delta V = \Delta V_1 + \Delta V_2$, while the charges are equal, $Q = Q_1 = Q_2$.

7.3 Energy stored in a capacitor

- To charge up a capacitor, one needs to do work against the electric field associated with the charge already deposited on the plates.
- If q is the current charge on the capacitor, then the associated potential difference of the plates is q/C. To move additional charge dq against that potential requires work $dW = dq\, q/C$.
- The total work W is given by the integral

$$W = \int_0^Q dq\, \frac{q}{C} = \frac{1}{2}\frac{Q^2}{C} \tag{27}$$

- This work is the electrostatic potential energy U_E stored in the capacitor, and it can be written in several equivalent ways:

$$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}Q\Delta V \tag{28}$$

- For a parallel plate capacitor, which has $C = \epsilon_0 A/d$ and $\Delta V = Ed$, one can show that the (volume) energy density $u_E \equiv U_E/\text{volume}$ is given by $u_E = \frac{1}{2}\epsilon_0 E^2$. This expression is actually valid for *any* electrostatic field, not just for the field between the plates of a parallel-plate capacitor.

8 Dielectrics

8.1 Electric dipoles

- An *electric dipole* $\vec{p}$ consists of two charges $\pm q$ separated by a distance $d = 2a$. The magnitude of $\vec{p}$ is $p = 2a|q| = |q|d$. The direction of $\vec{p}$ is given by the vector that points from $-q$ to $+q$.
- Electric dipoles can be induced in an insulator by an external electric field. The external field separates the centers of negative (electrons) and positive (nuclei) charge in an atom or molecule. The induced dipole moment is often proportional to the magnitude of the applied field, $\vec{p} = \alpha\vec{E}$.
- Electric dipoles can also exist in the absence of an external field—e.g., for *polar* molecules like H_2O, whose centers of positive and negative charge are already separated due to the shape of the molecule.
- An electric dipole in a uniform electric field $\vec{E}$, feels no net force since $\vec{F}_+ = -\vec{F}_-$, but does experience a net torque $\vec{\tau} = \vec{p} \times \vec{E}$. The torque acts so as to align $\vec{p}$ with $\vec{E}$.
- An electric dipole in an electrostatic field has an associated potential energy $U_E = -\vec{p} \cdot \vec{E}$, which is the work required to rotate the dipole from some initial configuration to some final configuration, $W = \int_{\mathrm{i}}^{\mathrm{f}} \tau \, d\theta$.
- Note that $U_E = -\vec{p} \cdot \vec{E}$ increases from $U_E = -pE$ for $\theta = 0$ (aligned), to $U_E = 0$ for $\theta = \pi/2$ (perpendicular), to $U_E = +pE$ for $\theta = \pi$ (anti-aligned).

8.2 Dielectrics

- A *dielectric* is an insulator placed between the conductors of a capacitor.
- Alignment of polarized molecules within the dielectric creates an induced electric field that *opposes* the applied electric field associated with the free charges $\pm Q$ on the two conductors.
- This induced electric field reduces the total electric field between the conductors, leading to a reduced value of the potential diffence ΔV, and a corresponding increase in $C = Q/\Delta V$.
- The increase in capacitance is encoded in the (dimensionless) dielectric constant $\kappa \geq 1$, defined by $C = \kappa C_0$, where C_0 is the capacitance in the absence of a dielectric.
- Equivalently, the permitivity of free space ϵ_0 should be replaced by the permitivity of the dielectric, $\epsilon = \kappa\epsilon_0$.
- κ is related to the free and induced charge densities σ and σ_{ind} via

$$\frac{\sigma}{\kappa} = \sigma - \sigma_{\mathrm{ind}} \tag{29}$$

- For vacuum, $\kappa = 1$; for air, $\kappa = 1.00059$; for paper, $\kappa = 3.7$; for a conductor, $\kappa = \infty$ (see Table 26.1).

9 Current and resistance

9.1 Current

- Current is the rate of flow of charge through some surface:

$$I_{\text{avg}} = \Delta Q/\Delta t \quad \text{or} \quad I = dQ/dt \quad \text{(instantaneous)} \tag{30}$$

where ΔQ is the net charge that crosses the surface (area A) in time interval Δt.

- The unit of current is an amp, 1 A $\equiv$ 1 C/s. The direction of current is the direction of flow of the positive charges. Current density J is defined as $J \equiv I/A$.
- For charges to flow in a conductor, we need a non-zero electric field inside the conductor. (We are no longer dealing with electrostatics.) The non-zero field creates a potential difference ΔV between two ends of the conductor.

9.2 Microscopic model of current flow

- Consider a cylindrical conductor with cross-sectional area A, charge carriers with number density n, each with charge q.
- In the absence of an applied electric field, there is no net flow of the charge carriers (e.g., free electrons in the conductor move in random directions with velocity $\vec{v}_{\text{th}}$ due to thermal effects).
- In the presence of an applied electric field, the charge carriers accelerate, acquiring a drift velocity v_{d}, which is several orders of magnitude smaller than the thermal velocity v_{th}. (NOTE: $v_{\text{th}} \sim 10^6$ m$/s$ for electrons at room temperature.)
- Take the length of the cylinder to be $\Delta x = v_{\text{d}}\Delta t$, so that in time interval Δt all of the charges in the cylinder have passed through one end of the cylinder. Then

$$I_{\text{avg}} = \frac{\Delta Q}{\Delta t} = \frac{qnA\Delta x}{\Delta t} = qnAv_{\text{d}} \quad \Rightarrow \quad J = qnv_{\text{d}} \tag{31}$$

- Example 27.4: Calculate the drift velocity of free electrons in a copper wire with cross-sectional area $A = 3.3 \times 10^{-6}$ m^2, $I = 10$ A, mass density $\rho = 8.92$ g/cm^3. Assume that there is one free electron per copper atom. Also, recall that the molar mass of copper is $M = 63.5$ g/mole, and $N_A = 6.02 \times 10^{23}$ is the number of of atoms/mole. The result is $v_{\text{d}} = 2.23 \times 10^{-4}$ m/s, which means that it take more than 1 hr for a free electron to travel 1 meter!
- So it's not the speed of the free electrons that make a light turn on almost instantaneously when you flip a switch. Rather it's the electrostatic force between neighboring electrons which "smooths" out their distribution, so that there isn't an excess or lack of charge carriers in any part of the circuit.

9.3 Resistance

- The drift velocity of charges carriers is produced by a force. If that force is proportional to the applied electric field, then the current density $J = qnv_{\text{d}}$ is proportional to the field, $\vec{J} = \sigma\vec{E}$, where σ is the *conductivity* of the conductor (property of the material).
- Materials for which $\vec{J} = \sigma\vec{E}$ are said to obey *Ohm's law.* (NOTE: Ohm's law really isn't a law; it's an empirical property of certain materials.)
- For a cylindrical piece of conductor with conductivity σ, cross-sectional area A, length l, and applied field $\vec{E}$, we have

$$J = \sigma E \quad \Leftrightarrow \quad I/A = \sigma\Delta V/l \quad \Leftrightarrow \quad R = \rho l/A \tag{32}$$

where $R \equiv \Delta V/I$ is the *resistance* of object and $\rho \equiv 1/\sigma$ is the *resisitivity* of the material.

