5. Fourier Analysis and Synthesis of Waveforms

PURPOSE AND BACKGROUND

The simplest sound is a pure sine wave with a single frequency and amplitude. Most sound sources and instruments do not produce such simple waves. Usually their sound contains many sine waves with higher frequencies, called *harmonics.* These act together according to the *superposition principle* to produce a complex tone. This addition of sine waves with suitable amplitudes and phases is called *Fourier synthesis of sound*. The opposite, the decomposition of sound into its sine-wave components, is called *Fourier analysis*. Periodic sound can be synthesized or analyzed with a sufficient number of sine waves. A *pure* tone is a sine wave with a single frequency. Many sine waves added together form a complex tone and waveform periodic in time. This laboratory is about the analysis and synthesis of sound and how electronic *synthesizers* can mimic real instruments.

I Fourier Synthesis of Waveforms

For our experiment, we will use the Fourier Series Applet, which is available online at `https://www.falstad.com/fourier/`. Listen to several available waveforms (e.g., sine, triangular, square, and sawtooth waves) at a fundamental frequency of $f_1 = 500$ Hz. To do so, you will need to adjust the "Playing Frequency" slider as close as you can to 500 Hz, and then check the "Sound" box to listen to the sounds.

Questions

1. Draw sketches of the four waveforms.
2. Which waveform most resembles a pure sine wave?
3. Which waveform least resembles a pure sine wave?
4. Which tone sounds least like the pure sine wave?

Complex waveforms are produced by adding sine waves of different frequencies and amplitudes. The tone heard in all four cases has the same *pitch* or fundamental frequency $f_1 = 500$ Hz. For a pure tone (sine wave), the fundamental is the only frequency present. For complex tones, sine waves with integer multiples of the fundamental frequency and suitable amplitudes are added together. For example, the next integer multiples of the fundamental $f_1 = 500$ Hz are $f_2 = 2f_1 = 1000$ Hz, $f_3 = 3f_1 = 1500$ Hz, and so on.

These higher frequencies are called *overtones* or *harmonics*. Just like the fundamental, each overtone has a single frequency. A complex waveform can be produced with the fundamental plus higher harmonics of suitable amplitudes. This process is called *superposition* of waves or, mathematically speaking, *Fourier synthesis* of waves. Conversely, you can take a complex waveform apart by decomposing it with a spectrum analyzer into its individual harmonics. This is called *Fourier analysis* of waves.

A Sawtooth Waveform

The harmonics of the sawtooth wave follow a simple pattern. All harmonics exist from $N = 1$ to $N = \infty$, with amplitudes given by $A_N = A_1/N$, where A_1 is the amplitude of the fundamental frequency. Thus all integer multiples of the fundamental frequency contribute to the waveform. Since in practice we cannot add an infinite number of harmonics, we shall use only the first five non-zero harmonics and add them up.

Using the online app, start by clicking the "Sine" box. You should see a single white dot, sticking up above the rest, at a height corresponding to the amplitude of the first harmonic. The second harmonic $N = 2$, $f_2 = 1000$ Hz should have an amplitude $A_2 = A_1/2$ for a sawtooth wave. Add this harmonic to the fundamental by adjusting the height of the second white dot to half the height of the first white dot. Take a look at and listen to the waveform generated.

Find the frequencies of the next three higher harmonics and their relative amplitudes in percent. Complete the entries in Table 1.

Table 1: Sawtooth waveform

N	f_N	A_N
1	500Hz	100%
2	1000Hz	50%

Continue adding harmonics (3rd, 4th, 5th, etc.) by appropriately adjusting the heights of the white dots. Note the changes in the tone and the waveform. With each addition of a harmonic, the waveform should look more and more like a sawtooth.

B Square Wave

The square or rectangular waveform is similar to the sawtooth in that the amplitudes of the harmonics follow the $A_N = A_1/N$ dependence. However, the major difference is that only the *odd* harmonics $N = 1$, $N = 3$, $N = 5$, etc., contribute.

Use this information and complete the entries in Table 2 for the square wave.

Table 2: Square wave

N	f_N	A_N
1	500 Hz	100%
3	1500 Hz	33.33%

Synthesize a square wave using the online app by starting as before with just a "Sine", and then successively adding the higher harmonics with the amplitudes given in Table 2. Note the changes in tone and shape of the waveform as more harmonics are added.

C Triangular Waveform

The triangular wave is similar to the square wave in that it too consists of odd harmonics only. However, the amplitudes no longer follow the $A_N = A_1/N$ dependence, but rather a $A_N = \pm A_1/N^2$ dependence. The sign of the amplitude alternates +, $-$, +, $-$, etc., for the 1st, 3rd, 5th, 7th harmonics, etc. For instance, given an amplitude of the first harmonic of 100%, the amplitude of the third harmonic now is $A_3 = -A_1/3^2 = -11.11\%$.

Complete the entries in Table 3 for the triangular waveform.

Table 3: Triangular waveform

N	f_N	A_N
1	500 Hz	100%
3	1500 Hz	−11.11%

Use the completed Table 3 to synthesize a triangular waveform using the online app and listen to the result.

Questions

1. Of the three waveforms, which had the least noticeable contributions from its overtones to the overall form and tone?
2. Which of the three waveforms had the most noticeable contributions from its overtones?
3. How could you get sharper "edges" on the square and sawtooth waveform than those created with just five non-zero harmonics?
4. Why can you hear a 1 Hz square wave?