- R has units of ohms, 1 $\Omega \equiv$ 1 V/A, and ρ has units of $\Omega \cdot$ m. Typical values of ρ are $\sim 10^{-8}$ $\Omega \cdot$ m for metals, $\sim 10^{13}$ $\Omega \cdot$ m for insulators, and ~ 1 to 10^3 for semiconductors.

10 Resistance, superconductivity, electrical power

10.1 Classical model of electrical conduction in metals

- Paul Drude ($\sim$ 1900) came up with a *classical* model of electrical conduction in metals, which treats the free electrons as a classical *gas*.
- In the absence of an applied electric field, the free electrons move in random directions with thermal velocity $\vec{v}_{\rm th}$, colliding with atomic nuclei, fixed in the metal. In the presence of an applied electric field $\vec{E}$, the free electrons accelerate in the direction opposite the field, with $\vec{a} = q\vec{E}/m_e$, so $\vec{v}_{\rm f} = \vec{v}_{\rm i} + \vec{a}\Delta t$.
- Since $\vec{v}_{\rm i,avg} = \vec{0}$, the average of $\vec{v}_{\rm f}$ over all free electrons gives the drift velocity $\vec{v}_{\rm d} \equiv \vec{v}_{\rm f,avg} = (q\vec{E}/m_e)\tau$, where $\tau \equiv l_{\rm avg}/v_{\rm th}$ is the average time between collisions.
- Thus,
$$\vec{J} = qn\vec{v}_{\rm d} = (q^2 n\tau/m_e)\vec{E} \quad \Rightarrow \quad \sigma = q^2 n\tau/m_e\,, \quad \rho = m_e/(q^2 n\tau) \tag{33}$$
- Problems: (i) the predicted resistivity values are a factor $10\times$ smaller than the observed values; (ii) the predicted temperature dependence is $\rho \propto \sqrt{T}$ as opposed to the observed dependence $\rho \propto T$.
- Reason: classical model (gas of electrons). Proper treatment requires quantum mechanics.

10.2 Temperature dependence of resistivity

- Observed temperature dependence of resistivity: $\rho = \rho_0\,[1 + \alpha(T - T_0)]$, where ρ_0 is the resistivity at temperature T_0, and α is the temperature coefficient of resistivity (units: $1/^\circ$C).
- NOTE: $\alpha \sim 10^{-3}/^\circ$C for most conductors. It is possible for $\alpha < 0$ for semiconductors (increasing T may increase density of charge carriers).

10.3 Superconductors

- The resistance of certain materials goes to zero below a *critical temperature* $T_{\rm c}$.
- In 1911, Kamerlingh-Onnes discovered that Mercury was superconducting below 4.2 K (recall that 0 K, which is *absolute zero*, corresponds to -273.15 °C). In 1987, the Nobel Prize in Physics was awarded to Bednorz and Müller for discovering superconductivity in copper-oxide materials at relatively high temperatures ($\sim$ 100 K).
- Current continues to flow in a superconductor even if the potential difference is removed. No energy is dissipated as heat in a superconducting wire.
- Application: Superconducting magnets made with coils of superconducting wire produce stronger magnetic fields than conventional, non-superconducting, magnets. Superconducting magnetis are often used in MRI and NMR machines in hospitals.

10.4 Electrical power

- Recall that power is the time rate of change of energy production or consumption, $P = dE/dt$.
- In an electric circuit, a charge q loses energy $q\Delta V$ as it moves through a circuit from the positive terminal of a battery to the negative terminal. It gains energy $q\Delta V$ as it moves from the negative terminal to the positive terminal *through* the battery. (Chemical reactions provide the energy.)
- Thus, the power delivered to a resistor R with potential difference ΔV is
$$P = q\Delta V/\Delta t = I\Delta V = I^2 R = (\Delta V)^2/R \tag{34}$$
- The power delivered to a resistor is dissipated as heat, so-called I^2R *losses*.

11 Direct current circuits

11.1 Electromotive force

- Electromotive force (or EMF) is the potential difference associated with the force per unit charge that drives charges from the negative terminal of a battery (solar cell, electric generator, etc.) to the positive terminal.
- EMF is what's constant in a battery (e.g., a 9-volt transistor battery). EMF is typically denoted by $\mathcal{E}$.
- The output voltage ΔV of a *real* battery depends on the amount of current i flowing out of the battery, according to $\Delta V = \mathcal{E} - ir$, where r is the *internal* resistance of the battery.
- Example 28.1 shows that as a battery ages, the internal resistance r increases, which causes ΔV and the power delivered to a load resistor R to become much smaller fractions of $\mathcal{E}$ and the total power produced by the battery, $P_r + P_R$. The increased power delivered to the internal resistance causes the battery to heat up.
- Example 28.2 shows that the *maximum* power delivered to a load resistor R occurs when its value agrees with the internal resistance r.

11.2 Series and parallel combinations of resistors

- For a series combination of resistors R_1 and R_2:

  $$R_{\rm eq} = R_1 + R_2 \tag{35}$$

  The proof of this relation uses the fact that the currents are equal for both resistors, $I_1 = I_2 = I$, while the potential differences add, $\Delta V = \Delta V_1 + \Delta V_2$.
- For a parallel combination of resistors R_1 and R_2:

  $$\frac{1}{R_{\rm eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{36}$$

  The proof of this relation uses the fact that the currents add, $I = 1_1 + I_2$, while the potential differences are equal $\Delta V = \Delta V_1 = \Delta V_2$. Note that $R_{\rm eq} < R_{\rm smallest}$ for a parallel combination.
- Application: Christmas tree lights

  (i) Old-fashioned Christmas tree lights were connected in series, which meant that if one bulb burned out, all the other bulbs would go out (no current to those bulbs).

  (ii) Newer Christmas tree lights were connected in parallel to solve that problem. But a parallel connection draws a lot of current, which makes these lights a fire hazard.

  (iii) "Miniature" Christmas tree lights are wired again in series, but each bulb has an insulating *jumper* that becomes conductive when that bulb burn out. So the remaining bulbs stay lit.

11.3 Kirchhoff's rules

- Kirchhoff's rules are used to analyze circuits that cannot be reduced using simple parallel and series combinations of resistors.
- Kirchoff's rules are a consequence of conservation of charge and conservation of energy:

  (i) Junction rule: The sum of all the currents into a node is zero.