II Fourier Analysis of Waveforms

The "FFT Analyzer Tool" in the Electroacoustic Toolbox analyzes an incoming signal with a mathematical operation called a *Fast Fourier Transform* (FFT) to identify the different frequencies in the signal. The display is a frequency spectrum—see Figure 1. For a sine wave, the FFT tool will show a frequency spectrum with one peak for the only frequency present (the fundamental), with the amplitude being the height of the peak. For non-sinusoidal periodic waveforms you will see many peaks. The location of the peaks and their relative amplitudes follow the theoretical expressions that we used above to synthesize sawtooth, square, and triangular waveforms.

A Musical Synthesizers

Modern keyboards are capable of simulating sounds from real instruments quite well. They work on the basis of *Fourier analysis and synthesis*. Every tone from a given instrument has its own *timbre* and Fourier spectrum. The fundamental frequency determines the *pitch* of the tone. Often the fundamental does not have the highest amplitude. Some higher harmonics may be stronger. Nonetheless, the ear discerns the frequency of the fundamental as the pitch of the tone. Musical instruments produce sound with complex Fourier spectra. These change with every note. For example, the Fourier spectra of "middle C" ($f = 261.63$ Hz) from a violin and a viola or bassoon look quite different.

B Real and Synthesized Sound of a Didgeridoo

An example of the spectrum from a didgeridoo and the corresponding synthesized tone is shown in Figure 1. The synthesized tone sounds similar to the actual one, but not quite the same.

Questions

1. Why does the synthesized tone not sound exactly like the real tone from a didgeridoo?

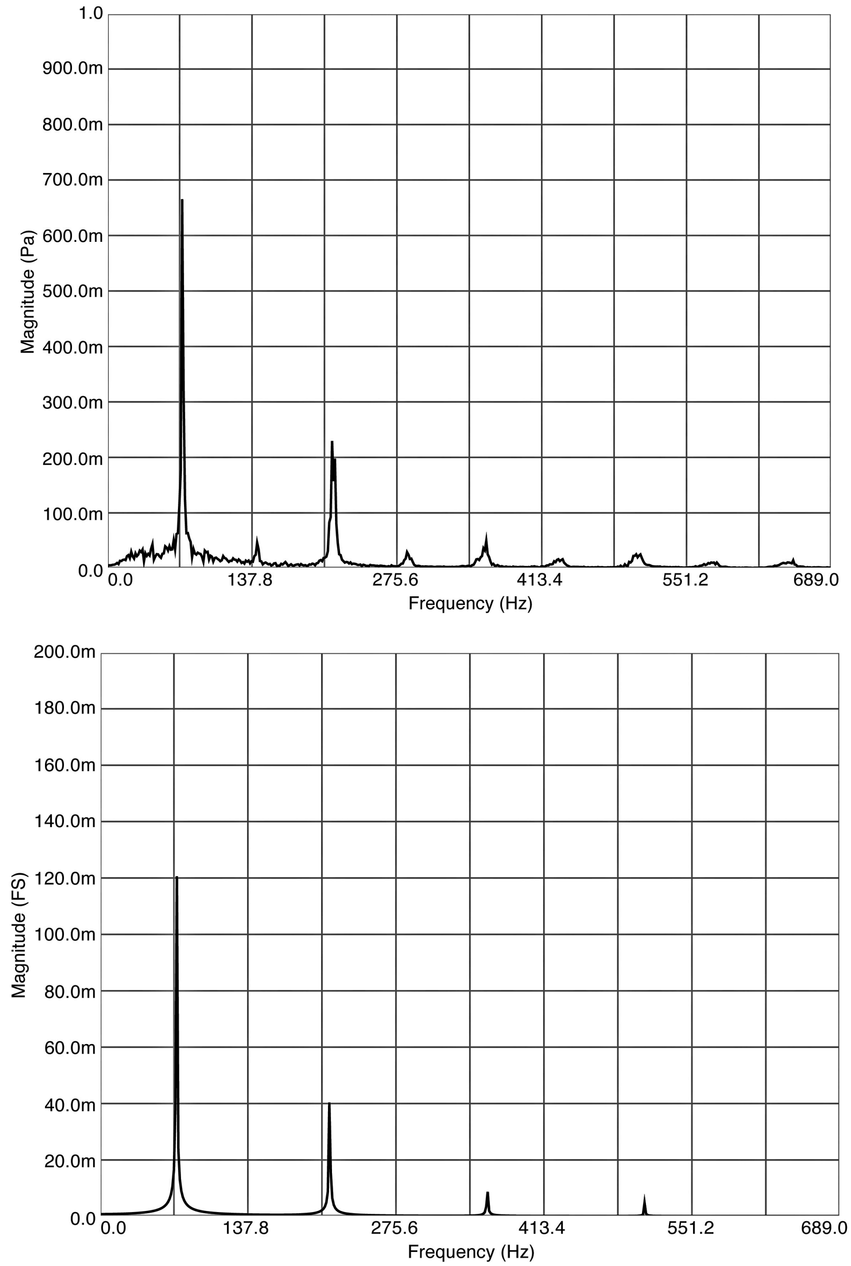

Figure 1: Top: Actual sound spectrum of the note D2 from a didgeridoo. The odd harmonics dominate, as expected for a "closed tube". Bottom: Synthesized sound spectrum, using only the first four odd harmonics $N = 1, 3, 5, 7$.