  (ii) Loop rule: The sum of all the potential differences around a closed loop is zero.
- When applying Kirchhoff's rules, it is important to follow certain sign conventions: (i) if the source of a potential difference (e.g., a battery) is traversed from the $-$ to $+$ terminal, ΔV is assigned a positive sign; (ii) if current i flows through resistor R in the same direction that the loop is traversed, the potential difference is assigned a negative sign, $-IR$, a so-called "voltage drop".

12 Direct current circuits (continued)

12.1 Resistor-Capacitor (RC) circuits

- RC circuits contain resistors, capacitors and possibly sources of EMF.
- Charging a capacitor: When a capacitor C is charged through a resistor R by an EMF $\mathcal{E}$, the charge increases exponentially while the current decreases exponentially:

$$q(t) = Q_{\max}\left(1 - e^{-t/RC}\right), \qquad i(t) = I_{\max}\, e^{-t/RC} \tag{37}$$

  where $Q_{\max} = \mathcal{E}C$ and $I_{\max} = \mathcal{E}/R$. The combination $\tau \equiv RC$ in the exponential is called the *RC time constant*, since it has units of time.
- These results follow from applying the loop rule to the RC circuit: $\mathcal{E} - iR - q/C = 0$ or $\mathcal{E} - R\,dq/dt - q/C = 0$, and then solving the differential equation for $q(t)$.
- Discharging a capacitor: When a capacitor C is discharged through a resistor R, both the charge on the capacitor and the current through the circuit decrease exponentially:

$$q(t) = Q_{\max}\, e^{-t/RC}, \qquad i(t) = I_{\max}\, e^{-t/RC} \tag{38}$$

  where $I_{\max} = Q_{\max}/RC$.
- These results follow from applying the loop rule to the RC circuit: $q/C - iR = 0$ or $q/C + R\,dq/dt = 0$, and then solving the differential equation for $q(t)$. (NOTE: $i = -dq/dt$ since the charge q on the capacitor is *decreasing*.)

12.2 Household wiring and electrical safety

- Some important numbers:

  - The EMF of a wall outlet is 120 volts (AC, but that really doesn't matter for this analyis).

  - 14-gauge household wire typically carries up to ~ 20 A of current. It has a resistance of 2.5 Ω/1000 ft, so 100 feet of that wire has a resistance $R_{\mathrm{w}} \sim 0.25\ \Omega$.

  - A household appliance (e.g., hair dryer) has $P \sim 1200$ W implying $I = 10$ A and $R_{\mathrm{load}} = 12\ \Omega$.

  - A human being has a resistance $R_h \sim 100{,}000\ \Omega$ (dry skin), but $R_h \sim 1000\ \Omega$ (wet skin).

  - Summary: $R_{\mathrm{w}} \sim 0.1\ \Omega \ll R_{\mathrm{load}} \sim 10\ \Omega \ll R_{\mathrm{h}} \sim 1000\ \Omega$.
- Current kills, not potential difference. $I = 0.01$ A will give a painful shock; $I = 0.1$ A is potentially lethal. (120 V and $R_{\mathrm{h}} \sim 1000\ \Omega$ gives $I \sim 0.1$ A, which can kill you.)
- Modern household wire consists of 3 wires: a "hot" wire encased in black plastic (120 V), a "neutral" wire encased in white plastic (typically grounded, so 0 V), and a "ground" wire, usually bare or encased in green plastic, which should be connected to the Earth (i.e., "ground").
- Circuit breakers typically consist of fuses (older houses) or spring-loaded switches, which either "burn out" or "trip" if too much current (e.g., greater than 20 A) flows in the circuit, thus preventing fires.
- NEVER replace an old-fashioned fuse with a penny or any other piece of metal that won't burn out if too much current is flowing in the circuit.
- Household appliances that have a three-prong plug have the ground wire connected to the chassis. If the appliance is connected to a three-prong outlet and the hot wire accidently touches the chassis of the appliance, then the total resistance is approximately that of the wire, which will cause the circuit breaker to trip, which prevents you from being electrocuted even if you are touching the appliance.
- If the appliance doesn't have a ground wire, and the hot wire accidently touches the chassis, then the total resistance of the circuit will be that of the load. The circuit breaker won't trip since the current is normal (i.e., less than 20 A). But you will have 120 V across you, which will produce 0.1 A of current through you if you have wet skin. This can electrocute you.

13 Magnetic fields

13.1 Basic properties

- Two magnetic poles, N and S, similar to $+$ and $-$ electrical charges. Like poles repel, unlike poles attract; isolated magnetic poles do not exist in nature (unlike isolated electric charges).
- Iron filings are attracted to both poles of a magnet. They become magnetized.
- Iron filings align themselves with magnetic field lines. Magnetic field lines form closed loops (as opposed to starting or terminating at N and S poles).
- The Earth has an intrinsic magnetic field (due to motion of charged molten material in its core) with its South magnetic pole located near Earth's North geographic pole. A compass needle is a small magnet which aligns its North pole with Earth's South magnetic pole (points toward geographic North).
- All magnetic fields are ultimately due to the motion of electric charges (e.g., electron orbital motion and instrinsic spin).

13.2 Magnetic force on a charged particle

- The magnetic force on a charged particle is
$$\vec{F}_B = q\vec{v} \times \vec{B} \tag{39}$$
Magnitude: $F_B = |q|vB\sin\theta$, where θ is the angle between the velocity $\vec{v}$ of the charged particle and the magnetic field $\vec{B}$. Direction: right-hand rule.
- If the particle is at rest ($v = 0$) or if $\vec{v}$ is parallel to $\vec{B}$, then $\vec{F}_B = \vec{0}$. Measuring $\vec{F}_B$ for a given $\vec{v}$ determines only the components of $\vec{B}$ perpendicular to $\vec{v}$.
- Since $\vec{F}_B$ is perpendicular to $\vec{v}$, the magnetic force does no work on a charged particle. (It can only change the direction of $\vec{v}$, not its magnitude).
- Units of $\vec{B}$: 1 Tesla $\equiv$ 1 N/A $\cdot$ m. The Gauss defined by 1 Tesla $\equiv 10^4$ Gauss is also sometime used.
- Earth's magnetic field $\sim$ 0.5 Gauss; MRI magnet $\sim$ 1.5 T; superconducting magnet $\sim$ 30 T; magnetic field of a neutron star $\sim 10^8$ T.

13.3 Motion of a charged particle in a uniform magnetic field

- The motion of a charged particle in a uniform magnetic field with $\vec{v}$ perpendicular to $\vec{B}$ is *uniform circular motion* with $v =$ const. The speed, radius, magnetic field, and charge-to-mass ratio of the particle are related by
$$v = \frac{|q|rB}{m} \quad \Leftrightarrow \quad \omega = \frac{|q|B}{m} \quad \Leftrightarrow \quad T = \frac{2\pi m}{|q|B} \tag{40}$$
where ω and T are the *cyclotron* (angular) frequency and period, respectively.
- If $\vec{v}$ has a component parallel to $\vec{B}$, then the motion is a helix.
- Applications of the motion of a charged particle in a uniform magnetic field include:

  (i) velocity selector: formed from perpendicular uniform electric and magnetic fields. The velocity of a charged particle is constant provided $v = E/B$.

  (ii) mass spectrometer: a velocity selector followed by another uniform magnetic field $\vec{B}_0$. It is used to select different particle masses by the radii of their circular motion $|q|/m = E/BB_0r$. J.J. Thomson (1897) determined the charge-to-mass ratio of the electron this way.

  (iii) a cyclotron: the first particle accelerator (E.O. Lawrence, M.S. Livingston, 1934) constructed from a uniform magnetic field and two "dees", having alternating potential difference. A charged particle is accelerated to larger and larger velocities with correspondingly larger and larger radii.

14 Magnetic fields (continued)

14.1 Magnetic force on a current-carrying wire

- The magnetic force on a current-carrying wire is

$$\vec{F}_B = I\vec{L} \times \vec{B} \tag{41}$$

  which is a simple extension of the magnetic force on a single charged particle q to nAL charged particles moving with drift velocity $\vec{v}_\mathrm{d}$ in a wire (recall $\vec{I} = qn\vec{v}_\mathrm{d}A$).

- For a wire which changes its direction $\vec{F}_B = \int d\vec{F}_B = I\int d\vec{s} \times \vec{B}$.

- There is no net force on a current-carrying loop in a uniform magnetic field (Example 29.4): $\oint d\vec{F}_B = \vec{0}$.

14.2 Torque on a current-carrying loop

- Consider a rectangular loop of wire (sides a and b) in a uniform magnetic field $\vec{B}$, with $\vec{B}$ parallel to the sides having length b and perpendicular to the area vector $\vec{A}$.

- Then the total magnetic force on the wire is zero, but there is a non-zero net torque about an axis perpendicular to $\vec{B}$ (passing through the center of the loop):

$$\tau = IAB = \mu B \tag{42}$$

  where $\mu \equiv IA$ is the *magnetic dipole moment* of the current loop.

- More generally, $\vec{\tau} = \vec{\mu} \times \vec{B}$, where $\vec{\mu} \equiv I\vec{A}$. (Recall: $\vec{\tau} = \vec{p} \times \vec{E}$ for an electric dipole in a uniform electric field.)

- The torque is in a direction to rotate the magnetic dipole moment $\vec{\mu}$ in the direction of the magnetic field $\vec{B}$.

- Similar to $U_E = -\vec{p}\cdot\vec{E}$, there is a potential energy associated with a current loop in a uniform magnetic field, $U_B = -\vec{\mu}\cdot\vec{B}$.

- Application: motors (but need to alternate the direction of the current flow in order to keep the loop turning.)

14.3 Hall effect

- The Hall effect (Edwin Hall, 1879) is an experimental way of determining the sign of the charge carriers.

- Consider a rectangular slab of material with height d, thickness t, carrying a current I along its length. Apply a uniform magnetic field $\vec{B}$ perpendicular to the current in the direction of the thickess of the slab.

- Then the magnetic force on the current carriers sets up a potential difference ΔV_H, called the Hall potential, given by $\Delta V_H = v_\mathrm{d}Bd$ where v_d is the drift velocity of the charges. (Recall: $I = |q|nv_\mathrm{d}A$, where n is the number density of charge carriers and $A = td$ is the cross-sectional area of the current flow.)

- The sign of the Hall potential indicates the sign of the charge carriers.

- NOTE: The quantum Hall effect refers to the fact that ΔV_H is quantized. (K. von Klitzking, Nobel Prize in 1985).

15 Sources of a magnetic field

15.1 Biot-Savart law

- Oersted (in 1819) discovered that a current-carrying wire produces a magnetic field that circles around the wire in accordance with the right-hand rule (thumb points in the direction of the current; fingers curl in the direction of the magnetic field).
- Biot and Savart subsequently wrote down an expression for the (infinitesimal) magnetic field $d\vec{B}$ at point P produced by an (infinitesimal) current element $I\,d\vec{s}$:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{s}\times\hat{r}}{r^2}\,, \quad \text{where} \quad \mu_0 = 4\pi\times 10^{-7}\ \frac{\mathrm{T\cdot m}}{\mathrm{A}} \tag{43}$$

  The unit vector $\hat{r}$ points from the current element $I\,d\vec{s}$ to P, with r being the distance between them. To obtain the total field produced by a full wire, one must integrate: $\vec{B} = \int_{\text{wire}} d\vec{B}$.
- Examples:

  - field at a distance a above a finite segment of a straight wire, with endpoints making angles θ_1, θ_2 with respect to vertical: $B = (\mu_0/4\pi a)(\sin\theta_2 - \sin\theta_1)$.

  - field at a perpendicular distance a away from infinitely-long, straight wire: $B = \mu_0 I/2\pi a$.

  - field at the center of an arc $\Delta\theta$ of a circle of radius a: $B = \mu_0 I\Delta\theta/4\pi a$.

  - field at the center of a circular loop: $B = \mu_o I/2a$.

  The direction of $\vec{B}$ for these examples is given by the right-hand rule as discussed previously.

15.2 Force between two current-carrying wires

- Consider two infinitely-long, straight current-carrying wires (currents I_1, I_2) parallel to one another and separated by perpendicular distance a. Then the force per unit length between the wires is

$$\frac{F}{l} = \frac{\mu_0\,I_1 I_2}{2\pi a} \tag{44}$$

- The above result can be obtained by finding the magnetic field $\vec{B}_1$ produced by wire 1 at the location of wire 2, and then calculating the force that $\vec{B}_1$ exerts on the current I_2.
- If the currents in the two wires flow in the same direction, then the force is attractive; if the currents flow in opposite directions, then the force is repulsive.

15.3 Ampère's law

- Ampére's law is similar to Gauss's law for electric fields, but it involves magnetic fields, enclosed currents, and closed loops (instead of electric fields, enclosed charges, and closed surfaces):

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0\,I_{\text{enc}} \tag{45}$$

  where I_{enc} is the (net) current passing through any surface spanning the closed curve C. The current gets a positive sign if it passes in the same direction as the normal vector to the surface, defined by the RHR.
- One can prove this result for a single current-carryng wire by using the expression for the magnetic field obtained using the Biot-Savart law, and then showing that the integral on the LHS is equal to μ_0 times I, if the wire passes through the closed loop, and 0 otherwise.
- The result extends to multiple currents using the superposition principle.
- Ampère's law can be used to used to calculate the magnetic field for highly-symmetric current configurations (examples in next lecture).

16 Sources of a magnetic field (continued)

16.1 Examples of fields calculated using Ampère's law

- Cylindrical wire (uniform current I, radius R): $B = \mu_0 I r/2\pi R^2$ for $r < R$; $B = \mu_0 I/2\pi r$ for $r > R$; For both cases $\vec{B}$ circles around the axis of the wire.
- Torus (current I, N turns, cross-sectional radius a, inner and outer radii b and c): $B = \mu_0 N I/2\pi r$ for $b < r < c$. The direction of $\vec{B}$ is circumferential. Outside the magnetic field is weak but non-zero, circling around the cross-section of the torus.
- Solenoid (current I, n turns per unit length, radius a): Inside $B = \mu_0 n I$. The direction of $\vec{B}$ is along the axis of the solenoid. Outside the magnetic field is weak but non-zero, circling around the cross-section of the solenoid.

16.2 Gauss's law for magnetic fields

- One can define magnetic flux Φ_B through a surface S analogous to electric flux Φ_E:

$$\Phi_B \equiv \int_S \vec{B} \cdot \hat{n}\, dA \tag{46}$$

- But since magnetic field lines form closed loops (because isolated magnetic poles do not exist), the magnetic flux through a *closed* surface S is zero:

$$\oint_S \vec{B} \cdot \hat{n}\, dA = 0 \tag{47}$$

  Recall that for electric fields, Gauss's law is $\oint_S \vec{E} \cdot \hat{n}\, dA = q_{\rm enc}/\epsilon_0$.

16.3 Magnetic properties of materials

- The magnetic properties of materials are determined by the magnetic dipole moments associated with atomic current loops (orbital or spin angular momentum of electrons). A proper treatment of magnetism at the atomic level requires quantum mechanics.
- A classical treatment of orbital angular momentum of an electron yields $\mu_{\rm orb} = eL/2m_{\rm e}$, with $\vec{\mu}_{\rm orb}$ pointing opposite $\vec{L}$. Quantum mechanically, $\mu_{\rm orb}$ is quantized, with values given by integer multiples of $\sqrt{2}e\hbar/2m_{\rm e}$, where $\hbar = 1.05 \times 10^{-34}$ J $\cdot$ s is Planck's constant.
- A magnetic moment associated with the spin angular momentum of an electron doesn't make sense classically since an electron is a *point particle*. Quantum mechanically an electron has an intrinsic spin angular momentum $\pm\hbar/2$ and corresponding spin magnetic moment $\mu_{\rm spin} = \pm e\hbar/2m_{\rm e}$. Since electrons are paired with opposite spins (Pauli exclusion principle), the spin magnetic moment is non-zero only for atoms containing an *odd* number of electrons.
- Ferromagnetic materials (iron, cobalt, nickel) have strong coupling between neigboring magnetic dipole moments associated with unpaired spins forming magnetic *domains*. These domains increase in size and align themselves in the direction of an external magnetic field. The induced magnetization remains even if the external field is removed. For $T > T_{\rm curie}$ (770 °C for iron), the magnetization is lost due to thermal motion of the atoms.
- Paramagnetic materials have magnetic dipole moments associated with unpaired electron spins that align themselves in the direction of an external magnetic field. The coupling between neighboring dipole moments is weak, so thermal effects easily disrupt the alignment of the spin magnetic moments.
- Diamagnetic materials (water, copper, gold, silver bismuth) have magnetic dipole moments associated with the orbital angular momentum of electrons that change in a direction *opposite* to an external magnetic field. The induced magnetization thus repels the external field. Magnetic levitation is possible with diamagnetic superconductors.

17 Faraday's law of induction

17.1 Faraday's law of induction

- Three demonstrations:

  (i) Move a permanent magnet toward or away from a coil of wire. A current is induced in the coil when the magnet is moving (CW or CCW current depending on whether the magnet is moving toward or away from the wire, consistent with Lenz's law).

  (ii) Move a coil of wire toward or away from a permanent magnet. A current is induced in the moving coil (CW or CCW current depending on whether the coil is moving toward or away from the magnet, consistent with Lenz's law).

  (iii) Link two coils of wire by an iron core, with one of the coils connected to a battery and a switch. Close the switch, and a current is induced in the other coil. Open the switch, and a current is induced in the other coil in the opposite direction.

- Demos (i) and (ii) show that only *relative* motion is important. Demo (iii) shows that you can get an induced electric field in the absence of physical motion.

- Bascially, the above demos illustrate that a *changing magnetic field produces (induces) an electric field.*

- Mathematically, the induced emf in the coil is given by

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}, \qquad \Phi_B \equiv \int_S \vec{B}\cdot\hat{n}\,dA \tag{48}$$

  where N is the number of loops and Φ_B is magnetic flux passing through any surface S spanning a loop. The minus sign comes from Lenz's law (see below).

- Applications: (i) GFCI (ground-fault circuit interruptor) outlets; (ii) pick-up of an electric guitar.

17.2 Motional EMF

- Demonstration (ii) is an example of *motional emf*; namely, the charge carriers in a moving conductor feel a magnetic force.

- The induced emf for this case is given by $\mathcal{E} = \oint_C \vec{f}_{\rm mag}\cdot d\vec{s}$, where $\vec{f}_{\rm mag} \equiv \vec{v}\times\vec{B}$ is the magnetic force per unit charge on the charge carriers in the moving conductor.

- Connect the moving conductor to an external circuit with a resistor, and current will flow with the power dissipated in the resistor equal to the power associated with the applied force, which moves the wire in the external magnetic field. (The applied force will be balanced by an equal and opposite magnetic force once an induced current in the moving conductor starts to flow.)

17.3 Lenz's law

- Lenz's law: *The direction of the induced current* opposes *the change of magnetic flux through the circuit.*

- Said another way, Nature tries to preserve the status quo. (Use the right-hand rule to determine whether a CW or CCW induced current will oppose the change of flux.)

- Lenz's law is a consequence of conservation of energy.

- Demos: (i) Dropping a magnet down a conducting tube (it takes a long time to fall); (ii) a jumping hoop (a small conducting loop encircling an electromagnet flies off when the electromagnetic is switched on); (iii) a pendulum bob made of a copper sheet is damped when it swings through a permanent magnetic field (due to eddy currents induced in the copper sheet).

18 Induced electric fields; motors and generators

18.1 Induced electric fields

- Recall that emf is defined by $\mathcal{E} \equiv \oint_C \vec{f}_\mathrm{s} \cdot d\vec{s}$, where f_s is the force per unit charge exerted by the source, which can be chemical reactions in a battery, a solar cell, etc.
- For motional emf, $\vec{f}_\mathrm{s} = \vec{f}_\mathrm{mag}$, the magnetic force per unit charge on a moving charge.
- For a changing magnetic field $\vec{f}_\mathrm{s} = \vec{E}$, an induced electric field that acts on charges according to $\vec{F} = q\vec{E}$, just like electric fields produced by electric charges. But this induced electric field is *non-conservative* since $\oint_C \vec{E} \cdot d\vec{s} \neq 0$.
- So electric fields can be produced in two ways: (i) by electric charges (which gives rise to a conservative field), or (ii) by changing magnetic fields (which gives rise to a non-conservative field).

18.2 Generators and motors

- A generator converts mechanical energy into electrical energy. A motor is a generator run in reverse; it converts electrical energy into mechanical energy.
- AC (alternating current) generator: Rotate a loop of wire in a permanent magnetic field and get an AC current out. (Mathematically: $\Phi_B = BA\cos\omega t$ implies $\mathcal{E} = -N d\Phi_B/dt = NBA\omega \sin\omega t$.)
- An AC generator uses a *slip ring* to connect the ends of the rotating loop to brushes connected to an external circuit. A DC generator uses a *split ring* instead of a slip ring; the split ring switches the polarity every half cycle so that $\mathcal{E} = NBA\omega|\sin\omega t|$.
- A motor is constructed by sending a current through a coil of wire in a permanent magnetic field. The current in the coil produces a magnetic field that interacts with the external magnetic field—it feels a torque in the external field—and the coil rotates on account of that torque.
- Demos: (i) Hand-powered AC generator; (ii) DC motor constructed from electromagnets; (iii) Do-it-yourself DC motor constructed from a D-cell battery, a small magnet, paper clips, and a coil of (magnet) wire stripped on one side.

18.3 Back-emf

- When a motor is turned on and starts running, there is a *back emf* (and current) induced in the coil due to its motion, which opposes the applied emf (and current) that originally got it moving.
- The back emf can be a sizable fraction of the applied emf when the motor is running at full speed, which decreases the current to the load, $I = (\mathcal{E} - \mathcal{E}_\mathrm{back})/R$.
- For a heavy load, the motor turns more slowly. The back emf is then proportionally smaller, leading to a larger current to the load. If the motor "jams", the back emf is zero. This leads to a potentially large current that can "burn out" the motor.

19 Inductance

19.1 Self inductance

- When a circuit is first connected to a battery with emf $\mathcal{E}$, the changing magnetic flux through the circuit (due to the increasing current) induces a back emf

$$\mathcal{E}_L \equiv -L\,\frac{di}{dt} \tag{49}$$

which opposes the change in the flux. The proportionality constant L is called the *inductance* (or *self inductance*) of the circuit; it has units of Henries (H), where 1 H = 1 V $\cdot$ s/A.

- Since we can also write $\mathcal{E} = -N\,d\Phi_B/dt$, we have $L\,di/dt = N\,d\Phi_B/dt$, or $L = N\Phi_B/i$, which depends only on the geometry of the circuit.

- Inductance corresponds to the *inertia* (or opposition) of a circuit to changes in its current. Inductance is analogous to mass, which is inertia to changes in the velocity of an object $\vec{F} = m\,d\vec{v}/dt$.

- Examples:

  (i) solenoid (length l, cross-sectional area A, number of turns per length n): $L = \mu_0 n^2 Al$.

  (ii) coaxial cable (length l, inner radius a, outer radius b): $L/l = (\mu_0/2\pi)\ln(b/a)$.

19.2 Mutual inductance

- Consider two loops of wire with numbers of loops N_1, N_2 and currents i_1, i_2. Then a changing current i_1 in loop 1 induces a changing flux Φ_{12} in loop 2, leading to $\mathcal{E}_2 \equiv -M_{12}\,di_1/dt$, where $M_{12} = N_2\Phi_{12}/i_1$. (Similarly, $\mathcal{E}_1 = -M_{21}\,di_2/dt$, where $M_{21} = N_1\Phi_{21}/i_2$.)

- It turns out that $M_{12} = M_{21} \equiv M$, called the *mutual inductance* of the two loops. The mutual inductance depends only the geometry (size, relative orientation, etc.) of the two loops.

- Examples

  (i) Two coaxial solenoids with the longer solenoid having length l, cross-sectional area A, and number of turns N_1, and the shorter solenoid having number of turns N_2: $M = \mu_0 N_1 N_2 A/l$. (For this problem, it is simplest to calculate M_{12} since the flux through solenoid 2 due to solenoid 1 is uniform; this is not so for the flux through solenoid 1 due to solenoid 2.)

  (ii) Transformer with number of turns N_1, N_2 for the input and output: $\mathcal{E}_1/N_1 = \mathcal{E}_2/N_2$. (Thus a transformer *steps-up* the voltage if $N_2 > N_1$.)

19.3 RL circuits

- Consider a battery with emf $\mathcal{E}$ connected to a resistor R and inductor L in series.

- Then applying the loop rule to the potential differences around the circuit leads to a differential equation $\mathcal{E} - iR - L\,di/dt = 0$, which has solution

$$i(t) = i_{\rm max}\left(1 - e^{-t/\tau}\right) \tag{50}$$

where $i_{\rm max} = \mathcal{E}/R$ and $\tau \equiv L/R$ is the time constant for the LR circuit.

- The current doesn't increase instantaneously due to the inductance in the circuit.

- After obtaining maximum current through the inductor, one can remove the battery leaving just the resistor and the inductor. Then the current will decrease exponentially:

$$i(t) = i_{\rm max}\,e^{-t/\tau} \tag{51}$$

- NOTE: If you pull out the power chord of a running vacuum cleaner you will see a spark at the outlet, as the inductance of the motor is trying to keep the current following, in accordance with Lenz's law.

20 Energy stored in an inductor, LC circuits

20.1 Energy stored in an inductor

- Energy is stored in the current flowing in an inductor, similar to energy being stored in the charge on a capacitor.
- The power delivered to an inductor is given by $P = i\mathcal{E}_L = iL\,di/dt$. Integrating the power with respect to time to find the energy, we have

$$U_B = \int_0^t P(t)\,dt = \int_0^i iL\,di = \frac{1}{2}Li^2 \tag{52}$$

  The subscript B indicates that the stored energy is associated with a magnetic field. (NOTE: The above result is similar to $U_E = \frac{1}{2}C(\Delta V)^2 = q^2/2C$ for the energy stored in a capacitor.)
- Consider a solenoid ($B = \mu_0 ni$, $L = \mu_0 n^2 lA$). Substituting these expressions into $U_B = \frac{1}{2}Li^2$, we obtain $u_B = \frac{1}{2\mu_0}B^2$ for the *energy density* (energy/volume). (NOTE: The above result is similar to $u_E = \frac{1}{2}\epsilon_0 E^2$ for the energy density in the electric field, obtained from $U_E = \frac{1}{2}C(\Delta V)^2$ for a parallel-plate capacitor, $C = \epsilon_0 A/d$, $E = \Delta V/d$.)
- Although the above expressions for the energy density u_B and u_E were obtained for very specific cases, i.e., solenoid and parallel-plate capacitor, these expressions hold for *general* electric and magnetic fields.

20.2 LC circuits

- Consider a fully-charged capacitor C connected to an inductor L. When the switch is closed, charge will start to flow in the circuit, giving rise to a current $i = -dq/dt$ (minus sign since charge leaving the capacitor corresponds to a positive current).
- Applying the loop rule leads to the differential equation $q/C - L\,di/dt = 0$, or equivalently

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0 \tag{53}$$

- This equation is similar to the simple harmonic oscillator equation for a mass m attached to a spring with spring constant k in the absence of friction: $m\,d^2x/dt^2 = -kx$, which has oscillatory solutions $x(t) = x_{\max}\cos(\omega t + \phi)$, with angular frequency $\omega \equiv \sqrt{k/m}$.
- Thus, for the LC circuit, charge (and current) will oscillate with angular frequency $\omega \equiv 1/\sqrt{LC}$.

20.3 RLC circuits

- Adding a resistor in series to the LC circuit changes the differential equation to

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \tag{54}$$

  with solutions corresponding to damped oscillations:

$$q(t) = q_{\max}\,e^{-t/2\tau}\cos(\omega t + \phi)\,, \qquad \omega \equiv \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{R^2C}{4L}} \tag{55}$$

- Depending on the sign of the quantity under the square-root, the solutions correspond to either under-damped (+), over-damped (−), or critically-damped (0) motion.
- Energy is now dissipated in the resistor, analogous to a frictional force $F_f = -bv$ in a *damped* mass-spring system.
- The correspondence between the mechanical and electrical oscillators are: $L \leftrightarrow m$; $1/C \leftrightarrow k$; $R \leftrightarrow b$; $q \leftrightarrow x$; $i \leftrightarrow v$.

21 Maxwell's equations and electromagnetic waves

21.1 Displacement current

- As mentioned in previous lectures, Faraday discovered that a changing magnetic field induces an electric field (Faraday's law of induction).
- Maxwell realized that, likewise, a changing electric field induces a magnetic field. This observation requires adding a *displacement current*

$$I_{\rm d} \equiv \epsilon_0 \, d\Phi_E/dt \tag{56}$$

  to the conduction current $I_{\rm enc}$ in Ampère's law, where Φ_E is the electric flux through a surface S spanning a closed curve C.
- The displacement current is necessary for electric charge conservation, as illustrated by charging a parallel-plate capacitor. (One can consider a surface S spanning a closed curve C that does not interesect the wire but extends into the region between the plates of the capacitor.)

21.2 Maxwell's equations

- Including the effects of changing magnetic and electric fields, the full set of equations for electricity and magnetism (called Maxwell's equations) are:

$$\oint_S \vec{E} \cdot \hat{n} \, dA = \frac{q_{\rm enc}}{\epsilon_0} \quad \text{(Gauss's law)} \tag{57}$$

$$\oint_C \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law)} \tag{58}$$

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 \left(I_{\rm enc} + \epsilon_0 \frac{d\Phi_E}{dt} \right) \quad \text{(Ampere's law with Maxwell's displacement current)} \tag{59}$$

$$\oint_S \vec{B} \cdot \hat{n} \, dA = 0 \quad \text{(no magnetic monopoles)} \tag{60}$$

- Maxwell's equations and the Lorentz force law:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \, , \qquad \vec{F} = d\vec{p}/dt \tag{61}$$

  describe *all* of electrodynamics.

21.3 Electromagnetic waves

- Maxwell's equations in vacuum admit *wave* solutions—disturbances in the electric and magnetic fields that propagate through space with speed

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 2.998 \times 10^8 \text{ m/s} \equiv c \tag{62}$$

- The speed of electromagnetic waves in vacuum is the *same* as the speed of light c, which led Maxwell to infer that light is an example of an electromagnetic wave. (The relationship between light and electricity and magnetism was not known before.)
- Electromagnetic waves can propagate in vacuum (in the absence of charges and currents), because changes in the electric field induce magnetic fields, whose changes induce electric fields, etc.
- Visible light is just a small part of the electromagnetic *spectrum*, consisting of wavelengths between 400 nm (violet light) and 700 nm (red light). Infrared radiation, microwaves, and radio waves have longer wavelengths. Ultraviolet radiation, X-rays, and gamma rays have shorter wavelengths.

22 Nature of light, ray optics

22.1 Nature of light

- As shown in the previous lecture, light is an electromagnetic wave. But depending on the type of measurement you make, light will sometimes behave as a wave, and other times behave as a stream of particles.
- The wave nature of light is most evident in effects involving *interference*—i.e., the fact that waves can combine coherently or cancel out with other waves.
- The particle nature of light is most evident when discussing the interaction of light with electrons—e.g., the Compton effect (scattering of light by electrons) or the photoelectric effect (the ejection of electrons from a piece of metal when light is shined on it).

22.2 Measuring the speed of light

- Light travels with speed $c = 2.998 \times 10^8$ m/s in vacuum. It travel slower in other materials, e.g., $v = 3c/4$ in water.
- Galileo tried measuring the speed of light having two people uncover lanterns ~ 10 km apart. This attempt was unsucessful since human reaction time is much longer than the time it takes light to travel that distance.
- Romer estimated the speed of light to be 2.3×10^8 m/s by monitoring the time it took Jupiter's moon Io to pass behind it. If the Earth were moving toward Jupiter, the time for Io's passage was less than that when the Earth was moving away from Jupiter, similar to the Doppler effect for sound.
- Fizeau estimated the speed of light to be 3.1×10^8 m/s by using a rotating wheel with teeth and notches to let light pass from a source to a distant mirror and back.

22.3 Ray approximation

- The ray approximation for light assumes that light travels in a straight line in a uniform medium, in a direction perpendicular to the wavefronts.
- The ray approximation is a good approximation when the wavelength of the light is much smaller than the size of any obstacle or inhomogenity in the medium. (Otherwise, one must use wave optics.)
- For example, in the ray approximation, light casts sharp shadows as it passes through an opening.

22.4 Reflection

- When light is incident on a smooth surface between two different media, part of the light is reflected back into the first medium while the rest of it is transmitted into the second medium.
- For reflection off of a smooth surface, the angle of incidence equals the angle of reflection, $\theta_1 = \theta_1'$, where the angles are measured with respect to the normal to the surface.
- For a rough surface, parallel rays of light will be reflected in all different directions. Most objects are rough surfaces with respect to reflection.
- A *corner reflector*, made by placing three flat mirrors next to one another at right angles, has the property that the incident light is reflected back along the same direction from which it came. This is useful for reflectors on bicycles and cars.
- The Apollo 11 astronauts put corner reflectors on the Moon. These mirrors reflect back light sent from lasers here on Earth, used to monitor the Earth-Moon separation to a precision of ~ 15 cm.
- The law of reflection can be proved using *Fermat's principle of least time*, which says that travels between two points so as to minimize the travel time between the two points.

23 Ray optics (continued)

23.1 Refraction

- In addition to light rays reflecting off of a surface between two media, they can also be transmitted from one medium into the other.
- If $v_{1,2}$ denote the speeds of light in the two media, then the law of refraction (called Snell's law) is

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \qquad \text{or, equivalently} \qquad n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{63}$$

  where $\theta_{1,2}$ are the angles that the rays of light make with the normal to the surface between media 1,2.
- The second equation is expressed in terms of *indices of refraction*

$$n \equiv c/v \tag{64}$$

  for the two media. Since light travels fastest in vacuum, $n \geq 1$. For air, $n \approx 1.000293$; for water $n \approx 1.33$; for (crown) glass, $n \approx 1.5$.
- One can prove Snell's law using Fermat's principle of least time, similar to the proof for the law of reflection. (Lifeguard-and-drowning-swimmer analogy.)
- As light travels from a medium with a lower index of refraction to one with a higher index of refraction ($n_1 < n_2$), the rays are bent *toward* the normal (opposite for $n_1 > n_2$).
- Demo: A pencil appears to be bent if observed obliquely in a glass of water.

23.2 Dispersion

- Dispersion refers to the fact that the index of refraction of a material depends, in general, on the wavelength of the light propagating through it.
- Demo: "White" light passing through a prism is split into its spectrum of colors (violet light is refracted through a greater angle than red light, since violet light has a larger index of refraction).
- Dispersion is responsible for the production of a *rainbow* as "white" light from the Sun passes through a rain drop. (The light is actually refracted, then reflected, and then refracted again by the rain drop, causing the red band of the rainbow to be above the violet band).
- A *secondary* rainbow is formed when the light is reflected twice by the back surface of the rain drop.

23.3 Total internal reflection

- Total internal reflection occurs when light propagates from a medium with higher index of refraction to one with lower index of refraction ($n_1 > n_2$), provided the incident angle is greater than the *critical angle* θ_c defined by

$$\sin\theta_\mathrm{c} \equiv n_2/n_1 \tag{65}$$

- Note that for $\theta_1 = \theta_c$, the refracted angle $\theta_2 = 90°$, moving along the surface between the two media.
- Light can be made to follow the curved trajectory of a plastic fiber, provided the curvature of the fiber is not too sharp. This is the basic principle underlying light transfer using *fiber optics*.
- Fiber optics has applications for both the telecommunications industry and medical devices.

24 Image formation by flat and spherical mirrors

24.1 Flat mirrors

- Using angle of incidence equals angle of reflection, one can show that for a flat mirror, a virtual image is produced, which is upright, has the same size as the object ($h' = h$), and is as far behind the mirror as the object is in front of it ($q = -p$).
- NOTE:

  (i) The apparent left-right reversal (right hand of a person seen as the left hand of the image) is actually a front-back reversal.

  (ii) You only need a mirror half your height to see your full body, and it doesn't depend on how far away you are from the mirror.

  (iii) a double-mirror from from two flat mirrors intersecting at a right angle allows you to see yourself as others do (since the doubly-reflected image doesn't have any apparent left-right reversal).

24.2 Spherical mirrors

- Spherical mirrors are formed from spherical reflecting surfaces having radius of curvature R.
- They can either be *concave*, like a shaving mirror; or *convex*, like a passenger-side mirror on a car.
- Tracing rays of light from an object to the image using the law of reflection and simple geometry, one can show that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}\,, \qquad \frac{1}{f} \equiv \frac{2}{R} \tag{66}$$

  where p and q are the object and image location, respectively.
- The second equation relates the radius of curvature of the mirror to the *focal length* f. In the *paraxial* approximation (where rays of light don't deviate much from the main axis of the mirror), parallel rays of light converge to or diverge from a single focal point F.
- For spherical mirrors, parallel rays of light far from the main axis don't all meet at a single point. This is called *spherical aberration.*
- For parabolic mirrors, *all* parallel rays meet at a single point.
- The sign conventions for the mirror equation are:

  $p > 0$ if the object is in front of the reflecting surface of the mirror;

  $q > 0$ if the image is in front of the mirror;

  $h' > 0$ if the image is upright;

  $f, R > 0$ if the mirror is concave.

24.3 Ray tracing for concave and convex mirrors

- For a concave mirror, parallel rays of light meet at the focal point in front of the mirror.
- For a convex mirror, parallel rays of light diverge from the focal point in back of the mirror.
- By tracing rays of light reflecting off of a mirror, one can show that:

  (i) a concave mirror produces both real and virtual images (if $p > f$ and $p < f$, respectively).

  (ii) a convex mirror *always* produces virtual images that are upright and reduced in size.
- If $p = f$ for a concave mirror, no image is formed (formally, $q \to \infty$).
- Note that real images are always inverted (relative to the orientation of the object), while virtual images are always upright.

25 Image formation by refraction and thin lenses

25.1 Image formation by refraction

- Images can also be formed by refraction of rays of light from one medium into another.
- We will consider a spherical surface between two media with indices of refraction n_1 and n_2, respectively. R will denote the radius of curvature of the spherical surface.
- Tracing rays of light from the object point O to the image point I, using Snell's law and trigonometry in the paraxial approximation, one finds

  $$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R} \tag{67}$$

  where p and q denote the object and image location, respectively.
- The sign conventions for this equation are:

  $p > 0$ if the object is in medium 1, in front of the surface;

  $q > 0$ if the image is in medium 2, in back of the surface;

  $h' > 0$ if the image is upright;

  $R > 0$ if the center of curvature is in back of the surface.
- Application: For a flat surface, $R \to 0$ implying $n_1/p = -n_2/q$. Using this simplified formula, one can show that the apparent depth of an object in water is less than the actual depth by a factor of $n_2/n_1 = n_{\text{air}}/n_{\text{water}} = 3/4$. (See Example 36.7.)

25.2 Thin lenses

- For the case of thin lenses (made of e.g., glass, with index of refraction n) in air, one needs to consider the formation of images by refraction for both the front and back surfaces of the lens.
- Taking the image formed by refraction from surface 1 as the object for the image formed by refraction from surface 2, one obtains (in the limit that the thickness of the lens goes to zero):

  $$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}\,, \qquad \frac{1}{f} \equiv (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{68}$$

  where R_1 and R_2 are the radii of curvature for the two surfaces of the lens.
- The second equation is called the "lens maker's equation", where f is the focal length of the thin lens.
- The sign conventions for the thin lens equation are the same as those for images formed by refraction from a single surface, with the additional conventions that $R_{1,2} > 0$ if the center of curvature of surface 1,2 is in back of the lens; and $f > 0$ for a converging lens.

25.3 Ray tracing for converging and diverging lenses

- A *converging* lens is thicker in the middle than at the edges; parallel rays of light meet at the focal point in back of the converging lens, in the paraxial approximation.
- A *diverging* lens is thinner in the middle than at the edges; parallel rays of light diverge from the focal point in front of the diverging lens, in the paraxial approximation.
- By tracing rays of light through a lens, one can show that:

  (i) a converging lens produces both real and virtual images (if $p > f$ and $p < f$, respectively).

  (ii) a diverging lens *always* produces virtual images that are upright and reduced in size.
- Converging and diverging lenses behave similarly to concave and convex mirrors.